Chapter 10 Answers

1. Ten students sign up for a job opening, but only 1 of the students will be selected. The employer chooses randomly; all ten outcomes are equally likely. If person 3, 5, 7, or 9 gets the job, let \( X = 1 \); otherwise, \( X = 0 \). If person 1, 2, 3, 4, or 5 gets the job, let \( Y = 1 \); otherwise, \( Y = 0 \). Are \( X \) and \( Y \) independent random variables? Justify your answer.

Yes, \( X \) and \( Y \) are independent random variables. The sample space is \( \{1, 2, \ldots, 10\} \). The random variable \( X \) is an indicator for the event \( \{3, 5, 7, 9\} \), which we shall call event \( A \); the random variable \( Y \) is an indicator for the event \( \{1, 2, 3, 4, 5\} \), which we shall call event \( B \). So it is enough to prove that events \( A \) and \( B \) are independent. We see that \( P(A) = \frac{4}{10} = \frac{2}{5} \) and \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{3, 5\})}{P(\{1, 2, 3, 4, 5\})} = \frac{2/10}{5/10} = \frac{2}{5} \). Thus \( P(A \mid B) = P(A) \). So \( A \) and \( B \) are independent events. So \( X \) and \( Y \) are independent random variables.

2. Each day, Maude has a 1% chance of losing her cell phone (her behavior on different days is independent). Each day, Maude has a 3% chance of forgetting to eat breakfast (again, her behavior on different days is independent). Her breakfast and cell phone habits are independent.

Let \( X \) be the number of days until she first loses her cell phone. Let \( Y \) be the number of days until she first forgets to eat breakfast. (Here, \( X \) and \( Y \) are independent.) Find the joint mass of \( X \) and \( Y \).

The joint mass of \( X \) and \( Y \) is

\[
p_{X,Y}(x, y) = (0.99)^{x-1}(0.01)(0.97)^{y-1}(0.03),
\]

for all pairs of positive integers \( x \) and \( y \), and \( p_{X,Y}(x, y) = 0 \) otherwise.
3. A student flips a fair coin until heads appears. Let \( X \) be the numbers of flips until (and including) this first head. Afterwards, he begins flipping again until he gets another head. Let \( Y \) be the number of flips, after the first head, up to (and including) the second head. E.g., if the sequence of flips is TTTTTTHTTH then \( X = 7 \) and \( Y = 3 \).

Are \( X \) and \( Y \) independent? Justify your answer.

Yes, \( X \) and \( Y \) are independent random variables. Here is a simple argument to show that, using the fact that \( X \) and \( Y \) are independent if the joint mass of \( X \) and \( Y \) equals the product of the mass of \( X \) times the mass of \( Y \).

For any event consisting of just one outcome, namely, a sequence of \( x-1 \) tails, and then a head, and then \( y-1 \) tails, and then a head, the probability of such an event is \((1/2)^{x-1}(1/2)^{y-1}(1/2)\). Thus the joint mass of \( X \) and \( Y \) is
\[
p_{X,Y}(x,y) = (1/2)^{x-1}(1/2)^{y-1}(1/2),
\]
for all positive integers \( x \) and \( y \); and \( p_{X,Y}(x,y) = 0 \) otherwise. The mass of \( X \) is
\[
p_X(x) = (1/2)^{x-1}(1/2),
\]
for all positive integers \( x \); and \( p_X(x) = 0 \) otherwise. The mass of \( Y \) is
\[
p_Y(y) = (1/2)^{y-1}(1/2),
\]
for all positive integers \( y \); and \( p_Y(y) = 0 \) otherwise. So, indeed, the joint mass of \( X \) and \( Y \) is the product of the masses of \( X \) and \( Y \), i.e.,
\[
p_{X,Y}(x,y) = p_X(x)p_Y(y),
\]
for all \( x \) and \( y \). Thus \( X \) and \( Y \) are independent.

4. Same scenario as problem 3. Let \( Z \) be the total number of flips until (and including) the second head. So \( Z = X + Y \); e.g., in the example given, \( Z = 10 \). Are \( X \) and \( Z \) independent? Justify your answer.

No, \( X \) and \( Z \) are not independent random variables. Any little example can be used to show this.

For instance, \( P(X = 10) > 0 \), (indeed, \( P(X = 10) = (1/2)^9(1/2) = 1/1024 \), but this exact value is not even needed...we will only need the fact that \( P(X = 10) \) is positive—not zero—for this little argument). On the other hand, \( P(X = 10 \mid Z = 3) = 0 \), because it is impossible to take 10 flips to reach the first head, if we are given that it only takes 3 flips to reach the second head. Thus \( p_X(10) \neq p_{X \mid Z}(10 \mid 3) \). So the mass of \( X \) is not equal to the conditional mass of \( X \) given \( Z \). Thus \( X \) and \( Z \) are dependent random variables.