STAT553 HW08 **KEY**

Ex8c.1. (a)

$$
\begin{aligned}
&\sum_i\sum_j\sum_k(\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\\
=&JK\sum_i(\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\\
=&JK\sum_i(\bar{Y}_{i\cdot\cdot}^2-2\bar{Y}_{i\cdot\cdot}\bar{Y}_{\cdot\cdot\cdot}+\bar{Y}_{\cdot\cdot\cdot}^2)\\
=&JK\sum_i\bar{Y}_{i\cdot\cdot}^2-2\bar{Y}_{\cdot\cdot\cdot}\sum_i\bar{Y}_{i\cdot\cdot}+I\bar{Y}_{\cdot\cdot\cdot}^2\\
=&JK\sum_i\bar{Y}_{i\cdot\cdot}^2-I\bar{Y}_{\cdot\cdot\cdot}^2\\
=&\sum_i\frac{Y_{i\cdot\cdot}^2}{JK}-\frac{Y_{\cdot\cdot\cdot}^2}{IJK}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_i\sum_j\sum_k(\bar{Y}_{ij\cdot}-\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot j\cdot}+\bar{Y}_{\cdot\cdot\cdot})^2\\
=&\sum_i\sum_j\sum_k\left[(\bar{Y}_{ij\cdot}-\bar{Y}_{i\cdot\cdot})-(\bar{Y}_{\cdot j\cdot}-\bar{Y}_{\cdot\cdot\cdot})\right]^2\\
=&K\sum_i\sum_j(\bar{Y}_{ij\cdot}-\bar{Y}_{i\cdot\cdot})^2-2K\sum_j\left[(\bar{Y}_{\cdot j\cdot}-\bar{Y}_{\cdot\cdot\cdot})\sum_i(\bar{Y}_{ij\cdot}-\bar{Y}_{i\cdot\cdot})\right]+IK\sum_j(\bar{Y}_{\cdot j\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\\
=&K\sum_i\sum_j(\bar{Y}_{ij\cdot}-\bar{Y}_{i\cdot\cdot})^2-2KI\sum_j(\bar{Y}_{\cdot j\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2+IK\sum_j(\bar{Y}_{\cdot j\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2\\
=&K\sum_i\sum_j\bar{Y}_{ij\cdot}^2-JK\sum_i\bar{Y}_{i\cdot\cdot}^2-IK\sum_j\bar{Y}_{\cdot j\cdot}^2+IJK\bar{Y}_{\cdot\cdot\cdot}^2\\
=&\sum_i\sum_j\frac{Y_{ij\cdot}^2}{K}-\sum_i\frac{Y_{i\cdot\cdot}^2}{JK}-\sum_j\frac{Y_{\cdot j\cdot}^2}{IK}+\frac{Y_{\cdot\cdot\cdot}^2}{IJK}
\end{aligned}
$$

(b) Note that

$$
\begin{aligned}
E\bar{Y}_{ij\cdot}^2&=\text{Var}(Y_{ij\cdot})+(E\bar{Y}_{ij\cdot})^2=\frac{\sigma^2}{K}+(\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij})^2\\
E\bar{Y}_{i\cdot\cdot}^2&=\frac{\sigma^2}{JK}+(\mu+\alpha_i)^2\\
E\bar{Y}_{\cdot\cdot\cdot}^2&=\frac{\sigma^2}{IJK}+\mu^2
\end{aligned}
$$

Plug into results from part (a) we get the results.

2. First denote $Y_{ij}=\theta_i+\varepsilon_{ij}$ and $\Theta_i=\mu+r_i$.

(a) $Y_{ij} = \theta_i + N(0,1) = \mu + N(0,1) + N(0,1) = \mu_0 + N(0,1) + N(0,1) + N(0,1)$. So the marginal distribution of Y_{ij} is $N(\mu_0, 3)$.

(b) We show $\mathrm{Cov}(Y_{11}, Y_{12}) \neq \mathrm{Cov}(Y_{11}, Y_{21})$, thus the distribution is not exchangable.

$$\begin{aligned}\mathrm{Cov}(Y_{11}, Y_{12}) &= \mathrm{Cov}(\Theta_1 + \varepsilon_{11}, \Theta_1 + \varepsilon_{12}) \\ &= \mathrm{Var}(\Theta_1) \\ &= 2\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}(Y_{11}, Y_{21}) &= \mathrm{Cov}(M + r_1 + \varepsilon_{11}, M + r_2 + \varepsilon_{21}) \\ &= \mathrm{Var}(M) \\ &= 1\end{aligned}$$

(c) Looking at Y and Θ only, $Y_{ij} \sim N(\Theta_i, 1)$ and $\Theta_i \sim N(\mu_0, 2)$. Thus the posterior of Θ is

$$\Theta_i|Y_{ij} \quad \sim \quad N(\frac{2n}{2n+1}\bar{Y}_{i\cdot} + \frac{1}{2n+1}\mu_0, \frac{2}{2n+1})$$

Similarly, considering only Y and M, $\bar{Y}_i \sim N(M, 1 + \frac{1}{n})$ and $M \sim N(\mu_0, 1)$. Thus the posterior is

$$M|Y_{ij} \quad \sim \quad N(\frac{kn}{kn+n+1}\bar{Y}_{\cdot\cdot} + \frac{n+1}{kn+n+1}\mu_0, \frac{n+1}{kn+n+1})$$