Chapter 4 Answers

1. Consider Example 3.4 from Chapter 3, and also used in Example 2.8 of Chapter 2. We use “T” and “F” to denote when a song does or does not belong to a student’s favorite type of music. For each song, the probability that the song is one of her favorite type can be called \( p \), and the probability that the song is not one of her favorite type is \( 1 - p \). Define

\[
A = \{(F, F, F, x_4, \ldots, x_{10}) \mid x_j \in \{T, F\}\},
\]

\[
B = \{(x_1, T, x_3, T, x_5, T, x_7, T, x_9, T) \mid x_j \in \{T, F\}\},
\]

\[
C = \{(x_1, x_2, x_3, x_4, x_5, T, T, T, T) \mid x_j \in \{T, F\}\},
\]

so that, for instance, \( P(A) = (1 - p)^3 \), and \( P(B) = P(C) = p^5 \). We already saw that \( P(B \cap C) = p^7 \), and we noticed that \( P(A \cap B) = \emptyset \).

Find the following conditional probabilities:

We compute

\[
P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{p^7}{p^5} = p^2
\]

and

\[
P(C \mid B) = \frac{P(B \cap C)}{P(B)} = \frac{p^7}{p^5} = p^2
\]

We note that \( P(A \cap B) = \emptyset \) since the 2nd song cannot have both type “T” and “F”. So \( P(A \cap B) = 0 \). Thus

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{p^5} = 0
\]

and

\[
P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0}{(1 - p)^3} = 0
\]

Finally, we note that \( A \cap C = \{(F, F, F, x_4, x_5, T, T, T) \mid x_j \in \{T, F\}\} \), so \( P(A \cap C) = (1 - p)^3 p^5 \). So

\[
P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{(1 - p)^3 p^5}{p^5} = (1 - p)^3
\]

and

\[
P(C \mid A) = \frac{P(A \cap C)}{P(A)} = \frac{(1 - p)^3 p^5}{(1 - p)^3} = p^5
\]
2. Consider the events from Exercise 3 in Problem Set 3, i.e., events $A, B, C$ with the properties that

\[
\begin{align*}
P(A) &= .35 \\
P(B) &= .39 \\
P(C) &= .44 \\
P(A \cap B) &= .13 \\
P(A \cap C) &= .12 \\
P(B \cap C) &= .13 \\
P(A \cap B \cap C) &= .09 \\
P(A \cup B \cup C) &= .89 
\end{align*}
\]

Find the following conditional probabilities:

We compute

\[
P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{.13}{.44} \approx .30
\]

and

\[
P(C \mid B) = \frac{P(B \cap C)}{P(B)} = \frac{.13}{.39} \approx .33
\]

and

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.13}{.39} \approx .33
\]

and

\[
P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{.13}{.35} \approx .37
\]

and

\[
P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{.12}{.44} \approx .27
\]

and

\[
P(C \mid A) = \frac{P(A \cap C)}{P(A)} = \frac{.12}{.35} \approx .34
\]
3. Consider the solution to Question 1 on Problem Set 3: Let $A$ be the event that a song on Dr. Ward’s iTunes is either blues, jazz, or rock, i.e.,

$$A = \{ x \mid x \text{ is a blues, jazz, or rock song} \},$$

when one song is chosen at random. Let $B, J, R$ denote the events that the song is a blues, jazz, or rock song, respectively. We already noticed that $A = B \cup J \cup R$, and that $B, J, R$ are disjoint. We already proved $P(A) = P(B \cup J \cup R) = \frac{9153}{27333}$. Now find $P(B \mid A)$ and $P(J \mid A)$ and $P(R \mid A)$. Hint: these three answers should sum to 1.

We compute

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{330/27333}{9153/27333} = \frac{330}{9153}$$

and

$$P(J \mid A) = \frac{P(A \cap J)}{P(A)} = \frac{537/27333}{9153/27333} = \frac{537}{9153}$$

and

$$P(R \mid A) = \frac{P(A \cap R)}{P(A)} = \frac{8286/27333}{9153/27333} = \frac{8286}{9153}$$

4. This problem was given as “food for thought” the other day in class. Devise a scheme in which we can use a biased spinner—which shows “heads”, say, 77% of the time and “tails”, say, 23% of the time—to simulate a fair coin. It might take many repeated trials on the spinner to make this work. In other words, your scheme might require many of the biased spins on the spinner in order to simulate one coin flip, and that’s OK.

You don’t have to “prove” that your method works, but at least try to explain your method and to give some justification for it.

Here is one scheme:

If the first two flips are HT, then count the final result as “HEADS”.

If the first two flips are TH, then count the final result as “TAILS”.

If the first two flips are HH or TT, then ignore these two flips, and start over again.

Three things to observe:

1. The “HEADS” and “TAILS” final results are “equally likely” to occur.

2. The fact that we start over again if we get HH or TT on a pair should not be worrisome. The probability is 0 that we will go on forever in this way. We will eventually stop the process, with 100% certainty.

3. If this does not make sense at this point, do not worry. Students only needed to try to devise a method for making a “fair” coin when only a “biased” coin is available; we will discuss this reasoning more, at a later point in the course, when discussing geometric random variables.