RANDOM VARIABLE GENERATION

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All random variable generation techniques discussed here are based on the production of uniform random variables \( \text{Unif}[0,1] \), which is provided as a function in most programming languages. Uniform variables are generated using a deterministic technique called “pseudo-random number generator”.

**Inverse Transformation**

The basic idea of inverse transformation is that, given a uniform random variable generator, you can use it to generate a sequence of random variables whose distribution function is \( F \). The following theorem gives a constructive proof.

**Theorem 1.1.** If \( U \sim \text{Unif}[0,1] \), then the random variable \( X = F^-(U) \) has the distribution function \( F \), where \( F^- \) is the inverse function of \( F \) defined as

\[
F^{-}(p) = \inf\{x : F(x) \geq p\}, \quad 0 < p < 1.
\]

**Proof.** First, we have \( F^-(F(x)) \leq x \) for \(-\infty < x < \infty\) and \( F(F^{-1}(u)) \geq u \) for \( 0 < u < 1 \). Thus

\[
P(X \leq x) = P(F^-(U) \leq x) = P(U \leq F(x)) = F(x).
\]

Therefore, in order to generate a random variable \( X \sim F \), we can first generate \( U \sim \text{Unif}[0,1] \) and then apply the transformation \( X = F^-(U) \).

**Example 1.2.** (Exponential Random Variable) If \( X \sim \text{Exp}(1) \) we have \( F(x) = 1 - \exp(-x) \) which gives \( F^-(u) = -\log(1-u) \). So if \( U \sim \text{Unif}[0,1] \) we have the random variable \( X = -\log(1-U) \) has the exponential distribution. We only need to use \( X = -\log U \) since if \( U \sim \text{Unif}[0,1] \) we have \( 1-U \sim \text{Unif}[0,1] \) as well.

**Example 1.3.** (Bernoulli Random Variable) To generate a binary random variable \( X \sim \text{Bernoulli}(p) \) for \( 0 < p < 1 \) we can first generate \( U \sim \text{Unif}[0,1] \) and then set

\[
X = \begin{cases} 
0, & U \leq 1-p, \\
1, & U > 1-p.
\end{cases}
\]

In principle we can use \( U \sim \text{Unif}[0,1] \) to generate any random variable. However, practically it makes sense only if the computation of \( F^-(u) \) can be done efficiently.

**General Transformation**

If a distribution \( G \) is related to another distribution \( F \) in a simple way then we can generate random variables from \( Y \sim G \) based on transformations of random variables \( X \sim F \). Note that these methods are often specific to the distributions \( F \) and \( G \) and thus cannot generalize across distributions.

**Example 1.4.** (Box-Muller for Normal Random Variable) Let \( u_1 \) and \( u_2 \) be two IID samples from \( \text{Unif}[0,1] \), define

\[
x_1 = \sqrt{-2 \log u_1 \sin(2\pi u_2)}
\]

\[
x_2 = \sqrt{-2 \log u_1 \cos(2\pi u_2)}.
\]

Then \( x_1 \) and \( x_2 \) are two independent random samples from \( \text{N}(0,1) \) (prove it).
Mixture Distribution

In applications we often want to draw samples from a mixture of simple distributions. For example,

\[ X \sim \lambda \mathcal{N}(0,1) + (1 - \lambda) \mathcal{N}(0,10), \]

where \(0 < \lambda < 1\) is the mixture weight. The basic idea is to consider the pair of variables \((X, Z)\) where \(Z\) is a discrete random variable indicating which component is taken. Random samples from \(P(X)\) can be obtained by first drawing a sample \((X, Z)\) and then throwing \(Z\) away. In the above example, we can first draw \(Z \sim \text{Bernoulli}(\lambda)\) and then based on \(Z\)'s value, draw \(X|Z = 1 \sim \mathcal{N}(0,1)\) or \(X|Z = 0 \sim \mathcal{N}(0,10)\).

Accept-Reject Method

The accept-reject method is more general and can be applied to difficult cases where \(f\) does not have a simple form. The basic idea comes from the observation that if \(f\) is the target density, we have

\[ f(x) = \int_0^{f(x)} 1 \cdot du. \]

Thus, \(f\) can be thought as the marginal density of the joint distribution

\[ (X, U) \sim \text{Unif}\{(x, u) : 0 < u < f(x)\}, \]

where \(U\) is called an auxiliary variable.

**Theorem 1.5.** Let \(X \sim f(x)\) and let \(g(y)\) be a density function that satisfies \(f(x) \leq Mg(x)\) for some constant \(M \geq 1\). To generate a random variable \(X \sim f(x)\): (1) Generate \(Y \sim g(y)\) and \(U \sim \text{Unif}(0,1)\) independently; (2) If \(U \leq f(Y)/Mg(Y)\) set \(X = Y\); otherwise return to step (1).

**Proof.** The generated random variable \(X\) has distribution

\[
P(X \leq x) = P(Y \leq x|U \leq f(Y)/Mg(Y))
= \frac{P(Y \leq x, U \leq f(Y)/Mg(Y))}{P(U \leq f(Y)/Mg(Y))}
= \frac{\int_0^{f(x)/Mg(y)} \int_{\infty}^y f(y) 1 \cdot du \cdot g(y) dy}{\int_{-\infty}^{f(x)/Mg(y)} \int_{-\infty}^y f(y) 1 \cdot du \cdot g(y) dy}
= \int_{-\infty}^{f(x)} f(y) dy
\]

which is the desired distribution. \(\square\)

**Example 1.6.** (Beta Random Variable) To generate \(X \sim \text{Beta}(2,2)\), where

\[
\text{Beta}(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1.
\]

First note that \(\max_{x \in [0,1]} \{f(x|\alpha = 2, \beta = 2)\} = 1.5\). We can take \(M = 1.5\), \(Y \sim \text{Unif}[0,1]\), and \(U \sim \text{Unif}[0,1]\). If \(U \leq \frac{1}{1.5} f(Y)\) we take \(X = Y\).

**References**