STAT-598Y HW 3

DUE ON 10/22/09 IN CLASS

Problem. 1. Prove Theorem 9-1 in lecture notes.

Problem. 2. Prove Theorem 9-4 in lecture notes.

Problem. 3. Consider the hard-margin SVM primal problem

$$\min \|w\|^2 \ s.t. \ y_i(w^T x_i + b) \geq 1.$$

Write down the KKT condition, the Lagrangian and its dual problem. Give descriptions of the solution in terms of the dual variable α, similar to the ones given for soft-margin SVM in class.

Problem. 4. In this problem we illustrate the duality for a simple optimization problem with only inequality constraint. Consider the optimization problem

$$\min f(x) \ s.t. \ h(x) \leq 0$$

where $f(.)$ and $h(.)$ are convex functions. Furthermore, define an auxilary function $\omega(z) = \inf\{f(x) : h(x) \leq z\}$ for $z \geq 0$.

(a) Show that $\omega(z)$ is monotone in z.

(b) Define $\bar{f}(\lambda) = \inf_x\{f(x) + \lambda h(x)\}$ for $\lambda \geq 0$. Show that $\bar{f}(\lambda)$ is concave in λ.

(c) Prove that $\bar{f}(\lambda) = \inf_z\{\omega(z) + \lambda z\}$.

(d) Consider the set of linear functions of z with parameter λ: $l_\lambda(z) := -\lambda z + \bar{f}(\lambda)$ (when there are multiple constraints λ will be a vector and it becomes a hyperplane). Show that $l_\lambda(z) \leq \omega(z)$ for any $\lambda \geq 0$.

(e) Use (c) to prove that the intercept of $l_\lambda(z)$ with the vertical axis is upper bounded by $p^* = \omega(0)$.

Combining (a)-(e) we can see the geometric interpretation of the duality: the primal $p^* = \omega(0)$ is an upper bound of the intercepts of the set of hyperplanes $l_\lambda(z)$.

(f) In addition, show that $\omega(z)$ is convex in z (use the convexity condition of f and h).