Chapter 12 Answers

1. Jack and Jill are independently struggling to pass their last (one) class required for graduation. Jack needs to pass Calculus III, but he only has probability .30 of passing. Jill needs to pass Advanced Pharmaceuticals, but she only has probability .46 of passing. They work independently.

As in Problem Set 11, let $X$ be 0, 1, or 2, if (respectively) neither, one, or both of them graduate. Let $Y$ indicate if Jack graduates, so $Y = 1$ if Jack graduates, and $Y = 0$ otherwise. Let $Z$ indicate if Jill graduates, so $Z = 1$ if Jill graduates, and $Z = 0$ otherwise. Notice that $X = Y + Z$ always. Now find $E[X]$ using $E[Y]$ and $E[Z]$.

(The solution should be quick and easy, and you know from Problem Set 11 that $E[X]$ will be .76, of course!)

Since $Y$ is an indicator random variable, then $E[Y] = P(Y = 1)$. Similarly, $E[Z] = P(Z = 1)$. So we get


2. Four students order noodles at a certain local restaurant. Their orders are placed independently. Each student is known to prefer Japanese pan noodles 40% of the time (it is a very popular and tasty dish!). How many of them do we expect to order Japanese noodles?

Let $A_1, A_2, A_3, A_4$ be the events that (respectively) the first, second, third, or fourth person orders Japanese noodles. Let $X_1, X_2, X_3, X_4$ be indicator random variables for (respectively) $A_1, A_2, A_3, A_4$. Please justify your answer using the values of the $E[X_j]$’s.

Since each $X_j$ is an indicator function, then $E[X_j] = P(X_j = 1) = P(A_j) = .40$ for each $j$. So the expected number of students who eat Japanese pan noodles is


3. Three hundred little plastic yellow ducks are dumped in a pond; one of them contains a prize tied around its foot. Leonardo examines each duck until he discovers the prize. He discards each duck without a prize after he checks it, so that he never needs to check a duck more than one time!

How many ducks does he expect to check until he discovers the prize? Please carefully justify your answer.
Method #1. Let $A_j$ be the event that the $j$th duck checked is the one with the prize, so $P(A_j) = 1/300$ for each $j$. Let $X_j$ be the indicator for $A_j$, i.e., $X_j$ indicates whether the $j$th duck checked is the one with the prize, so $X_j = 1$ if the $j$th duck checked is the one with the prize, and $X_j = 0$ otherwise. So $E[X_j] = P(A_j) = 1/300$ for each $j$. Also, $X = 1X_1 + 2X_2 + 3X_3 + \cdots + 300X_{300}$. because (for instance) if the 3rd duck checked is the one with the prize, then $X_3 = 1$, and the other $X_j$’s are 0, so $X = 3X_3 = 3$, as desired. So

$$E[X] = E[1X_1 + 2X_2 + 3X_3 + \cdots + 300X_{300}] = 1E[X_1] + 2E[X_2] + 3E[X_3] + \cdots + 300E[X_{300}]$$

$$= 1(1/300) + 2(1/300) + 3(1/300) + \cdots + 300(1/300)$$

$$= (1/300)((301)(300)/2)$$

$$= 301/2$$

$$= 150.5.$$

So Leonardo expects to check 150.5 ducks to find the prize. As in Chapter 11, we used the helpful fact that $1 + 2 + \cdots + n = (n)(n+1)/2$.

Method #2. Let $A_j$ be the event that $j$ or more ducks need to be checked, so $P(A_j) = 1 - \frac{j-1}{300}$, because $A_j$ occurs if and only if the first $j-1$ ducks do not have the prize. Let $X_j$ be the indicator for $A_j$, i.e., $X_j$ indicates whether $j$ or more ducks need to be checked, so $X_j = 1$ if $j$ or more ducks need to be checked, and $X_j = 0$ otherwise. So $E[X_j] = P(A_j) = 1 - \frac{j-1}{300}$ for each $j$. Also, $X = X_1 + X_2 + \cdots + X_{300}$. because (for instance) if exactly 3 draws are needed, then $X_1, X_2, X_3$ are each equal to 1, and the other $X_j$’s are 0, so $X = X_1 + X_2 + X_3$, as desired. So

$$E[X] = E[X_1+X_2+\cdots+X_{300}] = E[X_1]+E[X_2]+\cdots+E[X_{300}] = \sum_{j=1}^{300} \left(1 - \frac{j-1}{300}\right) = 300 - \frac{\sum_{j=1}^{300} (j-1)}{300}$$

but $\sum_{j=1}^{300} (j-1) = 0 + 1 + 2 + \cdots + 299 = (300)(299)/2$, since $1 + 2 + \cdots + n = (n)(n+1)/2$. So

$$E[X] = 300 - \frac{(300)(299)/2}{300} = 300 - \frac{299}{2} = \frac{600 - 299}{2} = 301/2 = 150.5$$

Method #3. (It is 100 percent OK to skip this method—this method is only included here for readers who know and enjoy recursion; if the reader does not know recursion, it is safe to skip the method altogether!) We now consider various numbers of ducks at the start. Let $X_j$ denote the number of ducks needed to be checked to find the prize. Then we claim that $E[X_j] = (j+1)/2$ for all $j$, and we prove it by induction. If $j = 1$, then the first duck is the only duck, so it must have the prize, so $E[X_1] = (1+1)/2 = 1$; this shows the base case. Now we handle the inductive step: If $E[X_{j-1}] = j/2$, we show that $E[X_j] = (j+1)/2$. Indeed, check any one of the $j$ ducks. The one you pick is the duck with the prize, with probability $1/j$. The one you pick is not the duck with the prize, with probability $(j-1)/j$, and in such a case, this duck was checked, plus an additional $E[X_{j-1}]$ ducks will need to be checked, because the problem is essentially starting again with $j-1$ ducks. So

$$E[X_j] = \frac{1}{j} + \frac{j-1}{j} (1 + E[X_{j-1}]) = \frac{1}{j} + \frac{j-1}{j} \left(1 + \frac{j}{2}\right) = \frac{1}{j} + \frac{j-1}{j} + \frac{j-1}{2} = 1 + \frac{j-1}{2} = \frac{j+1}{2}.$$ 

In particular, when starting with 300 ducks, $E[X_{300}] = 301/2$. 

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4. The Super Breakfast Challenge (SBC) consists of bacon, eggs, oatmeal, orange juice, milk, and several other foods, and it costs $12.99 per person to order at a local restaurant. It is known to be very difficult to consume the entire SBC. Only 10% of people are able to eat all of the SBC. The other 90% of people will be unable to eat the whole SBC (it is too much food!).

A probability student hears about the SBC and goes to the local restaurant. He observes the number of customers, \( X \), that attempt to eat the SBC, until the first success. So if there are 4 failures and then 1 success (i.e., the outcome is \((F, F, F, F, T)\)), then \( X = 5 \).

Find the expected value of \( X \), i.e., the number of customers expected to try the SBC until the first success. Justify your answer completely, using one of the methods from Chapter 12.

Method #1. Let \( A_j \) be the event that \( j \) or more customers are required, so \( P(A_j) = (.90)^{j-1} \), because \( A_j \) occurs if and only if the first \( j - 1 \) customers cannot finish the breakfast. Let \( X_j \) be the indicator for \( A_j \), i.e., \( X_j \) indicates whether \( j \) or more customers are needed, so \( X_j = 1 \) if \( j \) or more customers are needed, and \( X_j = 0 \) otherwise. So \( E[X_j] = P(A_j) = (.90)^{j-1} \) for each \( j \). Also, \( X = X_1 + X_2 + X_3 + \cdots \) because (for instance) if exactly 3 draws are needed, then \( X_1, X_2, X_3 \) are each equal to 1, and the other \( X_j \)'s are 0, so \( X = X_1 + X_2 + X_3 \), as desired. So

\[
E[X] = E[X_1 + X_2 + X_3 + \cdots] = E[X_1] + E[X_2] + E[X_3] + \cdots = 1 + .90 + (.90)^2 + \cdots
\]

Thus

\[
E[X] = \sum_{j=0}^{\infty} (.90)^j = \frac{1}{1 - .90} = \frac{1}{.10} = 10
\]

Method #2. (Does not use indicators!) Let \( X \) be the number of flips that are necessary. With probability .10, the first customer is able to successful eat the SBC. With probability .90, the first customer is unable to eat the SBC, and the problem essentially starts over again: in this case, the original customer tried the SBC, plus \( E[X] \) more people will need to try the SBC. So we have

\[
E[X] = (.10)(1) + (.90)(1 + E[X]) = .10 + .90 + .90E[X] = 1 + .90E[X]
\]

Subtracting .90\(E[X]\) from both sides, we get .10\(E[X] = 1\), and we conclude that \( E[X] = 10 \).