Statistics 514: Design of Experiments

Topic 7c

Topic Overview

This topic will cover

- Sample Size Calculations

Sample Size Calculations

- *Multiple Comparisons* – *a posteriori* analysis (CI's and tests)
- *Sample Size Calculation* – *a priori* analysis
  - Might as well make assumptions, since...
  - Can't check them.
  - Try to minimize risk (Bayesian risk).
  - Run risk of "optimism" inherent in following assumptions too far.

Practical Answer

- How many samples?
- "As many as possible"

However, since experiments are often iterative (and budgeted and ethically responsible), we need to develop a strategy, either from the start or along the way.

Batch Example Revisited (Topic 6)

A supplier delivers several hundred batches of raw material to a company each year. The company is interested in a high yield from each batch of raw material (percent usable). Therefore, to investigate the consistency of this supplier, an experiment is done where five batches were selected at random and three yield determinations were made on each batch.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Between	147.74	4	36.93	20.5
Within	18.00	10	1.80	
Total	165.73	14		

- Assume five new batches are of interest
- Use model estimates to determine sample size

- Example question: want to detect deviation of 3 (percent usable) in **any** batch with 80% power.

Choice of Sample Size

Fixed Effects

- Must now consider all treatments ($a \geq 2$)
- Calculate power of joint test (F-test)
- For simplicity, assume n_i's constant

  Type II error – do not reject when you should

  $$\beta = Pr(F_0 < F_{\alpha,a-1,N-a} | \text{H}_0 \text{ false})$$

  Need to know distribution of F_0 when H_0 false.
  Can show $F_0 \sim F_{a-1,N-a}(\delta)$, where
  $\delta = n \sum \tau_i^2/\sigma^2$, non-centrality parameter
  Recall $E(MS_{Trt}) = \sigma^2 + n \sum \tau_i^2/(a-1)$

- OCC given in Chart V

  Plots β vs Φ
  $\Phi^2 = n \sum \tau_i^2/a\sigma^2 = \delta/a$

- Use SAS function `probf`

  $$\text{power} = 1 - \texttt{probf}(F_{\alpha,a-1,N-a}, a-1, N-a, \delta)$$

Determination of n

1. Choose treatment means ($\mu + \tau_i$)
   - Solve for τ_i's and compute Φ^2 or δ.
   - Difficult to select group of treatment means.
2. Use method similar to t-test approach
   - Rejection of any $|\tau_i - \tau_j| > D$
   - Use $\Phi^2 = nD^2/2a\sigma^2$ (i.e., $\{\tau_i\} = \{-D/2, 0, \ldots, 0, D/2\}$)
   - Power of test at least $1 - \beta$.
3. Specify a standard deviation percentage increase (P).
   - Under H_1, variance of random y_i is $\sigma_y^2 = \sigma^2 + \sum \tau_i^2/a$.
   - Randomly chosen τ_i has mean 0 and variance $\sum \tau_i^2/a$.

- $\sigma_y/\sigma = 1 + 0.01P$
- $\Phi^2 = ((1+0.01P)^2 - 1)n$

4. Use confidence interval approach
   - Specify length of $(1-\alpha)\%$ interval
   - Length/2 $= t_{\alpha/2,N-a}\sqrt{\frac{2MS_E}{n}}$
   - Based on estimate of MS_E, find n.
   - Simple but only looks at single interval.
   - Contrast $\{w_i\}$ under alternative, has distribution

$$F_{1,N-a}\left(\frac{(\sum w_i\tau_i)^2}{\sigma^2\sum w_i^2/n_i}\right)$$

Batch Example

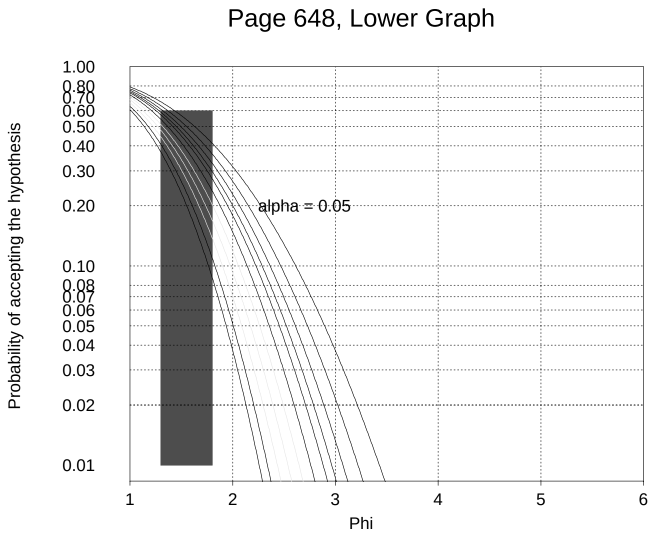
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Between	147.74	4	36.93	20.5
Within	18.00	10	1.80	
Total	165.73	14		

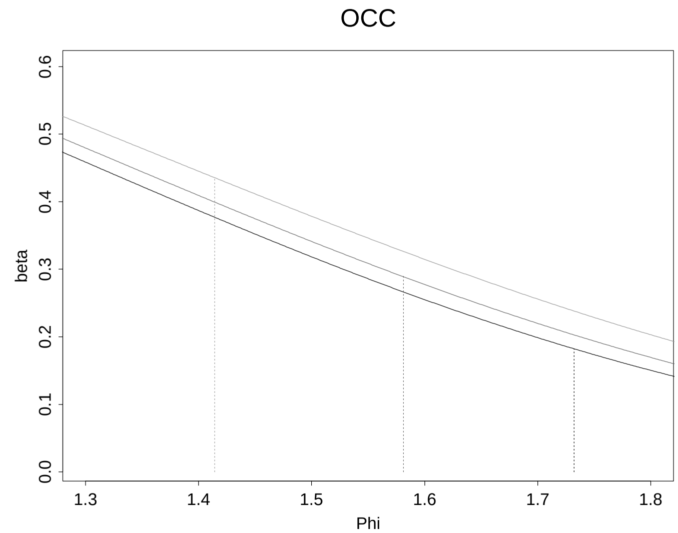
- Assume five new batch are of interest ($a = 5$)
- Want to detect deviation of 3 (percent usable) in **any** batch with 80% power. ($D = 3$ and $\beta_{\text{desired}} = 0.2$)

Use $\hat{\sigma}^2 = 1.8$ and $\alpha = 0.05$.

Using Table V: $\Phi^2 = 9n/2(5)(1.8) = 0.5n$

n	4	5	6
Φ	$\sqrt{2}$	$\sqrt{2.5}$	$\sqrt{3}$
$\text{df}_E = N-a$	15	20	25
β	45%	31%	18%





Using SAS: $\delta = a\Phi^2$

n	4	5	6
δ	10.0	12.5	15.0
$\text{df}_E = N - a$	15	20	25
β	43.5%	28.9%	18.2%

Appears $n = 6$ is proper choice

```
options nocenter ps=35 ls=72;
/* This is how you can calculate the power in one data set */

data params;
  input a alpha d var;
cards;
  5 .05 3.0 1.8
;
data new;
 set params;
 do n=2 to 10;
    df = a*(n-1);
    nc = n*d*d/(2*var);
    fcut  = finv(1-alpha,a-1,df);
    beta=probf(fcut,a-1,df,nc);
    output;
 end;

proc print;
var n nc beta;
run;
---------------------------------
```

```
Obs     n      nc       beta
 1      2     5.0    0.81008
 2      3     7.5    0.61721
 3      4    10.0    0.43549
 4      5    12.5    0.28897
 5      6    15.0    0.18227
 6      7    17.5    0.11017
 7      8    20.0    0.06421
 8      9    22.5    0.03626
 9     10    25.0    0.01992
```

Choice of Sample Size

Random Effects

- Can use central F distribution

$$\begin{aligned} (N-a)MS_E/\sigma^2 &\sim \chi^2_{N-a} \\ (a-1)MS_{Trt}/(\sigma^2 + n\sigma^2_\tau) &\sim \chi^2_{a-1} \\ \text{Thus, } F_0/(1 + n\sigma^2_\tau/\sigma^2) &\sim F_{a-1,N-a} \end{aligned}$$

  Can specify ratio σ^2_τ/σ^2
  Can specify percentage increase P

- OCC given in Chart VI

  Plots β against λ
  $\lambda^2 = 1 + n\sigma^2_\tau/\sigma^2$

- Use SAS function `probf`

$$\text{power } = 1 - \texttt{probf}(F_{\alpha,a-1,N-a}/\lambda^2, a-1, N-a)$$

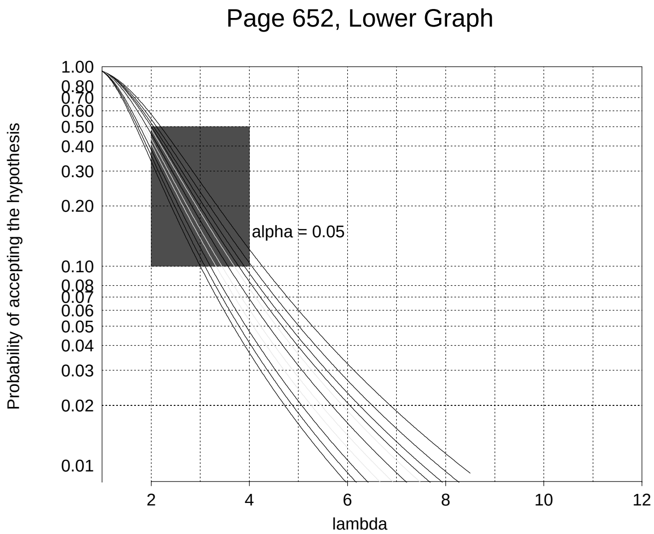
Batch Example Again

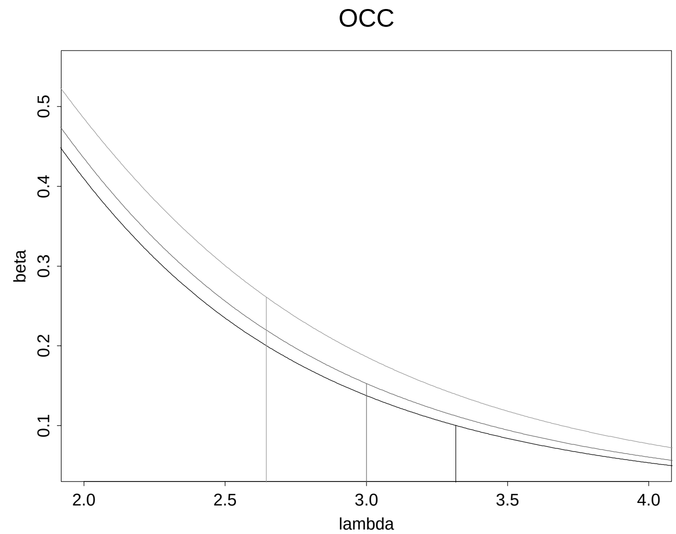
Consider new experiment a random effects problem

- Variance estimate still $\hat{\sigma}^2 = 1.8$.
- Want to detect situation when $\sigma^2_\tau \geq 3.6 = 2\sigma^2$.
- Set power to 80% and $\alpha = 0.05$.

Using Table VI: $\lambda = \sqrt{1 + 2n}$

n	3	4	5
λ	$\sqrt{7}$	$\sqrt{9}$	$\sqrt{11}$
df_E	10	15	20
β	28%	18%	15%





Using SAS: use λ^2

n	3	4	5
λ^2	7	9	11
df_E	10	15	20
β	26.1%	15.3%	10.0%

Appears $n = 4$ gives appropriate power.

```
/* This is how you can calculate the power in one data set */

data params;
  input a alpha ratiovar;
cards;
  5 .05 2.0
;

data new;
 set params;
 do n=2 to 10;
    df = a*(n-1);
    lambdasq = 1+ratiovar*n;
    fcut  = finv(1-alpha,a-1,df);
    beta=probf(fcut/lambdasq,a-1,df);
    output;
 end;
----------------------------------------------

Obs     n      beta

 1      2    0.52933
 2      3    0.26112
```

3	4	0.15292
4	5	0.10027
5	6	0.07081
6	7	0.05267
7	8	0.04072
8	9	0.03242
9	10	0.02643