

Lévy processes and Poisson random measures.

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Abstract

Some fundamental properties related to Lévy processes and poisson random measures are discussed. Topics include infinitely divisible distributions, Lévy-Khintchine formula, Lévy-Itô decomposition, and simulation.

1 Definition and first examples

Definition 1 A \mathbb{R}^d -valued process $X = \{X_t\}_{t \geq 1}$ is a **Lévy process** if

- (i) $X_0 = 0$;
- (ii) it has **independent increments**
- (iii) it has **stationary increments**;
- (iv) its paths are right-continuous with left-limits (rcll);
- (v) it has **no fixed jump-times**; that is, $\mathbb{P}(\Delta X_t \neq 0) = 0$, for any time t .

There are three fundamental examples of Lévy processes that deserve some revision: Brownian motion, Poisson process, and compound Poisson process.

Definition 2 A (standard) Brownian motion W is a real-valued process such that (i) $W_0 = 0$, (ii) it has independent increments, (iii) $W_t - W_s$ has normal distribution with mean 0 and variance $t - s$, for any $s < t$, and (iv) it has continuous paths.

It turns out that the only Lévy processes with continuous paths are of the form

$$X_t = \sigma W_t + bt,$$

for constants $\sigma > 0$ and b .

A Poisson process is another fundamental type of Lévy process that is often used as building blocks of other process.

Definition 3 A Poisson process N is an integer-valued process such that (i) $N_0 = 0$, (ii) it has independent increments, (iii) $N_t - N_s$ has Poisson distribution with parameter $\lambda(t - s)$, for any $s < t$, and (iv) its paths are rcll. The parameter λ is called the intensity of the process.

The process N_t is frequently used as a counting process of the number of events of certain type (say, car accidents) occurring by time t . That is,

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}_{T_i \leq t}, \quad (1)$$

where $\{T_i\}_{i \geq 1}$ represents the occurrence times of the event in question. Intuitively, if the events occur independently from one another, homogeneously in time, and with an intensity of λ events per unit time, then (1) is a Poisson process (a consequence of the Binomial approximation to the Poisson distribution).

It turns out that any Poisson process is of the form (1) with T_i such that the time span between events,

$$\tau_i := T_i - T_{i-1},$$

are independent with exponential distribution of parameter λ (mean of τ_i is $1/\lambda$).

To describe the next fundamental example, we need to introduce some terminology.

Terminology 1 Below, ρ and ν (and later on m) represent measure on \mathbb{R}^d . For those with limited background in real analysis, think of a measure ρ as a mapping that assigns a real number $\rho(A)$ to each set A of \mathbb{R}^d and that can be of two types:

(a) **Atomic:** There exists countable-many points $\{x_i\} \subset \mathbb{R}^d$ (called the atoms of the measure) such that

$$\rho(A) = \sum_i \mathbf{1}_{x_i \in A} = \text{Number of } x_i \text{ in the set } A.$$

(b) **Absolutely Continuous:** There exists a non-negative function s (called the density of the measure) such that

$$\rho(A) = \int_A s(x) dx = \text{Area or volume under } s \text{ over the set } A.$$

In that case, the integral of a function h with respect to the measure ρ , denoted by $\int h(x) \rho(dx)$, is defined as

$$\int h(x) \rho(dx) = \sum_i h(x_i),$$

if ρ is atomic, and as

$$\int h(x) \rho(dx) = \int h(x) s(x) dx,$$

if ρ is absolutely continuous with density s .

Intuitively a *compound Poisson process* $X := \{X_t\}_{t \geq 0}$ is like a Poisson process but with random jump sizes that are mutually independent and with common distribution. Suppose that N is a Poisson process with intensity λ and let $\{J_i\}_i$ are i.i.d random variable independent of N . Let ρ be the common distribution of J 's:

$$\rho(A) = \mathbb{P}(J_i \in A).$$

Then, the process

$$X_t := \sum_{i=1}^{N_t} J_i,$$

is said to be a compound Poisson process with jump distribution ρ and intensity λ . It turns out that a Poisson process is a Lévy process and moreover, it is the only Lévy process that has piece-wise constant paths with finitely-many jumps in any time interval $[0, T]$. Such a process is said to have *finite-jump activity*.

2 The marginal distributions of a Lévy process

The marginal distributions are **infinitely-divisible**. A random variable ξ is said to be *infinitely divisible* if for any n , one can construct n iid r.v. ξ_1, \dots, ξ_n such that

$$\xi \stackrel{\mathfrak{D}}{=} \xi_1 + \dots + \xi_n.$$

This class of distributions is closely related to limits in distribution of a array of row-wise i.i.d. r.v.:

Theorem 1 (Kallenberg [1]) ξ is infinitely divisible iff there exists $\{\xi_{n,k}\}_{k=1}^{k_n}$, i.i.d. for each n , such that

$$\sum_{k=1}^{k_n} \xi_{n,k} \xrightarrow{\mathfrak{D}} \xi, \quad \text{as } n \rightarrow \infty.$$

Infinitely divisible random variables are identified by three “parameters” via its characteristic function:

Theorem 2 (Lévy-Khinchine formula) ξ is infinitely divisible iff

$$\mathbb{E}e^{iu\xi} = \exp \left\{ -\frac{1}{2} \langle u, Au \rangle + i \langle b, u \rangle + \int_{\mathbb{R}^d} (e^{i\langle u, x \rangle} - 1 - i \langle u, x \rangle \mathbf{1}_{|x| \leq 1}) \nu(dx) \right\}, \quad (2)$$

for some symmetric nonnegative-definite matrix A , a vector $b \in \mathbb{R}^d$, and a measure ν on \mathbb{R}^d such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int (|x|^2 \wedge 1) \nu(dx) < \infty.$$

For simplicity, let us consider a real Lévy process X (the same reasoning applies to higher dimensions). In the light of the above, the time- t marginal distribution of X_t is identified by a triplet (σ_t^2, b_t, ν_t) . However, given that X has stationary and independent increment, it is not hard to see that

$$\mathbb{E}E^{iuX_t} = \{\mathbb{E}E^{iuX_1}\}^t.$$

Thus, $(\sigma_t^2, b_t, \nu_t) = t(\sigma_1^2, b_1, \nu_1)$, that we can denote by (σ^2, b, ν) for simplicity. In other words, a Lévy process is determined uniquely by a triplet (σ^2, b, ν) , called the Lévy triple of the process, such that

$$\mathbb{E}e^{iuX_t} = \exp \left\{ t \left(-\frac{1}{2}u^2\sigma^2 + ibu + \int_{\mathbb{R}^d} (e^{iux} - 1 - iux\mathbf{1}_{|x|\leq 1}) \nu(dx) \right) \right\}. \quad (3)$$

3 Poisson random measures

3.1 Basic definition and properties

Definition 4 Let S be a Borel subset of \mathbb{R}^d , let \mathcal{S} be the set of Borel subsets of S , and let m be a Borel measure on S that is σ -finite. Let (Ω, \mathcal{F}, P) be a probability space. A collection $\{M(B) : B \in \mathcal{S}\}$ of $\bar{\mathbb{Z}}_+$ -valued random variables is called a **Poisson random measure** (or process) on S with mean measure m if

- (1) for every B , $M(B)$ is a Poisson random variable with mean $m(B)$;
- (2) if B_1, \dots, B_n are disjoint, then $M(B_1), \dots, M(B_n)$ are independent;
- (3) for every sample outcome $\omega \in \Omega$, $M(\cdot; \omega)$ is a measure on \mathcal{S} .

It can be proved that (a.s.), $M(\cdot; \omega)$ is an atomic measure; that is, there exists countably many (random) points $\{\mathbf{x}_i\}_i \subset S$ such that

$$M(B) = \#\{i : \mathbf{x}_i \in B\}.$$

Equivalently,

$$M(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(\cdot).$$

Sometimes, the random points $\{\mathbf{x}_i\}_i$ is called a **Poisson point process** on the S with mean measure m .

3.2 Construction by conditioning

- **Key property:** Suppose that $m(B) < \infty$. Then, conditioning to $M(B) = k$, the points of $\{\mathbf{x}_i\}$ in B , denote by $\{\mathbf{x}_i^B\}$, are distributed according to the distribution $m(\cdot)/m(B)$, independently from each other:

$$\mathbb{P}\{\mathbf{x}_i^B \in C | M(B) = k\} = m(C)/m(B).$$

- Construction procedure:

1. Suppose that B_1, B_2, \dots is a partition of S such that $m(B_j) < \infty$.
2. Generate $n_j \sim \text{Poi}(m(B_j))$.
3. Independently, generate n_j -points, say $\{\mathbf{x}_i^j\}_{i=1}^{n_j}$, according to the distribution $m(\cdot)/m(B_j)$.
4. Define

$$M(B) = \#\{(i, j) : \mathbf{x}_i^j \in B\}.$$

- Example:

- Take \mathbb{R}_+ and $m(dt) = dt$. Then, if $T_1 < T_2 < \dots$ denotes the point process associated with a Poisson random measure on \mathbb{R}_+ with mean measure m , then $T_i - T_{i-1}$ are independent with common $\text{exp}(1)$ distribution (*Why?*). Such a Poisson process is said to be homogeneous and the sequence $\{T_i\}_i$ is also called the arrival times of the Poisson process.

3.3 Transformations of Poisson point processes

In this part we start with a Poisson random measure

$$M(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(\cdot),$$

be a Poisson random measure with mean measure m . We shall see that transformations of a Poisson point process is still a Poisson point process. The following is the simplest version:

Proposition 1 *Suppose that $T : S \rightarrow S' \subset \mathbb{R}^d$ be measurable. Then, the random measure associated with the transformed points $\mathbf{x}'_i := T(\mathbf{x}_i)$, namely*

$$M'(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}'_i}(\cdot).$$

is also a Poisson random measure with mean measure

$$M'(B) := M(\{\mathbf{x} : T(\mathbf{x}) \in B\}).$$

The following result shows that a *marked Poisson point process* is still a Poisson. Suppose that we associate a numeric mark or score \mathbf{x}'_i to each point \mathbf{x}_i of N . The marks are assigned independently from one another. The distribution of the mark can actually depend on the point \mathbf{x}_i . Concretely, let $\sigma(\mathbf{x}, d\mathbf{x}')$ be a probability measure on $S' \subset \mathbb{R}^d$, for each $\mathbf{x} \in S$. For each i , generate a r.v. \mathbf{x}'_i according $\sigma(\mathbf{x}_i, d\mathbf{x}')$ (*independently from any other variable*). Consider the so-called *marked Poisson process*

$$M'(\cdot) = \sum_{i=1}^{\infty} \delta_{(\mathbf{x}_i, \mathbf{x}'_i)}(\cdot),$$

Proposition 2 M' is a Poisson random measure with mean measure

$$m'(d\mathbf{x}, d\mathbf{x}') = \sigma(\mathbf{x}, d\mathbf{x}')m(d\mathbf{x}).$$

Proposition 3 Consider the following experiment. We classify the point of the Poisson process M into k different types. The probability that point \mathbf{x}_i is of type j is $p_j(\mathbf{x}_i)$, independently from any other classification. Let $\{\mathbf{y}_i^j\}$ be the points of $\{\mathbf{x}_i\}$ of type j and let M^j be the counting measure associated with $\{\mathbf{y}_i^j\}$:

$$M^j := \sum \delta_{\{\mathbf{y}_i^j\}}$$

Then, M^1, \dots, M^k are independent Poisson random measures with respective mean measures $m_1(d\mathbf{x}) := p_1(\mathbf{x})m(d\mathbf{x}), \dots, m_k(d\mathbf{x}) := p_k(\mathbf{x})m(d\mathbf{x})$. We say that the process M^1 is constructed from M by thinning.

Example 1 Suppose that we want to simulate a Poisson point process on the unit circle $S := \{(x, y) : x^2 + y^2 \leq 1\}$ with mean measure:

$$\xi(B) = \iint_{B \cap S} \sqrt{x^2 + y^2} dx dy.$$

A method to do this is based on the thinning method of the previous example. Suppose that we generate a “homogeneous” Poisson point process M on the square $R := \{(x, y) : |x| \leq 1, |y| \leq 1\}$ with intensity of λ points per unit area. That is, the mean measure of M is

$$m(B) = \frac{1}{4} \iint_B \lambda dx dy.$$

Let $\{(x_i, y_i)\}_i$ denote the atoms of the Poisson random measure M . Now, consider the following thinning process. We classify the point (x_i, y_i) of type 1 with probability $p(x_i, y_i) := \sqrt{x_i^2 + y_i^2}$ and of type 2 with probability $1 - p(x_i, y_i)$. Suppose that $\{(y_i^1, y_i^1)\}_i$ are the point of type 1. Then, this process is a Poisson point process with mean measure

$$m'(dx, dy) = \frac{1}{4} \lambda p(x, y) dx dy.$$

It suffices to take $\lambda = 4$.

3.4 Integration with respect to a Poisson random measure

Let M be a Poisson random measure as Definition 4. Since $M(\cdot; \omega)$ is an atomic random measure for each ω , say $M(\cdot; \omega) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i(\omega)}(\cdot)$, one can define the integral

$$M(f) := \int_S f(\mathbf{x}) M(d\mathbf{x}) = \sum_{i=1}^{\infty} f(\mathbf{x}_i),$$

for any measurable **nonnegative** deterministic function f . This is a $\bar{\mathbb{R}}_+ = \mathbb{R} \cup \{\infty\}$ -valued r.v. Moreover,

$$\begin{aligned}\mathbb{E} \left[e^{-\int f(\mathbf{x})M(d\mathbf{x})} \right] &= \exp \left\{ - \int (1 - e^{-f(\mathbf{x})}) m(d\mathbf{x}) \right\}, \\ \mathbb{E} \left[\int f(\mathbf{x})M(d\mathbf{x}) \right] &= \int f(\mathbf{x})m(d\mathbf{x}).\end{aligned}$$

Notice that if $B \in \mathcal{S}$ is such that $m(B) < \infty$, then

$$\int_B f(\mathbf{x})M(d\mathbf{x}) := \sum_{i:\mathbf{x}_i \in B} f(\mathbf{x}_i),$$

is a well-defined \mathbb{R}^d -valued r.v. for any measurable function $f : \mathcal{S} \rightarrow \mathbb{R}^d$. Its characteristic function is

$$\mathbb{E} \left[e^{i \langle \int_B f(\mathbf{x})M(d\mathbf{x}), \mathbf{u} \rangle} \right] = \exp \left\{ - \int_B (e^{i \langle f(\mathbf{x}), \mathbf{u} \rangle} - 1) m(d\mathbf{x}) \right\}.$$

Furthermore, if B_1, \dots, B_m are disjoint sets in \mathcal{S} with finite measure, then

$$\int_{B_1} f(\mathbf{x})M(d\mathbf{x}), \dots, \int_{B_m} f(\mathbf{x})M(d\mathbf{x}).$$

are independent (*Why? Hint: Consider linear combinations and its characteristic functions.*).

In the general case, the problem of determining conditions for the integral $\int_S f(\mathbf{x})M(d\mathbf{x})$ to be well-defined require some care. Let us assume that m is a radon measure (that is, $m(K) < \infty$, for any compact $K \subset S$). Then,

$$\int_S f(\mathbf{x})M(d\mathbf{x}) = \sum_{i=1}^{\infty} f(\mathbf{x}_i),$$

is well-defined for any bounded function $f : S \rightarrow \mathbb{R}$ of compact support. We say that the integral $\int_S f(\mathbf{x})M(d\mathbf{x})$ exists if there exists a random variable X such that

$$X \stackrel{\mathbb{P}}{=} \lim_{n \rightarrow \infty} \int_S f_n(\mathbf{x})M(d\mathbf{x})$$

for any sequence f_n of bounded functions with compact support such that $|f_n| \leq |f|$ and $f_n \rightarrow f$. In that case, $\int_S f(\mathbf{x})M(d\mathbf{x})$ is defined to be that common limit X . We define in a similar way the so-called *compensated integral of f* $\int_S f(\mathbf{x})(M - m)(d\mathbf{x})$. The following theorem gives conditions for the existence of the above integrals (see Kallenberg for a proof):

Proposition 4 *Let M be a Poisson random measure as Definition 4. Then,*

- (a) $M(f) = \int_S f(\mathbf{x})M(d\mathbf{x})$ exists iff $\int_S (|f(\mathbf{x})| \wedge 1)m(d\mathbf{x}) < \infty$
- (b) $(M - m)(f) := \int_S f(\mathbf{x})(M - m)(d\mathbf{x})$ exists iff $\int_S (|f(\mathbf{x})|^2 \wedge |f(\mathbf{x})|)m(d\mathbf{x}) < \infty$

Notice that $\int_S f(\mathbf{x})(M - m)(d\mathbf{x})$ might be well-defined as limit in probabilities, but

$$\int_S f(\mathbf{x})(M - m)(d\mathbf{x}) \neq \int_S f(\mathbf{x})M(d\mathbf{x}) - \int_S f(\mathbf{x})m(d\mathbf{x}).$$

4 Lévy-Itô Representation

Theorem 3 [13.4, Kallenberg] *Let $\{X(t)\}_{t \geq 0}$ be an rcll process in \mathbb{R}^d with $X(0) = 0$. Then, X has independent increments without fixed jumps times if and only if, a.s.*

$$\mathbf{X}(t) = \mathbf{b}_t + \mathbf{G}_t + \int_0^t \int_{\|\mathbf{x}\| \leq 1} \mathbf{x} (M - m)(ds, d\mathbf{x}) + \int_0^t \int_{\|\mathbf{x}\| > 1} \mathbf{x} M(ds, d\mathbf{x}), \forall t \geq 0, \quad (4)$$

for some continuous function \mathbf{b} with $\mathbf{b}_0 = 0$, some continuous centered Gaussian process \mathbf{G} with independent increments and $\mathbf{G}_0 = 0$, and some independent Poisson random measure M on $[0, \infty) \times \mathbb{R}_0^d$ with mean measure m satisfying

$$\int_0^t \int_{\mathbb{R}_0^d} (\mathbf{x}^2 \wedge 1) m(ds, d\mathbf{x}) < \infty, \quad t > 0. \quad (5)$$

The representation is almost surely unique, and all functions \mathbf{b} , processes G , and measures m with the stated properties may occur.

In terms of the random measure M_X of the jumps of \mathbf{X} , defined by

$$M_X((s, t] \times B) := \sum_{u \in (s, t]: \Delta X_u \neq 0} \mathbf{1}\{\Delta X_u \in B\},$$

the above theorem states that the M_X is a Poisson process with mean measure $m(dt, d\mathbf{x})$.

In the case of a Lévy process (that is, we also assume that \mathbf{X} has stationary increments), the previous theorem implies that the point process

$$\{(t, \Delta X_t) : \Delta X_t \neq 0\}$$

is a Poisson point process in $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$ with mean measure

$$m(dt, d\mathbf{x}) = \nu(d\mathbf{x})dt.$$

In the case of a one-dimensional Lévy process, the representation (4) takes the following form:

$$X(t) = bt + \sigma W_t + \int_0^t \int_{|x| \leq 1} x (M - m)(ds, dx) + \int_0^t \int_{|x| > 1} x M(ds, dx), \forall t \geq 0,$$

where W is a Wiener process.

Remark 1 [19.2, Sato] *In the equation (4), the second term can be understood as the limit*

$$\int_0^t \int_{\|\mathbf{x}\| \leq 1} \mathbf{x} (M - m)(ds, d\mathbf{x}) = \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\varepsilon < \|\mathbf{x}\| \leq 1} \mathbf{x} (M - m)(ds, d\mathbf{x}). \quad (6)$$

where the limit exists almost surely. Furthermore, the convergence is uniform in t on any bounded interval.

Example 2 One application of the Lévy-Itô decomposition is to determine conditions for certain path behavior. The following are some cases of interest:

1. *Path-continuity:* It is now clear that the only Lévy processes with continuous paths are essentially of the form $bt + \sigma W_t$.
2. *Finite-variation:* A necessary and sufficient condition is that $\sigma = 0$ and

$$\int_{|x| \leq 1} |x| \nu(dx) < \infty.$$

Notice that in that case one can write

$$X(t) = b_0 t + \int_0^t \int x M(ds, dx),$$

where $b_0 := b - \int_{|x| \leq 1} x \nu(dx)$ is called the drift of the Lévy process. A process of finite-variation can be written as the difference of non-decreasing processes. In the above representation, this processes will be $b_0 t + \int_0^t \int_{x > 0} x M(ds, dx)$, and $\int_0^t \int_{x < 0} x M(ds, dx)$ when $b_0 > 0$.

3. A non-decreasing Lévy process is called a subordinator. Necessary and sufficient conditions for X to be a subordinator are that $b_0 > 0$, $\sigma = 0$, and $\nu((-\infty, 0)) = 0$. Subordinators have received interest in recent years as they can be used as random clock or volatility processes.

5 Simulation of Lévy processes

5.1 Brief overview

Accurate path simulation of a pure jump Lévy processes $\mathbf{X} = \{\mathbf{X}(t)\}_{t \in [0, T]}$, regardless of the relatively simple statistical structure of their increments, present some challenging problems when dealing with *infinite activity* (namely, processes with infinite Lévy measure). Just try to conceive that in this case the jump times are in fact dense on $[0, \infty)$ (see Theorem 21.3 of [4]).

One of the most popular simulation schemes is based on the generation of *discrete skeletons*. Namely, the discrete skeleton of \mathbf{X} based on equally spaced time points is defined by

$$\tilde{\mathbf{X}}(t) = \sum_{k=1}^{\infty} \mathbf{X}(k-1/n) \mathbf{1}(t \in [(k-1)/n, k/n)) = \sum_{k=1}^{\infty} \Delta_k \mathbf{1}(t \geq k/n),$$

where $\Delta_k = \mathbf{X}(k/n) - \mathbf{X}(k-1/n)$ are i.i.d. with common distribution $\mathcal{L}(\mathbf{X}(1/n))$. The main drawback to the previous scheme is the fact that most often a r.v. with distribution $\mathcal{L}(\mathbf{X}(1/n))$ is not easily generated.

The second easiest scheme would be to approximate the Lévy process by finite activity Lévy processes. That is, suppose that \mathbf{X} is a pure-jump Lévy process, then, in light of the Lévy-Itô decomposition of sample paths, the process

$$\mathbf{X}_\varepsilon(t) \equiv t \left(\mathbf{b} - \int_{\|\mathbf{x}\| \geq \varepsilon} \mathbf{x} \nu(d\mathbf{x}) \right) + \sum_{s \leq t} \Delta \mathbf{X}(s) \mathbf{1}(\|\Delta \mathbf{X}(s)\| \geq \varepsilon) \quad (7)$$

converges uniformly on any bounded interval, and a.s. the limiting process coincides with the paths of \mathbf{X} (above, $\Delta \mathbf{X}(t) \equiv \mathbf{X}(t) - \mathbf{X}(t^-)$). The process $\sum_{s \leq t} \Delta \mathbf{X}(s) \mathbf{1}(\|\Delta \mathbf{X}(s)\| \geq \varepsilon)$ can be simulated using a *compound Poisson process* of the form $\sum_{k=1}^{N_t^\varepsilon} Y_k^\varepsilon$, where N_t^ε is a homogeneous Poisson process with intensity $\nu(\|\mathbf{x}\| \geq \varepsilon)$ and where $\{Y_k^\varepsilon\}_{k=1}^\infty$ are i.i.d with common distribution

$$\nu_\varepsilon(d\mathbf{x}) \equiv \mathbf{1}(\|\mathbf{x}\| \geq \varepsilon) \frac{\nu(d\mathbf{x})}{\nu(\|\mathbf{x}\| \geq \varepsilon)}.$$

Clearly, such a scheme is unsatisfactory because all jumps smaller than ε are totally ignored. An alternative method of simulation is based on time series representations of the form

$$\mathbf{X}(t) = \mathbf{b}t + \sum_{i=1}^{\infty} [\mathbf{H}(\Gamma_i, \mathbf{V}_n) \mathbf{1}(U_i \leq t) - t\mathbf{c}_i],$$

which will be explained next.

5.2 Simulations based on series representations

Throughout, $\mathbf{X} = \{\mathbf{X}(t)\}_{t \in [0, T]}$ is a Lévy process on \mathbb{R}^d with Lévy measure ν and without Gaussian part (which can be separately simulated). The results below are presented in Rosinski [3]. A series representation for \mathbf{X} can be derived from a series representations for the random measure associated with the jumps of \mathbf{X} . In general terms, suppose that the counting random measure M associated with the jumps of \mathbf{X} , namely

$$M(B) = \# \{t > 0 : (t, \Delta \mathbf{X}(t)) \in B\}, \quad (8)$$

has the representation

$$M(\cdot) = \sum_{i=1}^{\infty} \delta_{(U_i, \mathbf{J}_i)}(\cdot), \quad (9)$$

when restricted to $[0, T] \times \mathbb{R}_0^d$, where here $\{U_i\}_{i \geq 1}$ is a sequence of i.i.d uniform random variables on $[0, T]$ and $\{\mathbf{J}_i\}_{i \geq 1}$ is a certain sequence of random vectors. The random variables U 's govern the times of the jumps, while the \mathbf{J} 's give the size of the jumps. If $\{\mathbf{X}_t\}_{t \leq T}$ has finite variation with drift b_0 , one can write

$$\mathbf{X}(t) = \mathbf{b}_0 t + \int_0^t \int \mathbf{x} M(d\mathbf{x}, ds) = \mathbf{b}_0 t + \sum_{i=1}^{\infty} \mathbf{J}_i \mathbf{1}(U_i \leq t).$$

In the most general case where \mathbf{X} does not have bounded variation, it might be necessary to compensate the jumps by some constants as follows:

$$\mathbf{X}(t) = \mathbf{b}t + \sum_{i=1}^{\infty} [\mathbf{J}_i \mathbf{1}(U_i \leq t) - t\mathbf{c}_i], \quad 0 \leq t \leq T, \quad (10)$$

for suitable centers \mathbf{c}_i that compensate the jumps.

5.2.1 Series representation of Lévy jump measures

There are many ways to obtain (9) (we list a few below). The most general idea is the following. Think of the jumps \mathbf{J} 's as random responses to fictitious “shots” occurring in the past according to a homogeneous Poisson process. Let $0 < \Gamma_1 < \Gamma_2 < \dots$ be the arrival times of a homogeneous Poisson process on $(0, \infty)$ with constant intensity T . Denote \mathbf{J}_i the jump originating as consequence of a shot occurring Γ_i time units ago. The distribution of the jumps are dictated by a probability measure $\sigma(u; \cdot)$ on $\mathcal{B}(\mathbb{R}^d)$ in such a way that

$$\mathbb{P}\left(\mathbf{J}_i \in B \mid \{\Gamma_j\}_{j \geq 1}, \{\mathbf{J}_j\}_{j \neq i}\right) = \sigma(\Gamma_i, B), \quad B \in \mathcal{B}(\mathbb{R}^d). \quad (11)$$

It follows that the jumps $\mathbf{J}_1, \mathbf{J}_2, \dots$ form a Poisson process in \mathbb{R}^d with mean measure $T \int_0^\infty \sigma(u; B) du$, (*Why?*). Consequently, the point process $\sum_{i=1}^\infty \delta_{(U_i, \mathbf{J}_i)}(\cdot)$, where $\{U_i\}_i$ are i.i.d. uniform r.v. on $[0, T]$ and independent from $\{\mathbf{J}_i\}_i$, is a Poisson point process with mean measure of the form

$$m(dt, d\mathbf{x}) = \int_0^\infty \sigma(u; d\mathbf{x}) du dt.$$

Therefore, if the Lévy measure of the Lévy process has the representation:

$$\nu(B) = \int_0^\infty \sigma(u; B) du, \quad (12)$$

then the jump measure M , when restricted to $[0, T] \times \mathbb{R}^d$, has the same law as $\sum_{i=1}^\infty \delta_{(U_i, \mathbf{J}_i)}(\cdot)$.

It is not always easy to obtain (12). The following are typical methods: the inverse Lévy measure method, Bondesson’s method, and Thinning method (see Rosinski [3] for more details).

5.2.2 Series representation for Lévy processes

There are other considerations to think about for the representation (10) to hold true. It has to do with the probabilistic structure of the jumps $\{\mathbf{J}_i\}$. Roughly speaking, to avoid divergence problems and to guarantee the existence of compensating centers \mathbf{c}_i , it is necessary that the magnitude of the jumps decreases as the elapsed time between the jump and the shot increases (an appealing physical assumption as well). This is better explained if we assume that

$$\mathbf{J}_i = \mathbf{H}(\Gamma_i, \mathbf{V}_i),$$

for a sequence of random elements $\{\mathbf{V}_i\}_{i \geq 1}$ independent of $\{U_i, \Gamma_i\}_{i \geq 1}$. Notice that

$$\mathbf{H}(u, \mathbf{V}_i) \sim \sigma(u; d\mathbf{x}).$$

Then, it makes sense that $\|\mathbf{H}(r, \mathbf{v})\|$ should decrease in r for (10) to be true.

Let us summarize the conditions and give the main theorem for the simulation of Lévy processes.

Condition 1 *The jump measure of \mathbf{X} can be written as*

$$N(\cdot) \stackrel{\mathfrak{D}}{=} \sum_{i=1}^{\infty} \delta_{(U_i, \mathbf{H}(\Gamma_i, \mathbf{V}_i))}(\cdot). \quad (13)$$

for a function $\mathbf{H} : (0, \infty) \times S \rightarrow \mathbb{R}^d$, where S is an arbitrary measurable space. Here, $\{\Gamma_i\}_{i=1}^{\infty}$ is a homogeneous Poisson process on \mathbb{R}_+ with intensity T , $\{U_i\}_{i=1}^{\infty}$ is an independent random sample with uniform distribution on $[0, T]$, and $\{\mathbf{V}_i\}_{i=1}^{\infty}$ is an independent random sample $\{\mathbf{V}_i\}_{i=1}^{\infty}$ with common distribution F on the space S .

Remark 1 *Representation (13) can be obtained (in law) if the Lévy measure has the decomposition*

$$\nu(B) = \int_0^{\infty} \sigma(u; B) du, \quad (14)$$

where $\sigma(u; B) = \mathbb{P}[\mathbf{H}(u, \mathbf{V}) \in B]$.

Condition 2 *For any Poisson process $\{\Gamma_i^1\}_{i=1}^{\infty}$ on \mathbb{R}_+ with unit rate,*

$$\mathbf{A}(\Gamma_n^1) - \mathbf{A}(n) \rightarrow 0 \text{ a.s.}, \quad (15)$$

where

$$\mathbf{A}(s) \equiv \int_0^s \int_S \mathbf{H}(r, \mathbf{v}) \mathbf{1}(\|\mathbf{H}(r, v)\| \leq 1) F(d\mathbf{v}) dr. \quad (16)$$

The next lemma gives sufficient conditions for (15) (see pp. 409 [3]):

Lemma 1 *The limit in (15) holds true if either one of the following conditions is satisfied:*

- i) $\mathbf{a} \equiv \lim_{s \rightarrow \infty} \mathbf{A}(s)$ exists in \mathbb{R}^d ;
- ii) the mapping $r \rightarrow \|\mathbf{H}(r, v)\|$ is nonincreasing for each $v \in S$.

The following result establishes the series representations for Lévy processes.

Proposition 5 *If the conditions 1 and 2 are satisfied then, a.s.*

$$\mathbf{X}(t) \stackrel{\mathfrak{D}}{=} \mathbf{b}t + \sum_{i=1}^{\infty} [\mathbf{H}(\Gamma_i, \mathbf{V}_i) \mathbf{1}(U_i \leq t) - t\mathbf{c}_i], \quad (17)$$

for all $t \in [0, T]$, where $\mathbf{c}_i \equiv \mathbf{A}(i) - \mathbf{A}(i-1)$.

Proof: Notice that the Lévy-Itô representation (4) takes the form

$$\mathbf{X}(t) = \mathbf{b}t + \int_{[0,t] \times \mathbb{R}_0^d} \mathbf{x} \mathbf{1}(\|\mathbf{x}\| \leq 1) (\mathcal{J} - E\mathcal{J})(du, d\mathbf{x}) + \int_{[0,t] \times \mathbb{R}_0^d} \mathbf{x} \mathbf{1}(\|\mathbf{x}\| > 1) \mathcal{J}(du, d\mathbf{x}).$$

Define

$$M(\cdot) \equiv \sum_{i=1}^{\infty} \delta_{(U_i, \Gamma_i, \mathbf{V}_i)}(\cdot).$$

From Proposition 3.8. of [2], M is a (marked) Poisson process on $R \equiv [0, T] \times \mathbb{R}_+ \times \mathbb{R}_0^d$ with mean measure $du dr F(d\mathbf{v})$. By a “change of variables”,

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{b}t + \int_{R_t} \mathbf{H}(r, \mathbf{v}) \mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| \leq 1) (M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})) \\ &\quad + \int_{R_t} \mathbf{H}(r, \mathbf{v}) \mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| > 1) (M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})), \end{aligned}$$

where $R_t \equiv [0, t] \times \mathbb{R}_+ \times \mathbb{R}_0^d$. Define

$$\begin{aligned} \mathbf{X}_s(t) &= \mathbf{b}t + \int_{R_t^s} \mathbf{H}(r, \mathbf{v}) \mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| \leq 1) (M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})) \\ &\quad + \int_{R_t^s} \mathbf{H}(r, \mathbf{v}) \mathbf{1}(\|\mathbf{H}(r, \mathbf{v})\| > 1) (M(du, dr, d\mathbf{v}) - dudrF(d\mathbf{v})), \end{aligned}$$

where $R_t^s \equiv [0, t] \times [0, s] \times \mathbb{R}_0^d$. Using that the Poisson process M is an independently scatter measure (that is, $M(A_1), \dots, M(A_n)$ are mutually independent for disjoint sets A_1, \dots, A_n), we can verify in a standard way that $\mathbf{X}_s(t)$ has independent increments both with respect to $s \in [0, \infty)$ and $t \in [0, T]$. Also, notice that

$$\mathbf{X}_s(t) = \mathbf{b}t + \sum_{i: \Gamma_i \leq s} \mathbf{H}(\Gamma_i, \mathbf{V}_i) \mathbf{1}(U_i \leq t) - t\mathbf{A}(s), \quad (18)$$

implying that $\mathbf{X}_s(t)$ enjoys càdlàg paths in s for each t . We claim that almost surely,

$$\lim_{s \rightarrow \infty} \mathbf{X}_s(t) = \mathbf{X}(t),$$

for all $t \in [0, T]$. Fix $t \in [0, T]$ and take a sequence $s_n \uparrow \infty$. Since $\mathbf{X}(\cdot)$ has càdlàg paths, it suffices to check that $\lim_{n \rightarrow \infty} \mathbf{X}_{s_n}(t) = \mathbf{X}(t)$ a.s. Furthermore, since $\mathbf{X}_{s_n}(t) = \sum_{i=1}^n (\mathbf{X}_{s_i}(t) - \mathbf{X}_{s_{i-1}}(t))$ and since $\mathbf{X}(\cdot)$ has independent increment, it is enough to have convergence in distribution. The later can be deduced from arguments based on characteristic function. \square

Remark 2 *We finally point out that if condition (15) is true, and the representation (13) holds in distribution, then the representation (17) is valid in the sense of finite dimensional distributions.*

Remark 3 *The series (17) simplifies further when $\int_{\|\mathbf{x}\|\leq 1} \|\mathbf{x}\| \nu(d\mathbf{x}) < \infty$, namely, when \mathbf{X} has paths of bounded variation a.s. (see Theorem 21.9 of [4]). Concretely, a.s.*

$$\mathbf{X}(t) = (\mathbf{b} - \mathbf{a})t + \sum_{i=1}^{\infty} \mathbf{J}_i \mathbf{I}(U_i \leq t), \quad (19)$$

where

$$\mathbf{a} = \int_{\|\mathbf{x}\|\leq 1} \mathbf{x} \nu(d\mathbf{x}).$$

The vector $\mathbf{d} \equiv \mathbf{b} - \mathbf{a}$ is called the drift of the Lévy process.

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