Chapter 13 Answers

1. At a concert on campus, 20% of people purchase Zone A tickets, for $47 each. The other 80% of people purchase Zone B tickets, for $38 each. If five people are selected at random, what is the expected revenue from these five ticket sales?

Method #1. Let $X$ be the number of Zone A tickets among the five people. Then there are $5 - X$ Zone B tickets, so the revenue is $47X + 38(5 - X) = 9X + 190$. The expected value of $X$ is:

$$E[X] = (0)(.80^5) + (1)(5)(.20)(.80^4) + (2)(10)(.20^2)(.80^3) + (3)(10)(.20^3)(.80^2) + (4)(5)(.20^4)(.80) + (5)(.20^5) = 1,$$

and thus the expected revenue is $E[9X + 190] = 9E[X] + 190 = (9)(1) + 190 = 199$.

Method #2. Instead, let $X$ denote the total revenue. Let $X_j$ denote the revenue from the $j$th person. So $X = X_1 + X_2 + X_3 + X_4 + X_5$. The expected revenue from the $j$th person is $(.20)(47) + (.80)(38) = 39.8$. So the expected total revenue is $E[X] = E[X_1 + \cdots + X_5] = E[X_1] + \cdots + E[X_5] = 39.8 + 39.8 + 39.8 + 39.8 + 39.8 = 199$.

2. Four students order noodles at a certain local restaurant. Their orders are placed independently. Each student is known to prefer Japanese pan noodles 40% of the time (it is a very popular and tasty dish!). Let $X$ be the number of students who order Japanese pan noodles. What is the variance of $X$?

Method #1. Let $X$ be the number of the students who eat Japanese pan noodles. The mass of $X$ is:

$$p_X(0) = (.60)^4 = .1296$$
$$p_X(1) = (4)(.40)(.60)^3 = .3456$$
$$p_X(2) = (6)(.40)^2(.60)^2 = .3456$$
$$p_X(3) = (4)(.40)^3(.60) = .1536$$
$$p_X(4) = (.40)^4 = .0256$$

So

$$E[X] = (0)(.1296) + (1)(.3456) + (2)(.3456) + (3)(.1536) + (4)(.0256) = 1.6$$

and

$$E[X^2] = (0^2)(.1296) + (1^2)(.3456) + (2^2)(.3456) + (3^2)(.1536) + (4^2)(.0256) = 3.52$$
so the variance of \( X \) is

\[
\text{Var}(X) = E[X^2] - (E[X])^2 = 3.52 - (1.6)^2 = .96.
\]

Method #2. Let \( X \) be the number of the students who eat Japanese pan noodles. Let \( X_j \) be an indicator random variable for whether or not the \( j \)th student ate Japanese pan noodles, namely, \( X_j = 1 \) if the \( j \)th student ate the noodles, and \( X_j = 0 \) otherwise. So always \( X = X_1 + X_2 + X_3 + X_4 \). We have

\[
\]

Also (a bit creative, but we will do more of this):

\[
E[X^2] = E[(X_1 + X_2 + X_3 + X_4)(X_1 + X_2 + X_3 + X_4)],
\]

If we expand \( E[(X_1 + X_2 + X_3 + X_4)(X_1 + X_2 + X_3 + X_4)] \), we get 16 terms altogether. Four of the terms are of the form \( E[X_iX_j] \), but always \( X_iX_j = X_j \) since \( X_j \) is just 0 or 1, so \( E[X_iX_j] = E[X_j] = .40 \). The other twelve terms are of the form \( E[X_iX_j] \) for \( i \neq j \), so \( X_i \) and \( X_j \) do not affect each other and are independent, so \( E[X_iX_j] = P(X_iX_j = 1) = P(X_i = 1 \text{ and } X_j = 1) = P(X_i = 1)P(X_j = 1) = .40^2 \). Thus \( E[X^2] = (4)(.40) + (12)(.40^2) = 3.52 \). So we get \( \text{Var}(X) = 3.52 - (1.6)^2 = .96 \), as before.

3. Roll two dice; let \( X \) denote the maximum of the two values that appear. Find \( E[X] \).

The mass of \( X \) is:

\[
\begin{align*}
p_X(1) &= 1/36 \\
p_X(2) &= 3/36 \\
p_X(3) &= 5/36 \\
p_X(4) &= 7/36 \\
p_X(5) &= 9/36 \\
p_X(6) &= 11/36
\end{align*}
\]

So

\[
\]

and

\[
\]

so the variance of \( X \) is

\[
\text{Var}(X) = E[X^2] - (E[X])^2 = 791/36 - (161/36)^2 = 2555/1296 \approx 1.97.
\]
4. Three hundred little plastic yellow ducks are dumped in a pond; one of them contains a prize tied around its foot. Leonardo examines each duck until he discovers the prize. He discards each duck without a prize after he checks it, so that he never needs to check a duck more than one time!

Find the variance of the number of ducks he checks until he discovers the prize. Please carefully justify your answer.

Hint: You might find it useful to know, as discussed in Example 13.11, that

$$1^2 + 2^2 + \cdots + n^2 = \frac{(n)(n+1)(2n+1)}{6}.$$ 

Let $X$ be the number of ducks checked until the prize appears. We have $P(X = j) = 1/300$ for each $j$. So the expected value of $X$ is:

$$E[X] = (1)P(X = 1) + (2)P(X = 2) + (3)P(X = 3) + \cdots + (300)P(X = 300)
= (1)(1/300) + (2)(1/300) + (3)(1/300) + \cdots + (300)(1/300)
= (1 + 2 + \cdots + 300)(1/300)
= (1/300)((301)(300)/2)
= 301/2
= 150.5$$

The expected value of $X^2$ is

$$E[X^2] = (1^2)P(X = 1) + (2^2)P(X = 2) + (3^2)P(X = 3) + \cdots + (300^2)P(X = 300)
= (1^2)(1/300) + (2^2)(1/300) + (3^2)(1/300) + \cdots + (300^2)(1/300)
= (1^2 + 2^2 + \cdots + 300^2)(1/300)
= \frac{(300)(300 + 1)((2)(300) + 1)}{6} \left( \frac{1}{300} \right)
= 180901/6$$

We used the helpful fact that $1^2 + 2^2 + \cdots + n^2 = \frac{(n)(n+1)(2n+1)}{6}$. So the variance of $X$ is

$$Var(X) = 180901/6 - (301/2)^2 = 89999/12 \approx 7499.92.$$