STAT 490

Experimental Design

Lecture 18

Foldovers of Fractional Factorials

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Logistics

Homework 7 solutions have been posted on Blackboard.

Homework 8 is due at 4:00 PM in MATH 234 on 11-6-2015.

The graded Homework 6 and 7 can be picked up from 3:00 - 4:00 PM in MATH 234 on 11-6-2015.

Office hours will be held from 3:00 - 4:00 PM on 11-6-2015.

Recap: The 2^{K-P} Fractional Factorial Design

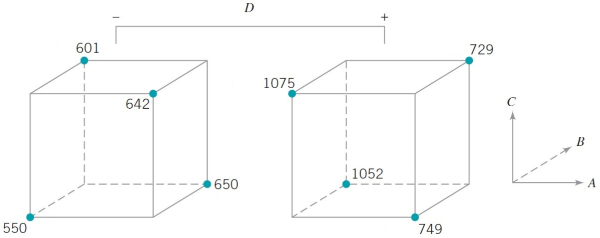
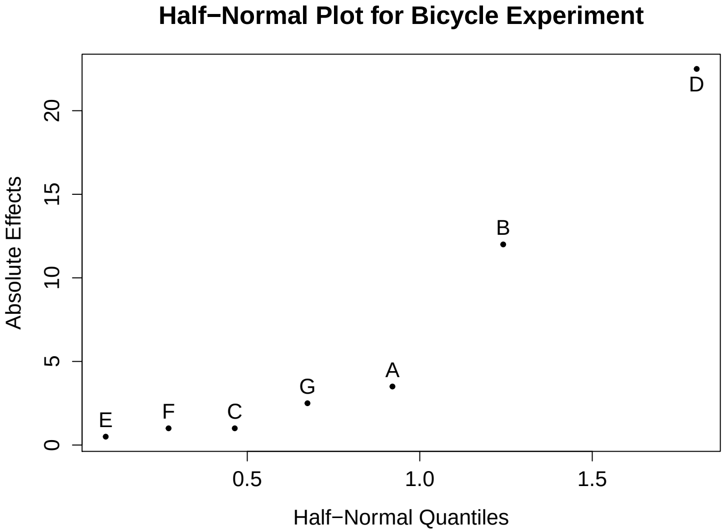


FIGURE 13.38 The 2^{4-1} design for Example 13.10.

Issue: Ambiguity in Effect Screening



Focus: Foldover to De-Alias Factorial Effects

Fractional factorial designs

- possess good run-size economy, and
- can yield unambiguous inferences on important effects in certain situations.

The inevitable price paid for using a fractional factorial design is aliasing of factorial effects.

An important problem is choosing follow-up runs to de-alias effects.

One simple approach is performing a foldover design.

The 2^{7-4} Bicycle Experiment (Box et al., 2005: p. 244)

Experimental Units
Eight days.

Covariates/Blocking Factors
None provided.

Treatment Factors
Seven, each with two levels.

Potential Outcome
Time taken to bike up the slope (in seconds).

The 2^{7-4} Bicycle Experiment (Box et al., 2005: p. 245)

Factor	Level -1	Level +1
1: Seat	Up	Down
2: Dynamo	Off	On
3: Handlebars	Up	Down
4: Gear	Low	Medium
5: Raincoat	On	Off
6: Breakfast	Yes	No
7: Tires	Hard	Soft

Initial 2^{7-4} Design for the Bicycle Experiment

$$\text{Design Matrix} = \begin{pmatrix} -1 & -1 & -1 & +1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -1 & -1 & +1 & -1 \\ +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & -1 \\ +1 & +1 & -1 & +1 & -1 & -1 & -1 \\ +1 & +1 & +1 & +1 & +1 & +1 & +1 \end{pmatrix}$$

$$\mathbf{4} = \mathbf{12},\ \mathbf{5} = \mathbf{13},\ \mathbf{6} = \mathbf{23},\ \mathbf{7} = \mathbf{123}$$

$$\Rightarrow I = \mathbf{124} = \mathbf{135} = \mathbf{236} = \mathbf{1237}$$

Ambiguity in Possible Active Effects

$$1 = 24 = 35 = 67$$

$$\mathbf{2} = \mathbf{14} = 36 = 57$$

$$3 = 15 = 26 = 47$$

$$\mathbf{4} = \mathbf{12} = 56 = 37$$

$$5 = 13 = 46 = 27$$

$$6 = 23 = 45 = 17$$

$$7 = 34 = 25 = 16$$

Follow-Up Runs: Foldover Column 4

$$\left(\begin{array}{ccccccc}
-1 & -1 & -1 & +1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 & -1 & -1 & +1 \\
-1 & +1 & -1 & -1 & +1 & -1 & +1 \\
-1 & +1 & +1 & -1 & -1 & +1 & -1 \\
+1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & -1 \\
+1 & +1 & -1 & +1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 \\
\hline
-1 & -1 & -1 & -1 & +1 & +1 & -1 \\
-1 & -1 & +1 & -1 & -1 & -1 & +1 \\
-1 & +1 & -1 & +1 & +1 & -1 & +1 \\
-1 & +1 & +1 & +1 & -1 & +1 & -1 \\
+1 & -1 & -1 & +1 & -1 & +1 & +1 \\
+1 & -1 & +1 & +1 & +1 & -1 & -1 \\
+1 & +1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & -1 & +1 & +1 & +1
\end{array}\right)$$

Observed Data for the Original and Follow-Up Runs

1	2	3	4	5	6	7	y_i^{obs}
-1	-1	-1	+1	+1	+1	-1	69
-1	-1	+1	+1	-1	-1	+1	71
-1	+1	-1	-1	+1	-1	+1	60
-1	+1	+1	-1	-1	+1	-1	59
+1	-1	-1	-1	-1	+1	+1	52
+1	-1	+1	-1	+1	-1	-1	50
+1	+1	-1	+1	-1	-1	-1	83
+1	+1	+1	+1	+1	+1	+1	88
-1	-1	-1	-1	+1	+1	-1	47
-1	-1	+1	-1	-1	-1	+1	53
-1	+1	-1	+1	+1	-1	+1	84
-1	+1	+1	+1	-1	+1	-1	87
+1	-1	-1	+1	-1	+1	+1	74
+1	-1	+1	+1	+1	-1	-1	78
+1	+1	-1	-1	-1	-1	-1	62
+1	+1	+1	-1	+1	+1	+1	60

$\widehat{\mathrm{ME}}(\mathbf{4})$ and $\widehat{\mathrm{INT}}(\mathbf{1},\mathbf{2})$ in the Full Design

1	2	3	4	5	6	7	y_i^{obs}
-1	-1	-1	+1	+1	+1	-1	69
-1	-1	+1	+1	-1	-1	+1	71
-1	+1	-1	-1	+1	-1	+1	60
-1	+1	+1	-1	-1	+1	-1	59
+1	-1	-1	-1	-1	+1	+1	52
+1	-1	+1	-1	+1	-1	-1	50
+1	+1	-1	+1	-1	-1	-1	83
+1	+1	+1	+1	+1	+1	+1	88
-1	-1	-1	-1	+1	+1	-1	47
-1	-1	+1	-1	-1	-1	+1	53
-1	+1	-1	+1	+1	-1	+1	84
-1	+1	+1	+1	-1	+1	-1	87
+1	-1	-1	+1	-1	+1	+1	74
+1	-1	+1	+1	+1	-1	-1	78
+1	+1	-1	-1	-1	-1	-1	62
+1	+1	+1	-1	+1	+1	+1	60

$$\widehat{\mathrm{ME}}(\mathbf{4}) = \frac{1}{8}\{69+71+83+88+84+87+74+78\} - \frac{1}{8}\{60+59+52+50+47+53+62+60\}$$

$$\widehat{\mathrm{INT}}(\mathbf{1},\mathbf{2}) = \frac{1}{8}\{69+71+83+88+47+53+62+60\} - \frac{1}{8}\{60+59+52+50+84+87+74+78\}$$

Foldover of a Single Column

The sole difference between the eight foldover runs and the initial eight runs are the entries in column **4**.

Signs in column **4** for the foldover runs are the opposite of the signs in **4** for the initial set of runs.

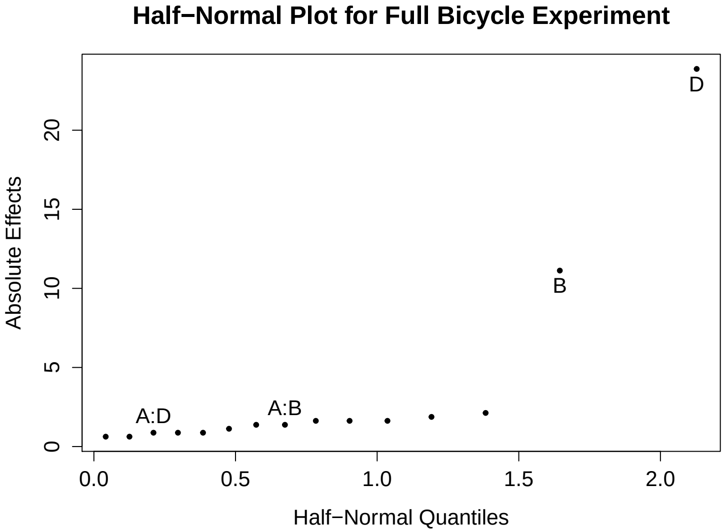
The foldover runs constitute a 2^{7-4} design with

$$\mathbf{4} = -\mathbf{12},\ \mathbf{5} = \mathbf{13},\ \mathbf{6} = \mathbf{23},\ \mathbf{7} = \mathbf{123}.$$

As $\mathbf{4} = \mathbf{12}$ in the initial eight runs, and $\mathbf{4} = -\mathbf{12}$, in the foldover runs, the estimators $\widehat{\mathrm{ME}(\mathbf{4})}$ and $\widehat{\mathrm{INT}(\mathbf{1}, \mathbf{2})}$ based on the entire design are no longer aliased, and are in fact orthogonal.

We can de-alias a particular main effect with all its interactions involving other effects by folding-over the column for the main effect.

Resolution of Ambiguity in Bicycle Experiment



New Blocking Factor Introduced by Foldover

1	2	3	4	5	6	7	B	y_i^{obs}
-1	-1	-1	+1	+1	+1	-1	-1	69
-1	-1	+1	+1	-1	-1	+1	-1	71
-1	+1	-1	-1	+1	-1	+1	-1	60
-1	+1	+1	-1	-1	+1	-1	-1	59
+1	-1	-1	-1	-1	+1	+1	-1	52
+1	-1	+1	-1	+1	-1	-1	-1	50
+1	+1	-1	+1	-1	-1	-1	-1	83
+1	+1	+1	+1	+1	+1	+1	-1	88
-1	-1	-1	-1	+1	+1	-1	+1	47
-1	-1	+1	-1	-1	-1	+1	+1	53
-1	+1	-1	+1	+1	-1	+1	+1	84
-1	+1	+1	+1	-1	+1	-1	+1	87
+1	-1	-1	+1	-1	+1	+1	+1	74
+1	-1	+1	+1	+1	-1	-1	+1	78
+1	+1	-1	-1	-1	-1	-1	+1	62
+1	+1	+1	-1	+1	+1	+1	+1	60

A new blocking factor **B**, corresponding to the two sets of runs, is introduced by performing a foldover.

Aliasing Between Treatment and Blocking Factors

The entire set of 16 runs constitutes a fractional factorial design with 7 treatment factors $\mathbf{1} - \mathbf{7}$, and one blocking factor $\mathbf{B}$.

The generators are

$$\mathbf{B} = -\mathbf{124},\ \mathbf{5} = \mathbf{13},\ \mathbf{6} = \mathbf{23},\ \mathbf{7} = \mathbf{123},$$

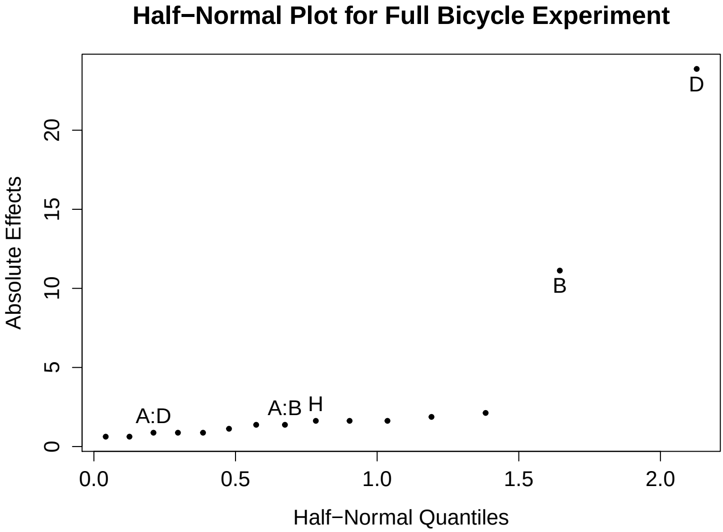
so that

$$\mathbf{I} = \mathbf{135} = \mathbf{236} = \mathbf{1237} = -\mathbf{124B}.$$

Analyzing the Bicycle Experiment with Blocks

```
> y = c(69,71,60,59,52,50,83,88)
> y = c(y,c(47,53,84,87,74,78,62,60))
>
> A_temp = c(rep(-1,4),rep(1,4))
> B_temp = rep(c(rep(-1,2),rep(1,2)),2)
> C_temp = rep(c(-1,1),4)
>
> A = rep(A_temp, 2)
>
> B = rep(B_temp,2)
>
> C = rep(C_temp,2)
>
> D = c(A_temp*B_temp,-A_temp*B_temp)
>
> E = rep(A_temp*C_temp,2)
>
> F = rep(B_temp*C_temp,2)
>
> G = rep(A_temp*B_temp*C_temp,2)
>
> H = c(rep(-1,8),rep(1,8))
>
> g = lm(y ~ A+B+C+D+E+F+G+A:B+A:D+B:D+C:D+D:E+D:F+D:G+H)
> observed_estimated_factorial_effects = 2*g$coeff[-1]
>
> half_Normal = function(x)
+ {
+   n = length(x)
+
+   halfn = 0.5 + 0.5*(1:n-0.5)/n
+
+   plot(qnorm(halfn), sort(abs(x)), type="p", pch=20,
+        xlab="Half-Normal Quantiles", ylab="Absolute Effects", main="Half-Normal Plot for Fu
ll Bicycle Experiment")
+
+   identify(qnorm(halfn), sort(abs(x)), names(sort(abs(x))))
+ }
>
> half_Normal(observed_estimated_factorial_effects)
```

Analyzing the Bicycle Experiment with Blocks



The 2^{7-4} Filtration Experiment (Box et al., 2005: p. 252)

Experimental Units
Eight runs.

Covariates/Blocking Factors
None provided.

Treatment Factors
Seven, each with two levels.

Potential Outcome
Filtration time (min).

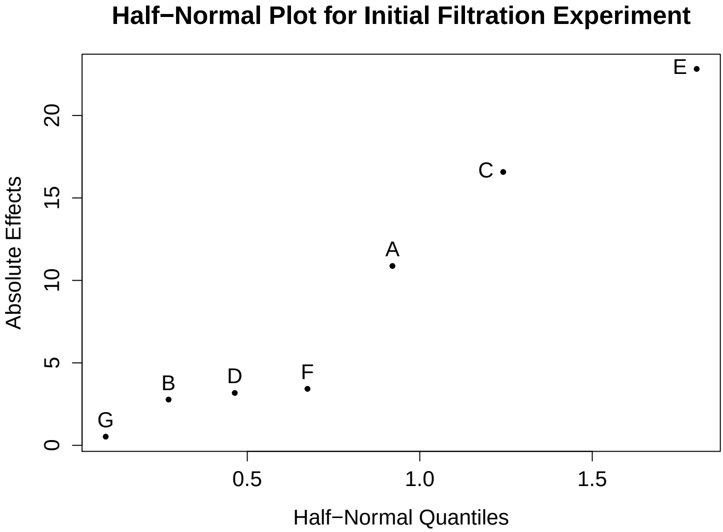
Treatment Factors (Box et al., 2005: p. 253)

Factor	Level -1	Level +1
1: Water supply	Town reservoir	Well
2: Raw material	On site	Other
3: Temperature	Low	High
4: Recycle	Yes	No
5: Caustic soda	Fast	Slow
6: Filter cloth	New	Old
7: Holdup time	Low	High

Observed Data for the Initial 2^{7-4} Design

1	**2**	**3**	**4**	**5**	**6**	**7**	y_i^{obs}
-1	-1	-1	+1	+1	+1	-1	68.4
-1	-1	+1	+1	-1	-1	+1	78.6
-1	+1	-1	-1	+1	-1	+1	66.4
-1	+1	+1	-1	-1	+1	-1	68.7
+1	-1	-1	-1	-1	+1	+1	77.7
+1	-1	+1	-1	+1	-1	-1	41.2
+1	+1	-1	+1	-1	-1	-1	81.0
+1	+1	+1	+1	+1	+1	+1	38.7

Ambiguity in Active Main Effects and Interactions



Follow-Up Runs: Foldover All Columns

$$
\begin{pmatrix}
-1 & -1 & -1 & +1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 & -1 & -1 & +1 \\
-1 & +1 & -1 & -1 & +1 & -1 & +1 \\
-1 & +1 & +1 & -1 & -1 & +1 & -1 \\
+1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & -1 \\
+1 & +1 & -1 & +1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 \\
\hline
+1 & +1 & +1 & -1 & -1 & -1 & +1 \\
+1 & +1 & -1 & -1 & +1 & +1 & -1 \\
+1 & -1 & +1 & +1 & -1 & +1 & -1 \\
+1 & -1 & -1 & +1 & +1 & -1 & +1 \\
-1 & +1 & +1 & +1 & +1 & -1 & -1 \\
-1 & +1 & -1 & +1 & -1 & +1 & +1 \\
-1 & -1 & +1 & -1 & +1 & +1 & +1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}
$$

Observed Data for the Original and Follow-Up Runs

1	2	3	4	5	6	7	y_i^{obs}
-1	-1	-1	+1	+1	+1	-1	68.4
-1	-1	+1	+1	-1	-1	+1	78.6
-1	+1	-1	-1	+1	-1	+1	66.4
-1	+1	+1	-1	-1	+1	-1	68.7
+1	-1	-1	-1	-1	+1	+1	77.7
+1	-1	+1	-1	+1	-1	-1	41.2
+1	+1	-1	+1	-1	-1	-1	81.0
+1	+1	+1	+1	+1	+1	+1	38.7
+1	+1	+1	−1	−1	−1	+1	66.7
+1	+1	−1	−1	+1	+1	−1	47.8
+1	−1	+1	+1	−1	+1	−1	86.4
+1	−1	−1	+1	+1	−1	+1	42.6
−1	+1	+1	+1	+1	−1	−1	65.0
−1	+1	−1	+1	−1	+1	+1	59.0
−1	−1	+1	−1	+1	+1	+1	61.9
−1	−1	−1	−1	−1	−1	−1	67.6

Foldover of All Columns

The difference between the eight foldover runs and the initial eight runs are the entries in columns $\mathbf{1-7}$.

Signs in columns $\mathbf{1-7}$ for the foldover runs are the opposite of the signs in $\mathbf{1-7}$ for the initial set of runs.

The foldover runs constitute a 2^{7-4} design with

$$\mathbf{4}=-\mathbf{12},\ \mathbf{5}=-\mathbf{13},\ \mathbf{6}=-\mathbf{23},\ \mathbf{7}=-\mathbf{123}.$$

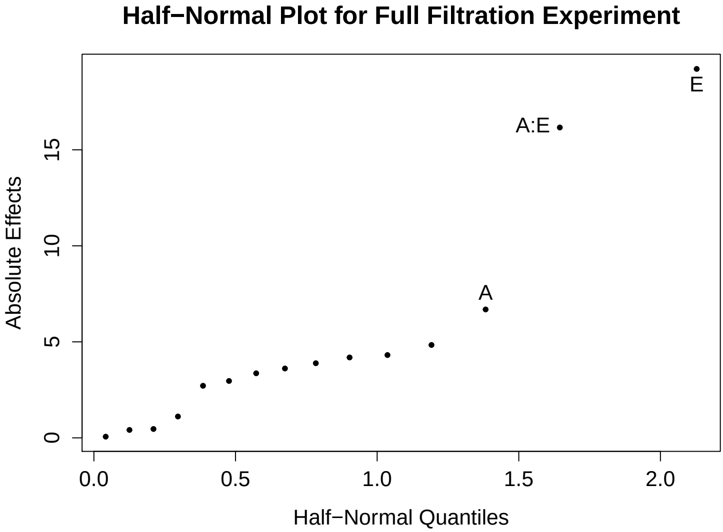
Combining the initial and foldover runs leads to a combination of their two defining relations to yield the defining relation

$$\mathbf{I}=\mathbf{1237}=\mathbf{1346}=\mathbf{1256}=\mathbf{2345}=\mathbf{3567}$$

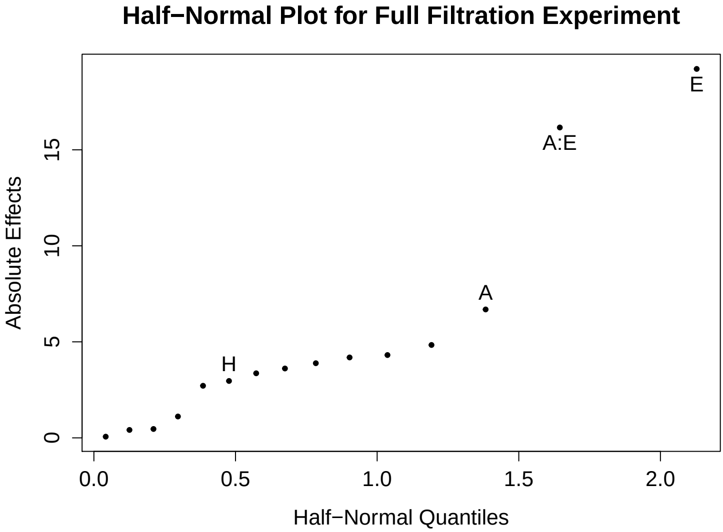
for the full design.

In the full design, no main effect is aliased with any two-factor interaction.

Resolution of Ambiguity in Filtration Experiment



Analyzing the Filtration Experiment with Blocks



Summary (Wu & Hamada, 2009: p. 228 - 229)

Foldover is effective for de-aliasing

- all the main effects, and
- one main effect and all the two-factor interactions involving this main effect.

The foldover technique requires that the number of follow-up runs equal the original run size, which can be wasteful if the number of effects to de-alias is much smaller than the run size.

Alternatives include

- targeted runs (Box et al., 2005: p. 257), and
- optimum follow-up runs (Wu & Hamada, 2009: p. 229 - 234).

Sequential Experimentation (BH^2, 2005: p. 251 - 252)

Structured experimentation can provide clear information on factorial effects.

Small experiments that successively reduce the number of possibilities are particularly effective.

While small experiments may not necessarily supply data to solve a problem, they can serve to eliminate a large number of possibilities and provide a basis for further conjecture.

The aim of follow-up runs is not uniqueness, but a strategy that is likely to converge to a useful solution.

The great value of experimental design is that it serves as a catalyst to the sequential process of scientific learning.

Before Thursday...

Read Sections 6.8 – 6.10 in BH^2.

Submit Homework 8 by by 4:00 PM in MATH 234 on 11-6-2015.

References I

Box G.E.P. and J.S. Hunter and W.G. Hunter (2005). *Statistics for Experimenters: Design, Innovation, and Discovery*, Wiley Inter-Science (2nd edition).

Wu, C.F.J. and Hamada M.S. (2009). *Experiments: Planning, Analysis, and Optimization*, Wiley Series in Probability and Statistics (2nd edition).