STAT553 HW03 **KEY**

1. (a) The m.g.f. of X is

$$\begin{aligned} E(e^{t'X}) &= \int \cdots \int \exp\{\sum t_i x_i\} \frac{1}{(2\pi)^{n/2}} \exp\{-\frac{1}{2}\sum x_i^2\} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int \cdots \int \prod_{i=1}^n \{\exp(t_i x_i) \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x_i^2)\} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \prod_{i=1}^n \{\int \exp(t_i x_i) \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x_i^2) \, \mathrm{d}x_i\} \\ &= \prod_{i=1}^n E(e^{t_i X_i}) \end{aligned}$$

is a product of m.g.f. of its components. Therefore the components are independent.

(b) Since Q has orthonormal columns, $Q'Q = I_p$. Then $Y = Q'X$ is MVN with mean 0 and variance matrix $Q'Var(X)Q = I_p$. Thus by a), the p components of Y are i.i.d. $N(0,1)$.

2. Conditioning on W,

$$\begin{aligned} P(XW \le a | W = 1) &= P(X \le a | W = 1) = \Phi(a) \\ P(XW \le a | W = -1) &= P(X \ge -a | W = -1) = 1 - \Phi(-a) = \Phi(a) \end{aligned}$$

Thus $P(XW \le a) = \Phi(a)$ unconditionally, which means $XW \sim N(0,1)$. The reason (X, XW) is not bi-variate normal is simple: given $X = x$, XW can only take 2 values x and $-x$. For a bi-variate normal the conditional distribution should be normal.

3. It is sufficient to show that $\bar{Y}$ is independent of $Y_i - \bar{Y}$ for any i. Since they are linear combinations of components of Y, they are bi-variate normal, thus it is sufficient to show their covariance is 0. Without loss of generality, we just show the case when $i = 1$.

$$\begin{aligned} Cov(\bar{Y}, Y_1 - \bar{Y}) &= \left(\frac{1}{n}, \cdots, \frac{1}{n}\right) Var(Y) \begin{pmatrix} 1 - \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix} \\ &= \frac{1}{n}\mathbf{1}'\sigma^2((1-\rho)I + \rho\mathbf{1}\mathbf{1}')(\delta_1 - \frac{1}{n}\mathbf{1}) \\ &= \frac{1}{n}\sigma^2((1-\rho)\mathbf{1}' + \rho n\mathbf{1}')(\delta_1 - \frac{1}{n}\mathbf{1}) \\ &= \frac{1}{n}\sigma^2(1 + (n-1)\rho)\mathbf{1}'(\delta_1 - \frac{1}{n}\mathbf{1}) \\ &= \frac{1}{n}\sigma^2(1 + (n-1)\rho)(1 - \frac{1}{n}n) \\ &= 0 \end{aligned}$$

in which $\delta_1 = (1, 0, \cdots, 0)'$.

4. The joint distribution of $Y = (Y_1, Y_2)'$ is a bi-variate normal. Thus its only necessary to calculate its expectation and variance. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$EY = A\mu = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$Var(Y) = A\Sigma A' = \begin{pmatrix} 10 & 2 \\ 2 & 1 \end{pmatrix}$$

i.e. the joint distribution of Y is

$$Y \sim MVN(\begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 & 2 \\ 2 & 1 \end{pmatrix})$$

5. (a) Marginal distribution of Y_1 is $N(5, 2)$.

   (b) Joint distribution of Y_1 and Y_2 is bi-variate normal with mean $(5, 6)'$ and variance matrix
   $$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$
   For the above 2 questions just take corresponding row and column of the mean/variance matrix.

   (c) Notice $X = Y_3 - \frac{1}{2}Y_1 - \frac{2}{3}Y_2$ is independent of Y_1 and Y_2, by checking the covariance:
   $$\begin{aligned} & Cov(Y_3 - \frac{1}{2}Y_1 - \frac{2}{3}Y_2, Y_1) \\ = \ & 1 - \frac{1}{2} \times 2 - \frac{2}{3} \times 0 \\ = \ & 0 \end{aligned}$$
   And the variance is
   $$\begin{aligned} Var(X) &= Var(Y_3 - \frac{1}{2}Y_1 - \frac{2}{3}Y_2) \\ &= 4 + \frac{1}{4} \times 2 + \frac{4}{9} \times 3 \\ &\quad -2 \times \frac{1}{2} \times 1 - 2 \times \frac{2}{3} \times 2 + 2 \times \frac{1}{2} \times \frac{2}{3} \times 0 \\ &= 13/6 \end{aligned}$$

So given (Y_1, Y_2), the distribution of X is $N(\frac{1}{2}, 13/6)$. Thus the distribution of Y_3 given (Y_1, Y_2) is

$$Y_3 \sim N(\frac{1}{2} + \frac{1}{2}Y_1 + \frac{2}{3}Y_2, 13/6)$$

(d) $X = Y_3 - \frac{1}{2}Y_1$ is independent of Y_1. $Var(X) = 7/2$. So $X \sim N(9/2, 7/2)$. Thus given Y_1, $Y_3 \sim N(9/2 + \frac{1}{2}Y_1, 7/2)$.

(e) $X_1 = Y_1 - \frac{1}{4}Y_3$, $X_2 = Y_2 - \frac{1}{2}Y_3$ is jointly independent of Y_3. The joint distribution is

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim MVN(\begin{pmatrix} 3.25 \\ 2.4 \end{pmatrix}, \begin{pmatrix} 1.75 & -0.5 \\ -0.5 & 2 \end{pmatrix})$$

This means given Y_3, the conditional distribution of (Y_1, Y_2) is

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim MVN(\begin{pmatrix} 3.25 + \frac{1}{4}Y_3 \\ 2.4 + \frac{1}{2}Y_3 \end{pmatrix}, \begin{pmatrix} 1.75 & -0.5 \\ -0.5 & 2 \end{pmatrix})$$

(f) correlation $\rho_{12} = Cov(Y_1, Y_2)/\sqrt{Var(Y_1)Var(Y_2)} = 0$.

(g) This distribution of Z is bi-variate normal with expectation and variance matrix given below:

$$EZ = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \begin{pmatrix} -15 \\ -18 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Var(Z) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 11 \\ 11 & 15 \end{pmatrix}$$

(h) Write $Z = AY + b$. The m.g.f. of (Y, Z) is

$$\begin{aligned} M_{Y,Z}(t, r) &= E\exp\{t'Y + r'Z\} \\ &= E\exp\{t'Y + r'(AY + b)\} \\ &= E\exp\{(t' + r'A)Y + r'b\} \\ &= M_Y(t + A'r)\exp\{r'b\} \\ &= \exp\left((t' + r'A)\mu + \frac{1}{2}(t' + r'A)\Sigma(t + A'r) + r'b\right) \\ &= \exp\left(t'\mu + r'EZ + t'\Sigma t + r'Var(Z)r + 2t'\Sigma A'r\right) \end{aligned}$$

6.

$$\begin{aligned} Q &= 2(y_1 - 2)^2 + y_2^2 + y_3^2 + 2(y_1 - 2)y_2 \\ &= \begin{pmatrix} y_1 - 2 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 - 2 \\ y_2 \\ y_3 \end{pmatrix} \end{aligned}$$

so the mean and variance matrix is

$$\mu = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7. First we write Q as

$$Q = \sum_i \lambda_i Z_i^2 \tag{1}$$

where Z_i's are i.i.d. standard normal. The construction is as follows:

Let $X = \Sigma^{-1/2} Y$. Then $X \sim MVN_n(0, I_n)$. Hence

$$Q = Y'AY = X'\Sigma^{1/2} A \Sigma^{1/2} X$$

Let the spectral decompositon of $\Sigma^{1/2} A \Sigma^{1/2} = P'\Lambda P$, where P is an orthogonal matrix and $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$. Let $Z = PX$, which is also $MVN(0, I_n)$, then

$$Q = X'P'\Lambda P X = Z'\Lambda Z = \sum_i \lambda_i Z_i^2$$

Now knowing the m.g.f. of a $\chi^2(1)$ is $M_{\chi^2}(t) = (1-2t)^{-1/2}$, it's easy to calculate that the m.g.f. of Q is

$$\begin{aligned}
M_Q(t) &= \prod_i M_{\chi^2}(\lambda_i t) \\
&= \prod_i (1 - 2t\lambda_i)^{-1/2} \\
&= |I_n - 2t\Lambda|^{1/2} \\
&= |I_n - 2tP\Sigma^{1/2} A \Sigma^{1/2} P'|^{-1/2} \\
&= |P\Sigma^{1/2}(I_n - 2tA\Sigma)\Sigma^{-1/2}P'|^{-1/2} \\
&= \left(|P\Sigma^{1/2}| \cdot |I_n - 2tA\Sigma| \cdot |\Sigma^{-1/2}P'|\right)^{-1/2} \\
&= |I_n - 2tA\Sigma|^{-1/2}
\end{aligned}$$

The calculation utilized the fact that $|AB| = |A| \cdot |B|$ and $|A| \cdot |A^{-1}| = 1$.

For the case when A is a projection with rank r and $\Sigma = I$, the spectral decompositon of $\Sigma^{1/2} A \Sigma^{1/2} = A$ has eigenvalues consisting of r 1's and $n - r$ 0's. Thus the result is obvious from the calculation.

8. B_2 and B_3 are written in a similar form as B_1.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

Then $B_1 = v_1 v_1'/\|v_1\|^2, B_2 = v_2 v_2'/\|v_2\|^2, B_3 = v_3 v_3'/\|v_3\|^2$.

(a) $Y'B_iY$ are actually square of some normal r.v.

$$Y'B_iY = \frac{\|v_i'Y\|^2}{\|v_i\|^2} \quad \sim \quad \chi^2(1)\sigma^2$$

b and c) $(v_1'Y, v_2'Y, v_3'Y)$ is multi-variate normal. So to prove the quadratic forms are independent, it is sufficient to show that components of this multi-variate normal are independent. This is easily done by checking the covariance.

2b.3.

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

Thus (Z_1, Z_2) is bi-variate normal with mean and variance

$$EZ = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad Var(Z) = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

2b.9. (X_1, X_2, X_3) is a multivariate normal with mean 0 and variance matrix I_3.

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

So (Y_1, Y_2, Y_3) is multivariate normal with mean $A0 = 0$ and variance matrix $AIA' = AA'$. Thus it's sufficient to check that A is orthogonal.

Misc2.4. Since X and Y are independent, we get a $2n$-dimensional r.v. with a $2n$-dimensional MVN distribution by combining X and Y together.

$$aX + bY = (aI_n, bI_n) \begin{pmatrix} X \\ Y \end{pmatrix}$$

is a linear combination of components thus is a MVN.

Misc2.5. Without loss of generality, we assume a is a unit vector. Let $A = (a_1, a_2, \cdots, a_n)'$ be an orthogonal matrix with $a_1 = a$ being its first row. Then AY is $MVN(0, I_n)$, i.e.

$$a_i'Y \sim N(0,1) \qquad i.i.d$$

So given $a_1'Y = 0$,

$$Y'Y = (AY)'AY = \sum_{i=1}^{n} (a_i'Y)^2 = \sum_{i=2}^{n} (a_i'Y)^2 \sim \chi^2(n-1)$$

Misc2.10. A proof for a general n-dimensional case is given.

Let $\Sigma = R'R$ be the Cholesky decomposition, R being upper triangular. Let $Z = (R^{-1})'Y$. Then Z is MVN with mean 0 and variance

$$Var(Z) = (R^{-1})'\Sigma R^{-1} = (R^{-1})'R'RR^{-1} = I$$

So the components of Z are i.i.d. standard normal. Notice $r_{11} = (R)_{11} = \sqrt{\sigma_{11}}$ and $Z_1 = r_{11}^{-1}Y_1$. Then

$$\begin{aligned} & Y'\Sigma^{-1}Y - \sigma_{11}^{-1}Y_1^2 \\ = \ & Y'R^{-1}(R')^{-1}Y - Z_1^2 \\ = \ & Z'Z - Z_1^2 \end{aligned}$$

which is obviously $\chi^2(n-1)$.

Misc2.16. Let

$$A = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}$$

Then the quadratic form equals to $\|AY\|^2 = Y'A'AY$. Then by theorem 1.6, (4th central moment is 3)

$$Var(Y'A'AY) = 2tr((A'A)^2) = 2\sum b_{ij}^2$$

where b_{ij} is the i, j element of $B = A'A$. Since

$$B = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}$$

The final result is $12n - 16$.