Chapter 3 Answers

1. Consider the songs from solution 2 in Chapter 2 (from Dr. Ward’s iTunes). Let \( A \) be the event that a song is either blues, jazz, or rock, i.e.,
\[
A = \{ x \mid x \text{ is a blues, jazz, or rock song} \},
\]
when one song is chosen at random. What is \( P(A) \)?
(Also: do we need to use inclusion-exclusion, yes or no? Explain very briefly; a one-sentence explanation will do.)

Let \( B, J, R \) denote the events that the song is a blues, jazz, or rock song, respectively. Since a song is classified according to only 1 type of music in our model, then \( B, J, R \) are disjoint. Thus
\[
P(B \cup J \cup R) = P(B) + P(J) + P(R).
\]
Looking up the values from Solution 2 in the problem set of Chapter 2, we conclude
\[
P(B \cup J \cup R) = \frac{330}{27333} + \frac{537}{27333} + \frac{8286}{27333} = \frac{9153}{27333}.
\]
We do not have any intersections between the events to worry about, so we did not worry about using inclusion-exclusion. It is not necessary to use inclusion-exclusion. (It would be an OK method to use, and we would get the same answer, but it is not necessary, and it would be more complicated, of course!)

2. Consider events \( A, B, C \) with the properties that
\[
P(A) = P(B) = P(C) = .38,
\]
and
\[
P(A \cap B) = P(A \cap C) = P(B \cap C) = .12,
\]
and
\[
P(A \cap B \cap C) = .05,
\]
then what is \( P(A \cup B \cup C) \)?

We use inclusion-exclusion:
\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\]
\[
= .38 + .38 + .38 - .12 - .12 - .12 + .05
\]
\[
= .83
\]
3. Consider events $A, B, C$ with the properties that
\[
\begin{align*}
P(B) & = .39 \\
P(C) & = .44 \\
P(A \cap B) & = .13 \\
P(A \cap C) & = .12 \\
P(B \cap C) & = .13 \\
P(A \cap B \cap C) & = .09 \\
P(A \cup B \cup C) & = .89
\end{align*}
\]
then what is $P(A)$?

We use inclusion-exclusion:
\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).
\]
Equivalently,
\[
.89 = P(A) + .39 + .44 - .13 - .12 - .13 + .09,
\]
so $P(A) = .35$.

4. It is always true, for any events $A, B$, that $P(A \cup B) \leq P(A) + P(B)$. Why? Explain briefly with words or a very clear picture.

We can write $A \cup B$ as $A \cup (B \setminus A)$, since the outcomes in $A \cup B$ are either in $A$ or in the part of $B$ not found in $A$. Also, $A$ and $B \setminus A$ are disjoint. So
\[
P(A \cup B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A) \leq P(A) + P(B),
\]
where the inequality is true since $B \setminus A \subset B$, so $P(B \setminus A) \leq P(B)$. This should be immediately clear with a picture in a Venn diagram.

Is it always true that $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$? If so, explain why, either using words or a very clear picture. If not, please give a counterexample.

Yes, this is always true. We can write $A \cup B \cup C$ as $A \cup (B \setminus A) \cup (C \setminus (A \cup B))$, since the outcomes in $A \cup B \cup C$ are either:

1. in $A$
2. in the part of $B$ not found in $A$
3. in the part of $C$ not found in $A \cup B$

Also, $A$ and $B \setminus A$ and $C \setminus (A \cup B)$ are disjoint. So
\[
P(A \cup B \cup C) = P(A \cup (B \setminus A) \cup (C \setminus (A \cup B)))
\]
\[
= P(A) + P(B \setminus A) + P(C \setminus (A \cup B))
\]
\[
\leq P(A) + P(B) + P(C)
\]
where the inequality is true since $B \setminus A \subset B$, so $P(B \setminus A) \leq P(B)$, and since $C \setminus (A \cup B) \subset C$ so $P(C \setminus (A \cup B)) \leq P(C)$. This should be immediately clear with a picture in a Venn diagram.
**Supplemental Comment about Problem Set 3, Question 4.**

In general, the following is true:

For any events $A_1, A_2, \ldots$, 

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} P(A_j).$$

The argument follows the same lines discussed in Problem 4. First consider $\bigcup_{j=1}^{\infty} A_j$. The $A_j$’s might have many overlaps. So we want to write this union in a different way, so that there are not overlaps among the pieces. To do this, we write 

$$\bigcup_{j=1}^{\infty} A_j = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2)) \cup (A_4 \setminus (A_1 \cup A_2 \cup A_3)) \cup (A_5 \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)) \cdots$$

$$= \bigcup_{j=1}^{\infty} (A_j \setminus (A_1 \cup \cdots \cup A_{j-1})).$$

So 

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = P\left(\bigcup_{j=1}^{\infty} (A_j \setminus (A_1 \cup \cdots \cup A_{j-1}))\right).$$

The sets in the union on the right-hand-side, however, are disjoint. So we can take the sum of the parts 

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j \setminus (A_1 \cup \cdots \cup A_{j-1})).$$

Also, $A_j \setminus (A_1 \cup \cdots \cup A_{j-1}) \subset A_j$, so $P(A_j \setminus (A_1 \cup \cdots \cup A_{j-1})) \leq P(A_j)$. So we conclude that 

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} P(A_j).$$

The argument above also works if we only have finitely-many events. To show this for $A_1, A_2, \ldots, A_n$, just let $A_{n+1} = A_{n+2} = \cdots = \emptyset$, so $P(A_{n+1}) = P(A_{n+2}) = \cdots = 0$. Then we have $\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{n} A_j$ and also $\sum_{j=1}^{\infty} P(A_j) = \sum_{j=1}^{n} P(A_j)$. So we get the following immediately:

For any $n$ events $A_1, A_2, \ldots, A_n$, 

$$P\left(\bigcup_{j=1}^{n} A_j\right) \leq \sum_{j=1}^{n} P(A_j).$$

Question 4 from the problem set was asking about the $n = 3$ and $n = 4$ cases, but now we see that this (much more general) result is true!