Jump-diffusion models driven by Lévy processes

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Abstract

For more than a decade, there has been an explosion of continuous-time financial models in a quest to incorporate the so-called *stylized features of asset prices* like fat-tails, high kurtosis, volatility clustering, and leverage. Modeling driven by "memoryless homogeneous" jump processes (Lévy processes) constitutes one of the most plausible directions in this enterprise. The basic principle is to replace the underlying Brownian motion of the Black-Scholes model with a type of jump-diffusion process. In this chapter, the basic methods and tools behind jump-diffusion models driven by Lévy processes are covered, providing an accessible overview of the probabilistic concepts and results of Lévy processes, coupled with their financial applications and relevance. The material will be drawn upon recent monographs (c.f. [17], [46], [45]) and recent papers in the field.

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1 An overview of financial models with jumps

The seminal Black-Scholes model [9] provides a framework to price options based on the fundamental concepts of hedging and absence of arbitrage. The underlying stock price process is the geometric Brownian motion (GBM), originally proposed by Samuelson [44], which postulates that the time-t price of the stock is given by

$$S_t = S_0 e^{\sigma W_t + \mu t}, \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is the standard Brownian motion. This model is quite plausible since Brownian motion is the model of choice to describe the evolution of a random measurement which value is the result of a large-number of small shots occurring through time with high-frequency. This is indeed the situation with stock prices which value is the result of a high number of agents posting bid and ask prices almost at all times.

Almost from its inception, the limitations of the GBM were well recognized. In the statistical world, time series of log returns, say $\log\{S_\Delta/S_0\}, \ldots, \log\{S_{k\Delta}/S_{(k-1)\Delta}\}$, exhibit *leptokurtic*[1] distributions which are inconsistent with the Gaussian distribution postulated by the GBM. The volatility, as measured for instance by the square root of the realized variance of log returns, exhibits *clustering* and *leverage* effects, which contradicting the random-walk property of a GBM. Specifically, when plotting the time series of log returns against time, there are periods of high variability followed by low variability periods suggesting

[1] Fat tails with high kurtosis

that high volatility events “cluster” in time. Also, a tendency towards volatility growth is typically observed after a drop in prices suggesting that volatility is negatively correlated with returns. These and other *stylized properties* of asset returns are widely known in the financial community (see e.g. [16] and [8] for more information). In the risk-neutral world, it is well known that the Black-Scholes volatilities of call and put options are not flat neither with respect to the strike nor the maturity, as it should be under the model. Rather implied volatilities exhibit a smile or smirk curve shape.

In a quest to incorporate the stylized properties of asset prices, there have been many proposed models, most of them derived from natural variations of the Black-Scholes model. The basic idea is to replace the Brownian motion in (1), with another related process such as a Lévy process, a Wiener integral $\int_0^t \sigma_s dW_s$, or a combination of both, leading to a *jump-diffusion model* or a *semimartingale* model. The simplest jump-diffusion model is of the form

$$S_t := S_0 e^{bt+\sigma W_t + Z_t}, \tag{2}$$

where $Z := \{Z_t\}_{t\geq 0}$ is a “pure-jump” Lévy process. Even this simple extension is able to accommodate several stylized features such as heavy tails, high-kurtosis, and asymmetry. There are several reasons in the support of “jumps” in the dynamics of the stock prices. On one hand, there exists event-driven information coming up at discrete unpredictable times that produce “sudden” price shifts (*jumps*). Second, in fact stock prices are made up discrete trades through time arriving at very high frequency. Hence, processes exhibiting infinitely many jumps in any finite time horizon $[0, T]$ are arguably better approximations to such high-activity stochastic processes.

Merton [34], following Press [39], proposed one of the earliest models of the form (20), taking a compound Poisson process Z with normally distributed jumps. However, earlier Mandelbrot [33] had already proposed a pure-jump model driven by a stable Lévy process Z. Merton's model is considered to exhibit light tails as all exponential moments of the densities of $\log(S_t/S_0)$ are finite, while Mandelbrot's model exhibit very heavy tails with not even finite second moments. It was during the last decade that models exhibiting appropriate tail behavior were proposed. Among the better known models are the *variance Gamma model* of [12], the *CGMY model* of [10], and the *generalized hyperbolic motion* of [5, 6] and [19, 18]. We refer to Kyprianou et. al. [31, Chapter 1] and Cont and Tankov [17, Chapter 4] for a more extensive review of different types of geometric Lévy models in finance.

The geometric Lévy model (20) cannot incorporate volatility clustering and leverage effects due to the fact that log returns will satisfy the random walk property. To cope with this shortcoming, there have been two general classes of models driven by Lévy processes. The first approach due to Barndorff-Nielsen and Shephard (see e.g. [6]) proposes a stochastic volatility model of the form

$$S_t := S_0 e^{\int_0^t b_u du + \int_0^t \sigma_u W_u}, \tag{3}$$

where σ is a stationary non-Gaussian Ornstein-Uhlenbeck process

$$\sigma_t^2 = \sigma_0^2 + \int_0^t \alpha \sigma_s^2 ds + Z_{\alpha t},$$

driven by a subordinator Z (i.e. a non-decreasing Lévy process). These and other related models are surveyed in the two recent reviews Shephard [47] and Andersen and Benzoni [3]. The second approach, proposed by Carr et. al. [11, 13], introduces stochastic volatility via a random clock as follows:

$$S_t = S_0 e^{Z_{\tau(t)}}, \qquad \text{with} \qquad \tau(t) := \int_0^t r(u)du. \tag{4}$$

The process τ plays the role of a "business" clock which could reflect non-synchronous trading effects or a "cumulative measure of economic activity". Roughly speaking, the rate process r controls the volatility of the process; for instance, in time periods where r is high, the "business time" τ runs faster resulting in more frequent jump times. Hence, positive *mean-revering diffusion processes* $\{r(t)\}_{t\geq 0}$ are plausible choices to account for the volatility clustering effect.

To incorporate the leverage phenomenon, different combinations of the previous models have been considered leading to semimartingale models driven by Wiener and Poisson random measures. A very general model in this direction models the log return process $X_t := \log(S_t/S_0)$ as follows (c.f. [27], [48]):

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|x|\leq 1} \delta(s,x)\bar{M}(ds,dx) + \int_0^t \int_{|x|>1} \delta(s,x)M(ds,dx) \\ \sigma_t &= \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \int_{|x|\leq 1} \tilde{\delta}(s,x)\bar{M}(ds,dx) + \int_0^t \int_{|x|>1} \tilde{\delta}(s,x)M(ds,dx), \end{aligned}$$

where W is a $d-$dimensional Wiener process, M is the jump measure of an indepdent Lévy process Z, defined by

$$M(B) := \#\{(t, \Delta Z_t) \in B : t > 0 \text{ such that } \Delta Z_t \neq 0\},$$

and $\bar{M}(dt,dx) := M(dt,dx) - \nu(dx)dt$ is the compensate Poisson random measure of Z, where ν is the Lévy measure of Z. The integrands (b, σ, etc.) are random processes themselves, which could even depend on X and σ leading to a system of stochastic differential equations. One of the most active research fields in this very general setting is that of statistical inference methods based on high-frequency (intraday) financial data. Some of the methods studied include the prediction of the integrated volatility process $\int_0^t \sigma_s^2 ds$ or of the Poisson integrals $\int_0^t \int_{\mathbb{R}\setminus\{0\}} g(x)\mu(dx,ds)$ based on realized variations of the process (see e.g. [27, 28], [32], [50, 51], [36], [7]), testing for jumps ([7], [36], [1]), and estimation in the presence of "microstructure" noise ([2], [37, 38]).

In this work, the basic methods and tools behind jump-diffusion models driven by Lévy processes are covered. The chapter will provide an accessible overview of the probabilistic concepts and results, coupled with their financial application and relevance. Some of the topics include: Lévy processes and Poisson random measures, statistical estimation based on high- and low-frequency observations, a primer of stochastic calculus for jump processes, density transformation and risk-neutral change of measures, arbitrage-free option pricing and integro-partial differential equations. The material will be drawn upon recent monographs (c.f. [17], [46], [45]) and recent papers in the field.

2 Distributional properties and statistical estimation of Lévy processes

2.1 Definition and fundamental examples

A Lévy process is a probabilistic model for an unpredictable measurement X_t that evolves in time t, in such a way that the change of the measurement in disjoint time intervals of equal duration, say $X_{s+\Delta} - X_s$ and $X_{t+\Delta} - X_t$ with $s + \Delta \leq t$, are independent from one another but with identical distribution. For instance, if S_t represents the time-t price of an asset and X_t is the *log return during* $[0, t]$, defined by

$$X_t = \log\left(S_t/S_0\right),$$

then the previous property will imply that daily or weekly log returns will be independent from one another with common distribution. Formally, we have the following.

Definition 1 *A* ***Lévy process*** $X = \{X_t\}_{t\geq 1}$ *is a* $\mathbb{R}^d$*-valued stochastic process (collection of random vectors in* $\mathbb{R}^d$ *indexed by time) defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

(i) $X_0 = 0$*;*

(ii) X *has* ***independent increments****:* $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent for any* $0 \leq t_0 < \cdots < t_n$*;*

(iii) X *has* ***stationary increments****: the distribution of* $X_{t+\Delta} - X_t$ *is the same as* X_Δ*, for all* $t, \Delta \geq 0$*;*

(iv) its paths are right-continuous with left-limits (rcll);

(v) it has ***no fixed jump-times****; that is,* $\mathbb{P}(\Delta X_t \neq 0) = 0$*, for any time* t*.*

The last property can be replaced by asking that X is continuous in probability, namely, $X_s \xrightarrow{\mathbb{P}} X_t$, as $s \to t$, for any t. Also, if X satisfies all the other properties except (iv), then there exists a rcll version of the process.

There are three fundamental examples of Lévy processes that deserve some revision: Brownian motion, Poisson process, and compound Poisson process.

Definition 2 *A (standard) Brownian motion* W *is a real-valued process such that (i)* $W_0 = 0$*, (ii) it has independent increments, (iii)* $W_t - W_s$ *has normal distribution with mean* 0 *and variance* $t - s$*, for any* $s < t$*, and (iv) it has continuous paths.*

It turns out that the only real Lévy processes with continuous paths are of the form

$$X_t = \sigma W_t + bt,$$

for constants $\sigma > 0$ and b.

A Poisson process is another fundamental type of Lévy process that is often used as building blocks of other processes.

Definition 3 *A Poisson process N is an integer-valued process such that (i) $N_0 = 0$, (ii) it has independent increments, (iii) $N_t - N_s$ has Poisson distribution with parameter $\lambda(t-s)$, for any $s < t$, and (iv) its paths are rcll. The parameter λ is called the intensity of the process.*

The process N_t is frequently used as a counting process of events of certain type (say, car accidents) occurring by time t. That is,

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}_{T_i \leq t}, \tag{5}$$

where $\{T_i\}_{i\geq 1}$ represents the occurrence times of the event in question. Roughly speaking, if the events occur independently from one another, homogeneously in time, and with an intensity of λ events per unit time, then (5) is a Poisson process (a consequence of the Binomial approximation to the Poisson distribution); see e.g. Feller [20] for this heuristic construction of a Poisson process.

It turns out that any Poisson process is of the form (5) with T_i such that the time span between events,

$$\tau_i := T_i - T_{i-1},$$

are mutually independent with exponential distribution with mean $1/\lambda$ (so, the bigger the λ, the smaller the expected waiting time between events and the higher the intensity of events).

To introduce the next fundamental example, we recall the concept of probability distribution. Given a random vector J in $\mathbb{R}^d$, the distribution of J is the mapping ρ such that

$$\rho(A) := \mathbb{P}(J \in A),$$

for any $A \subset \mathbb{R}^d$. A *compound Poisson process* with jump distribution ρ and jump intensity λ is a process of the form

$$Z_t := \sum_{i=1}^{N_t} J_i,$$

where $\{J_i\}_{i\geq 1}$ are independent copies of J and N is a Poisson process with intensity λ that is independent of $\{J_i\}_i$. Intuitively a compound Poisson process $X := \{Z_t\}_{t\geq 0}$ is like a Poisson process but with random independent jumps. A compound Poisson process is the only Lévy process that has piece-wise constant paths with finitely-many jumps in any time interval $[0, T]$. Notice that the distribution of the compound Poisson process Z is characterized by the finite measure:

$$\nu(A) := \lambda\rho(A), \quad A \subset \mathbb{R}^d,$$

called the *Lévy measure* of Z. Furthermore, for any finite measure ν, there exists a compound Poisson process Z with Lévy measure ν; namely, the compound Poisson process with intensity of jumps $\lambda := \nu(\mathbb{R}^d)$ and jump distribution $\rho(dx) := \nu(dx)/\nu(\mathbb{R}^d)$.

For future reference, it is useful to notice that the characteristic function of Z_t is given by

$$\mathbb{E}e^{i\langle u, Z_t\rangle} = \exp\left\{ t \int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1\right) \nu(dx) \right\} \tag{6}$$

Also, if $\mathbb{E}|J_i| = \int |x|\rho(dx) < \infty$, then $\mathbb{E}Z_t = t\int x\rho(dx)$ and the so-called *compensated compound Poisson process* $\bar{Z}_t := Z_t - \mathbb{E}Z_t$ has characteristic function

$$\mathbb{E}e^{i\langle u,\bar{Z}_t\rangle} = \exp\left\{ t\int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\right)\nu(dx)\right\}. \tag{7}$$

Any Lévy process X can be approximated arbitrarily close by the superposition of a Brownian motion with drift, $\sigma W_t + bt$, and an independent compound Poisson process Z. The reminder $R_t := X_t - (\sigma W_t + bt + Z_t)$ is a pure-jump Lévy process with jumps sizes smaller than say an $\varepsilon > 0$, which can be taken arbitrarily small. The previous fundamental fact is a consequence of the Lévy-Itô decomposition that we review below.

2.2 Infinitely divisible distributions and the Lévy-Khinchine formula

The marginal distributions of a Lévy process X are *infinitely-divisible*. A random variable ξ is said to be *infinitely divisible* if for each $n \geq 2$, one can construct n i.i.d. r.v's . $\xi_{n,1},\ldots,\xi_{n,n}$ such that

$$\xi \stackrel{\mathfrak{D}}{=} \xi_{n,1} + \cdots + \xi_{n,n}.$$

That X_t is infinitely divisible is clear since

$$X_t = \sum_{k=0}^{n-1}\left(X_{(k+1)t/n} - X_{kt/n}\right),$$

and $\{X_{(k+1)t/n} - X_{kt/n}\}_{k=0}^{n-1}$ are i.i.d. The class of infinitely divisible distributions is closely related to limits in distribution of an array of row-wise i.i.d. r.v.'s:

Theorem 1 (Kallenberg [29]) *ξ is infinitely divisible iff for each n there exists i.i.d. random variables $\{\xi_{n,k}\}_{k=1}^{k_n}$ such that*

$$\sum_{k=1}^{k_n} \xi_{n,k} \xrightarrow{\mathfrak{D}} \xi, \quad \text{as} \quad n \to \infty.$$

In term of the characteristic function of ξ,

$$\varphi_\xi(u) := \mathbb{E}e^{i\langle u,\xi\rangle},$$

ξ is infinitely divisible if and only if $\varphi_\xi(u) \neq 0$, for all u, and the distinguished n^{th}-root $\varphi_\xi(u)^{1/n}$ is a characteristic function for each n (see Lemma 7.6 in [45]). This property of the characteristic function turns out to be sufficient to determine its form in terms of three "parameters" (A, b, ν), called the *Lévy triplet* of ξ (see e.g. Sato [45] for a proof):

Theorem 2 (Lévy-Khinchine formula) *ξ is infinitely divisible iff*

$$\mathbb{E}e^{i\langle u,\xi\rangle} = \exp\left\{ i\langle b,u\rangle - \frac{1}{2}\langle u,Au\rangle + \int \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle \mathbf{1}_{|x|\le 1}\right)\nu(dx)\right\}, \tag{8}$$

for some symmetric nonnegative-definite matrix A, a vector $b \in \mathbb{R}^d$, and a measure ν on $\mathbb{R}_0^d$ such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d\setminus\{0\}} (|x|^2 \wedge 1)\nu(dx) < \infty. \tag{9}$$

Moreover, all triplets (A, b, ν) with the stated properties may occur.

The following remarks are important:

Remark 1 *The previous result implies that the time-t marginal distribution of a Lévy process $\{X_t\}_{t\ge 0}$ is identified with a Lévy triplet (A_t, b_t, ν_t). Given that X has stationary and independent increments, it follows that $\mathbb{E}e^{i\langle u,X_t\rangle} = \left\{\mathbb{E}e^{i\langle u,X_1\rangle}\right\}^t$, for any rational t and by the right-continuity of X, for any real t. Thus, if (A, b, ν) is the Lévy triplet of X_1, then $(A_t, b_t, \nu_t) = t(A, b, \nu)$ and*

$$\varphi_{X_t}(u) := \mathbb{E}e^{i\langle u,X_t\rangle} = e^{t\psi(u)}, \qquad \textit{where} \tag{10}$$

$$\psi(u) := i\langle b,u\rangle - \frac{1}{2}\langle u,Au\rangle + \int \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle \mathbf{1}_{|x|\le 1}\right)\nu(dx). \tag{11}$$

The triple (A, b, ν) are called the Lévy triplet of the Lévy process X.

Remark 2 *The exponent in (11) is called the* Lévy exponent *of the Lévy process $\{X_t\}_{t\ge 0}$. We can see that its first term is the Lévy exponent of the Lévy process bt. The second term of $\psi(u)$ is the Lévy exponent of the Lévy process ΣW_t, where $W = (W^1, \dots, W^d)^T$ are $d-$independent Wiener processes and Σ is a $d \times d$ lower triangular matrix in the Cholesky decomposition $A = \Sigma\Sigma^T$. The last term in the Lévy exponent can be decomposed into two terms:*

$$\psi^{cp}(u) = \int_{|x|>1} \left(e^{i\langle u,x\rangle} - 1\right)\nu(dx), \quad \psi^{lccp}(u) = \int_{|x|\le 1} \left(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\right)\nu(dx).$$

The first term above is the Lévy exponent of a compound Poisson process X^{cp} with Lévy measure $\nu_1(dx) := \mathbf{1}_{|x|>1}\nu(dx)$ (see (7)). The exponent ψ^{lccp} corresponds to the limit in distribution of compensated compound Poisson processes*. Concretely, suppose that $X^{(\varepsilon)}$ is a compound Poisson process with Lévy measure $\nu_\varepsilon(dx) := \mathbf{1}_{\varepsilon<|x|\le 1}\nu(dx)$, then the process $X_t^{(\varepsilon)} - \mathbb{E}X_t^{(\varepsilon)}$ converges in distribution to a process with characteristic function* $\exp\{t\psi^{lccp}\}$ *(see (7)). Lévy-Khinchine formula implies that, in distribution, X is the superposition of four independent Lévy processes as follows:*

$$X_t \stackrel{\mathfrak{D}}{=} \underbrace{bt}_{\textit{Drift}} + \underbrace{\Sigma W_t}_{\textit{Brownian part}} + \underbrace{X_t^{cp}}_{\textit{Cmpnd. Poisson}} + \underbrace{\lim_{\varepsilon\searrow 0}\left(X_t^{(\varepsilon)} - \mathbb{E}X_t^{(\varepsilon)}\right)}_{\textit{Limit cmpstd cmpnd Poisson}}. \tag{12}$$

where equality is in the sense of finite-dimensional distributions. The condition (22) on ν guarantees the X^{cp} is indeed well defined and the compensated compound Poisson converges in distribution.

In the rest of this section, we go over some fundamental distributional properties of the Lévy process and their applications.

2.3 Short-term distributional behavior

The characteristic function (10) of X determines uniquely the Lévy triple (A, b, ν). For instance, the uniqueness of the matrix A is a consequence of the following limit:

$$\lim_{h \to 0} h \cdot \log \varphi_{X_t}\left(h^{-1/2} u\right) = -\frac{t}{2} \langle u, Au \rangle ; \tag{13}$$

see pp. 40 in [45]. In term of the process X, (13) implies that

$$\left\{ \frac{1}{\sqrt{h}} X_{ht} \right\}_{t \geq 0} \xrightarrow{\mathfrak{D}} \{\Sigma W_t\}_{t \geq 0}, \quad \text{as } h \to 0. \tag{14}$$

where $W = (W^1, \ldots, W^d)^T$ are $d-$independent Wiener processes and Σ is a lower triangular matrix such that $A = \Sigma\Sigma^T$.

From a statistical point of view, (14) means that, when $\Sigma \neq 0$, the short-term increments $\{X_{(k+1)h} - X_{kh}\}_{k=1}^n$, properly scaled, behave like the increments of a Wiener process. In the context of an exponential Lévy model, where the stock price process $\{S_t\}_{t \geq 0}$ is given by

$$S_t = S_0 e^{X_t},$$

for a real Lévy process X, (14) will imply that the log returns of the stock, properly scaled, are normally distributed when the Brownian component of the Lévy process X is not null. This is not consistent with empirical observations. Recently, Rosiński [43] proposes a pure-jump class, called tempered Lévy processes, such that

$$\left\{ \frac{1}{\sqrt{h}} X_{ht} \right\}_{t \geq 0} \xrightarrow{\mathfrak{D}} \{Z_t\}_{t \geq 0}, \quad \text{as } h \to 0, \tag{15}$$

where Z is a stable process with index $\alpha < 2$. This class includes the CGMY model of[10].

2.4 Moments and short-term moment asymptotics

Let $g : \mathbb{R}^d \to \mathbb{R}_+$ be a nonnegative locally bounded function and X be a Lévy process with Lévy triplet (A, b, ν). The g-moment of a random variable ξ is defined by $\mathbb{E}g(\xi)$. In the case of a compound Poisson process, it is clear that

$$\mathbb{E}g(X_t) < \infty, \text{ for any } t > 0 \text{ if and only if } \int_{|x|>1} g(x)\nu(dx) < \infty.$$

The previous relation holds for general Lévy processes if g is submultiplicative or subaddditive (see Kruglov [30] and Sato [45, Theorem 25.3]). Recall that a nonnegative locally bounded function g is submultiplicative (resp. subadditive) if there exists a constant $K > 0$

such that $g(x+y) \le Kg(x)g(y)$ (resp. $g(x+y) \le K(g(x)+g(y))$), for all x, y. Examples of this kind of functions are $g(x_1, \ldots, x_d) = |x_j|^p$, for $p \ge 1$, and $g(x_1, \ldots, x_d) = \exp\{|x_j|^\beta\}$, for $\beta \in (0, 1]$.

As a consequence, $X(t) := X_t$ has finite mean if and only if $\int_{|x|>1} |x|\nu(dx) < \infty$. In that case, by differentiation of the characteristic function, it follows that

$$\mathbb{E}X_j(t) = t\left(\int_{|x|>1} x_j \nu(dx) + b_j\right),$$

Similarly, $\mathbb{E}|X_t|^2 < \infty$ if and only if $\int_{|x|>1} |x|^2\nu(dx) < \infty$, in which case,

$$\mathrm{Cov}\left(X_j(t), X_k(t)\right) = t\left(A_{jk} + \int x_j x_k \nu(dx)\right).$$

The two equations above show the connection between the the Lévy triplet (A, b, ν), and the mean and covariance of the process. Notice that the variance rate $\mathrm{Var}(X_j(t))/t$ remains constant over time. It can also be shown that the kurtosis is inversely proportional to time t. In the risk-neutral world, these facts contradict the empirical evidence which suggest that both measurements increase with time t (see e.g. [11] and references therein).

The Lévy measure ν controls the short-term ergodic behavior of X. Namely, for any bounded continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$ vanishing on a neighborhood of the origin, it holds that

$$\lim_{t\to 0} \frac{1}{t}\, \mathbb{E}\varphi(X_t) = \int \varphi(x)\nu(dx); \tag{16}$$

cf. Sato [45, Corollary 8.9]. For a real Lévy processes X with Lévy triplet (σ^2, b, ν), (16) can be extended to incorporate unbounded functions and different behaviors at the origin. Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is $\nu-$continuous such that $|\varphi| \le g$ for a subadditive or submultiplicative function $g : \mathbb{R} \to \mathbb{R}_+$. Furthermore, fixing $I := \{r \ge 0 : \int \left(|x|^r \wedge 1\right)\nu(dx) < \infty\}$, assume that φ exhibits the following behavior as $x \to 0$:

(a) i. $\varphi(x) = o(|x|^2)$;
  ii. $\varphi(x) = O(|x|^r)$, for some $r \in I \cap (1, 2)$ and $\sigma = 0$;
  iii. $\varphi(x) = o(|x|)$, $1 \in I$ and $\sigma = 0$;
  iv. $\varphi(x) = O(|x|^r)$, for some $r \in I \cap (0, 1)$, $\sigma = 0$, and $\bar{b} := b - \int_{|x|\le 1} x\nu(dx) = 0$.

(b) $\varphi(x) \sim x^2$;

(c) $\varphi(x) \sim |x|$ and $\sigma = 0$.

Building on results in [50] and [28], [24] proves that

$$\lim_{t\to 0} \frac{1}{t}\, \mathbb{E}\varphi(X_t) := \begin{cases} \int \varphi(x)\nu(dx), & \text{if (a) holds,} \\ \sigma^2 + \int \varphi(x)\nu(dx), & \text{if (b) holds,} \\ |\bar{b}| + \int \varphi(x)\nu(dx), & \text{if (c) holds.} \end{cases} \tag{17}$$

Woerner [50] and also Figueroa-López [21] used the previous short-term ergodic property to show the consistency of statistics

$$\hat{\beta}^{\pi}(\varphi) := \frac{1}{t_n} \sum_{k=1}^{n} \varphi\left(X_{t_k} - X_{t_{k-1}}\right), \tag{18}$$

towards the integral parameter

$$\beta(\varphi) := \int \varphi(x)\nu(dx), \tag{19}$$

when $t_n \to \infty$ and $\max\{t_k - t_{k-1}\} \to 0$, for test functions φ as in (a). When $\nu(dx) = s(x)dx$, Figueroa-López [21] applied the estimators (18) to analyze the asymptotic properties of nonparametric *sieve-type* estimators $\hat{s}$ for s. The problem of model selection was analyzed further in [25, 22], where it was proved that sieve estimators $\widetilde{s}_T$ can match the rate of convergence of the minimax risk of estimators $\hat{s}$. Concretely, it turns out that

$$\limsup_{T\to\infty} \frac{\mathbb{E}\|s - \widetilde{s}_T\|^2}{\inf_{\hat{s}} \sup_{s\in\Theta} \mathbb{E}\|s - \hat{s}\|^2} < \infty,$$

where $[0, T]$ is the time horizon over which we observe the process X, Θ is certain class of smooth functions, and the infimum in the denominator is over all estimators $\hat{s}$ which are based on whole trajectory $\{X_t\}_{t\le T}$. The optimal rate of the estimator $\widetilde{s}_T$ is attained by choosing appropriately the dimension of the sieve and the sampling frequency in function of T and the smoothness of the class of functions Θ. In [23], the sieve estimators of [22] were also used to built confidence intervals (CI) and confidence bands for the Lévy density s.

2.5 Extraction of the Lévy measure

The Lévy measure ν can be inferred from the characteristic function $\varphi_{X_t}(u)$ of the Lévy process . Following Sato [45, pp. 40-41], one can recover $\langle u, Au\rangle$ from (13) and define

$$\Psi(u) := \log \varphi_{X_1}(u) + \frac{1}{2}\langle u, Au\rangle.$$

Then, one can verify that

$$\int_{[-1,1]^d} \left(\Psi(u) - \Psi(u+w)\right) dw = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\tilde{\nu}(dx), \tag{20}$$

where $\tilde{\nu}$ is the finite measure

$$\tilde{\nu}(dx) := 2^d \left(1 - \prod_{j=1}^{d} \frac{\sin x_j}{x_j}\right)\nu(dx).$$

Hence, ν can be recovered from the inverse Fourier transform of the left-hand side of (20).

The above method can be applied to devise non-parametric estimation of the Lévy measure by replacing the Fourier transform φ_{X_1} by its empirical version:

$$\hat{\varphi}_{X_1}(u) := \frac{1}{n}\sum_{k=1}^{n} \exp\left\{i\left\langle u, X_k - X_{k-1}\right\rangle\right\}.$$

given discrete observations $X_1, \ldots, X_n$ of the process. Recently, similar nonparametric methods have been proposed in the literature to estimate the Lévy density $s(x) = \nu(dx)/dx$ of a real Lévy process X (c.f. [35], [15], and [26]). For instance, based on the increments $X_1 - X_0, \ldots, X_n - X_{(n-1)}$, Neumann and Reiss [35] consider a nonparametric estimator for s that minimizes the distance between the "population" characteristic function $\varphi_{X_1}(\cdot; s)$ and the empirical characteristic function $\hat{\varphi}_{X_1}(\cdot)$. By appropriately defining the distance metric, Neumann and Reiss (2008) showed the consistency of the proposed estimators. Another approach, followed for instance by Watteel and Kulperger [49] and Comte and Genon-Catalot [15], relies on an "explicit" formula for the Lévy density s in terms of the derivatives of the characteristic function φ_{X_1}. For instance, under certain regularity conditions,

$$\mathcal{F}\left(xs(x)\right)(\cdot) = -i\frac{\varphi'_{X_1}(\cdot)}{\varphi_{X_1}(\cdot)},$$

where $\mathcal{F}(f)(u) = \int e^{iux} f(x)dx$ denotes the Fourier transform of a function f. Hence, an estimator for s can be built by replacing ψ by a smooth version of the empirical estimate $\hat{\varphi}_{X_1}$ and applying inverse Fourier transform $\mathcal{F}^{-1}$.

3 Path decomposition of Lévy processes

In this part, we show that the construction in (12) holds true almost surely. In other words, almost surely, any Lévy process X is the superposition of a drift bt, a Brownian component ΣW_t, a compound Poisson X_t^{cp}, and the limit of compensated Poisson processes. The fundamental tool of this result is a probabilistic characterization of the points $\{(t, \Delta X_t) : t \text{ s.t. } \Delta X_t \neq 0\}$ as a Poisson point process. Due to this fact, we first review the properties of Poisson random measures, which in any case are important building blocks of financial models.

3.1 Poisson random measures

Definition 4 *Let S be a Borel subset of $\mathbb{R}^d$, let $\mathcal{S}$ be the set of Borel subsets of S, and let m be a $\sigma-$finite measure on S. A collection $\{M(B) : B \in \mathcal{S}\}$ of $\bar{\mathbb{Z}}_+$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a* ***Poisson random measure*** *(or process) on S with mean measure m if*

(1) for every $B \in \mathcal{S}$, $M(B)$ is a Poisson random variable with mean $m(B)$;

(2) if $B_1, \ldots, B_n \in \mathcal{S}$ are disjoint, then $M(B_1), \ldots, M(B_1)$ are independent;

(3) for every sample outcome $\omega \in \Omega$, $M(\cdot;\omega)$ is a measure on $\mathcal{S}$.

It can be proved that (a.s.), $M(\cdot;\omega)$ is an atomic measure; that is, there exists countably many (random) points $\{\mathbf{x}_i\}_i \subset S$ such that

$$M(B) = \#\{i : \mathbf{x}_i \in B\} = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i}(\cdot).$$

Sometimes, the random points $\{\mathbf{x}_i\}_i$ is called a **Poisson point process** on the S with mean measure m. The following is a common procedure to construct a realization of the Poisson random measure by conditioning:

1. Suppose that $B_1, B_2, \ldots$ is a partition of S such that $m(B_j) < \infty$.
2. Generate $n_j \sim Poiss(m(B_j))$.
3. Independently, generate n_j-points, say $\{\mathbf{x}_i^j\}_{i=1}^{n_j}$, according to the distribution $m(\cdot)/m(B_j)$.
4. Define $M(B) = \#\{(i,j) : \mathbf{x}_i^j \in B\}$.

Transformation of Poisson random measures

Among the most useful properties is that certain transformations of a Poisson point process are still a Poisson point process. The following is the simplest version:

Proposition 1 *Suppose that $T : S \to S' \subset \mathbb{R}^{d'}$ be measurable. Then, the random measure associated with the transformed points $\mathbf{x}_i' := T(\mathbf{x}_i)$, namely $M'(\cdot) = \sum_{i=1}^{\infty} \delta_{\mathbf{x}_i'}(\cdot)$, is also a Poisson random measure with mean measure $m'(B) := m(\{\mathbf{x} : T(\mathbf{x}) \in B\})$.*

The following result shows that a *marked Poisson point process* is still a Poisson. Suppose that we associate a numeric mark or score $\mathbf{x}_i'$ to each point $\mathbf{x}_i$ of M. The marks are assigned independently from one another. The distribution of the mark can actually depend on the point $\mathbf{x}_i$. Concretely, let $\sigma(\mathbf{x}, d\mathbf{x}')$ be a probability measure on $S' \subset \mathbb{R}^{d'}$, for each $\mathbf{x} \in S$. For each i, generate a r.v. $\mathbf{x}_i'$ according $\sigma(\mathbf{x}_i, d\mathbf{x}')$ (*independently from any other variable*). Consider the so-called *marked Poisson process*

$$M'(\cdot) = \sum_{i=1}^{\infty} \delta_{(\mathbf{x}_i, \mathbf{x}_i')}(\cdot),$$

Proposition 2 *M' is a Poisson random measure with mean measure*

$$m'(d\mathbf{x}, d\mathbf{x}') = \sigma(\mathbf{x}, d\mathbf{x}')m(d\mathbf{x}).$$

Consider the following experiment. We classify the point of the Poisson process M into k different types. The probability that point $\mathbf{x}_i$ is of type j is $p_j(\mathbf{x}_i)$, independently from any other classification. Let $\{\mathbf{y}_i^j\}$ be the points of $\{\mathbf{x}_i\}$ of type j and let M^j be the counting measure associated with $\{\mathbf{y}_i^j\}$:

$$M^j := \sum \delta_{\{\mathbf{y}_i^j\}}$$

We say that the process M^1 is constructed from M by *thinning*.

Proposition 3 *$M^1, \ldots, M^k$ are independent Poisson random measures with respective mean measures $m_1(d\mathbf{x}) := p_1(\mathbf{x})m(d\mathbf{x}), \ldots, m_k(d\mathbf{x}) := p_k(\mathbf{x})m(d\mathbf{x})$.*

Example 1 *Suppose that we want to simulate a Poisson point process on the unit circle $S := \{(x,y) : x^2+y^2 \leq 1\}$ with mean measure:*

$$m'(B) = \iint_{B \cap S} \sqrt{x^2+y^2}\,dxdy.$$

A method to do this is based on the previous thinning method. Suppose that we generate a "homogeneous" Poisson point process M on the square $R := \{(x,y) : |x| \leq 1, |y| \leq 1\}$ with intensity of λ points per unit area. That is, the mean measure of M is

$$m(B) = \frac{1}{4}\iint_B \lambda dxdy.$$

Let $\{(x_i,y_i)\}_i$ denote the atoms of the Poisson random measure M. Now, consider the following thinning process. We classify the point (x_i,y_i) of type 1 with probability $p(x_i,y_i) := \sqrt{x_i^2+y_i^2}$ and of type 2 with probability $1-p(x_i,y_i)$. Suppose that $\{(y_i^1,y_i^1)\}_i$ are the point of type 1. Then, this process is a Poisson point process with mean measure

$$m'(dx,dy) = \frac{1}{4}\lambda p(x,y)dxdy.$$

It suffices to take $\lambda = 4$.

Integration with respect to a Poisson random measure

Let M be a Poisson random measure as Definition 4. Since $M(\cdot;\omega)$ is an atomic random measure for each ω, say $M(\cdot;\omega) = \sum_{i=1}^{\infty}\delta_{\mathbf{x}_i(\omega)}(\cdot)$, one can define the integral

$$M(f) := \int_S f(\mathbf{x})M(d\mathbf{x}) = \sum_{i=1}^{\infty} f(\mathbf{x}_i),$$

for any measurable **nonnegative** deterministic function f. This is a $\bar{\mathbb{R}}_+ = \mathbb{R} \cup \{\infty\}$-valued r.v. such that

$$\mathbb{E}\left[e^{-\int f(\mathbf{x})M(d\mathbf{x})}\right] = \exp\left\{-\int\left(1-e^{-f(\mathbf{x})}\right)m(d\mathbf{x})\right\}, \quad \mathbb{E}\left[\int f(\mathbf{x})M(d\mathbf{x})\right] = \int f(\mathbf{x})m(d\mathbf{x});$$

see e.g. [29, Lemma 10.2]. Also, if $B \in \mathcal{S}$ is such that $m(B) < \infty$, then

$$\int_B f(\mathbf{x}) M(d\mathbf{x}) := \sum_{i:\mathbf{x}_i \in B} f(\mathbf{x}_i),$$

is a well-defined $\mathbb{R}^d$−valued r.v. for any measurable function $f : \mathcal{S} \to \mathbb{R}^d$. Its characteristic function is given by

$$\mathbb{E}\left[e^{\mathbf{i}\left\langle \int_B f(\mathbf{x})M(d\mathbf{x}),\mathbf{u}\right\rangle}\right] = \exp\left\{\int_B \left(e^{\mathbf{i}\langle f(\mathbf{x}),\mathbf{u}\rangle} - 1\right) m(d\mathbf{x})\right\}.$$

Furthermore, if $B_1, \ldots, B_m$ are disjoints sets in $\mathcal{S}$ with finite measure, then

$$\int_{B_1} f(\mathbf{x})M(d\mathbf{x}), \ldots, \int_{B_m} f(\mathbf{x})M(d\mathbf{x}).$$

are independent (see e.g. [45, Proposition 19.5]).

In the general case, the problem of determining conditions for the integral $\int_S f(\mathbf{x})M(d\mathbf{x})$ to be well-defined require some care. Let us assume that m is a radon measure (that is, $m(K) < \infty$, for any compact $K \subset S$). Then, $\int_S f(\mathbf{x})M(d\mathbf{x}) = \sum_{i=1}^{\infty} f(\mathbf{x}_i)$, is well-defined for any bounded function $f : S \to \mathbb{R}$ of compact support. We say that the integral $\int_S f(\mathbf{x})M(d\mathbf{x})$ exists if there exists a random variable X such that

$$X \stackrel{\mathbb{P}}{=} \lim_{n\to\infty} \int_S f_n(\mathbf{x})M(d\mathbf{x})$$

for any sequence f_n of bounded functions with compact support such that $|f_n| \leq |f|$ and $f_n \to f$. In that case, $\int_S f(\mathbf{x})M(d\mathbf{x})$ is defined to be that common limit X. We define in a similar way the so-called *compensated integral of* f $\int_S f(\mathbf{x})(M - m)(d\mathbf{x})$. The following theorem gives conditions for the existence of the above integrals (see [29, Theorem 10.15]):

Proposition 4 *Let M be a Poisson random measure as Definition 4. Then,*

(a) $M(f) = \int_S f(\mathbf{x})M(d\mathbf{x})$ exists iff $\int_S (|f(\mathbf{x})| \wedge 1)m(d\mathbf{x}) < \infty$

(b) $(M - m)(f) := \int_S f(\mathbf{x})(M - m)(d\mathbf{x})$ exists iff $\int_S (|f(\mathbf{x})|^2 \wedge |f(\mathbf{x})|)m(d\mathbf{x}) < \infty$

3.2 The Lévy-Itô theorem

Theorem 3 ***[13.4, Kallenberg]*** *Let $\{X(t)\}_{t\geq 0}$ be an rcll process in $\mathbb{R}^d$ with $X(0) = 0$. Then, X has independent increments without fixed jumps times if and only if, there is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$,*

$$X_t(\omega) = b_t(\omega) + G_t(\omega) + \int_0^t \int_{|x|>1} x \; M(\omega; ds, dx) + \int_0^t \int_{|x|\leq 1} x \; (M - m)(\omega; ds, dx), \quad (21)$$

for any $t \geq 0$, and for some continuous function b with $b_0 = 0$, some continuous centered Gaussian process G with independent increments and $G_0 = 0$, and some independent Poisson random measure M on $[0, \infty) \times \mathbb{R}_0^d$ with mean measure m satisfying

$$\int_0^t \int_{\mathbb{R}_0^d} (|x|^2 \wedge 1)\, m(ds, dx) < \infty, \quad \forall t > 0. \tag{22}$$

The representation is almost surely unique, and all functions b, processes G, and measures m with the stated properties may occur.

The above theorem states that the *jump random measure* M_X of X, defined by

$$M_X((s,t] \times B) := \sum_{u \in (s,t]: \Delta X_s \neq 0} \mathbf{1}\{\Delta X_u \in B\},$$

is a Poisson process with mean measure $m(dt, dx)$. In the case of a Lévy process (that is, we also assume that X has stationary increments), the previous theorem implies that the random point process

$$\{(t, \Delta X_t) : t > 0 \text{ such that } \Delta X_t \neq 0\}$$

is a Poisson point process in $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$ with mean measure $m(dt, dx) = \nu(dx)dt$. In that case, the representation (21) takes the following form:

$$X(t) = bt + \Sigma W_t + \int_0^t \int_{|x|>1} x\, M(ds, dx) + \int_0^t \int_{|x|\leq 1} x\, (M - m)(ds, dx), \tag{23}$$

where W is a d-dimensional Wiener process. The third term is a compound Poisson process with intensity of jumps $\nu(|x| > 1)$ and jump distribution $\mathbf{1}_{|x|>1}\nu(dx)/\nu(|x| > 1)$. Similarly, the last term can be understood as the limit of compensated Poisson processes as follows:

$$\int_0^t \int_{|x|\leq 1} x(M - m)(ds, dx) = \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\varepsilon < x| \leq 1} x\, (M - m)(ds, dx). \tag{24}$$

Furthermore, the convergence is uniform on any bounded interval for t (c.f. [19.2, Sato]).

3.3 Sample path properties

One application of the Lévy-Itô decomposition is to determine conditions for certain path behavior. The following are some cases of interest (see Section 19 in [45] for these and other path properties):

1. Path-continuity: It is now clear that the only Lévy processes with continuous paths are essentially of the form $bt + \sigma W_t$.

2. Finite-variation: A necessary and sufficient condition is that $\sigma = 0$ and

$$\int_{|x|\leq 1} |x|\nu(dx) < \infty.$$

Notice that in that case one can write

$$X(t) = b_0 t + \int_0^t \int x \, M(ds, dx),$$

where $b_0 := b - \int_{|x|\leq 1} x\nu(dx)$ is called the *drift* of the Lévy process. A process of finite-variation can be written as the difference of non-decreasing processes. In the above representation, this processes will be $b_0 t + \int_0^t \int_{x>0} x \, M(ds, dx)$, and $\int_0^t \int_{x<0} x \, M(ds, dx)$ when $b_0 > 0$.

3. A non-decreasing Lévy process is called a *subordinator*. Necessary and sufficient conditions for X to be a subordinator are that $b_0 > 0$, $\sigma = 0$, and $\nu((-\infty, 0)) = 0$. Subordinators have received interest in recent years as they can be used as random clock or volatility processes.

4 Simulation of Lévy processes

4.1 Approximation by skeletons and compound Poisson processes

Accurate path simulation of a pure jump Lévy processes $X = \{X(t)\}_{t\in[0,1]}$, regardless of the relatively simple statistical structure of their increments, present some challenging problems when dealing with *infinite activity* (namely, processes with infinite Lévy measure). Just try to conceive that in this case the jump times are in fact dense on $[0, \infty)$ (see Theorem 21.3 of [45]).

One of the most popular simulation schemes is based on the generation of *discrete skeletons*. Namely, the discrete skeleton of X based on equally spaced observations is defined by

$$\widetilde{X}(t) = \sum_{k=1}^{\infty} X\left(\frac{k-1}{n}\right) \mathbf{1}_{\left[\frac{k-1}{n} \leq t < \frac{k}{n}\right]} = \sum_{k=1}^{\infty} \Delta_k \mathbf{1}_{\{t \geq \frac{k}{n}\}},$$

where $\Delta_k = X(k/n) - X((k-1)/n)$ are i.i.d. with common distribution $\mathcal{L}(X(1/n))$. The main drawback to the previous scheme is the fact that most often a r.v. with distribution $\mathcal{L}(X(1/n))$ is not easily generated.

A second approach is to approximate the Lévy process by a finite-jump activity Lévy processes. That is, suppose that X is a pure-jump Lévy process, then, in light of the Lévy-Itô decomposition of sample paths, the process

$$X_\varepsilon(t) \equiv t\left(b - \int_{|x|\geq\varepsilon} x\,\nu(dx)\right) + \sum_{s\leq t} \Delta X(s) \mathbf{1}_{\{|\Delta X(s)|\geq\varepsilon\}} \tag{25}$$

converges uniformly on any bounded interval to X a.s. (as usual $\Delta X(t) \equiv X(t) - X(t^-)$). The process $\sum_{s \leq t} \Delta X(s) \mathbf{1}_{\{|\Delta X(s)| \geq \varepsilon\}}$ can be simulated using a *compound Poisson process* of the form $\sum_{i=1}^{N_t^\varepsilon} J_i^\varepsilon$, where N_t^ε is a homogeneous Poisson process with intensity $\nu(|x| \geq \varepsilon)$ and where $\{J_i^\varepsilon\}_{i=1}^\infty$ are i.i.d with common distribution

$$\nu_\varepsilon(dx) \equiv \mathbf{1}_{\{|x| \geq \varepsilon\}} \frac{\nu(dx)}{\nu(|x| \geq \varepsilon)}.$$

Clearly, such a scheme is unsatisfactory because all jumps smaller than ε are totally ignored. An alternative method of simulation approximates the small jumps with a Wiener motion.

4.2 Approximation of the small jumps of a Lévy processes

Consider a Lévy process with Lévy triple (σ^2, b, ν). Define the following processes:

$$X_1^\varepsilon(t) := b_\varepsilon t + \sigma W_t + \int_0^t \int_{|x| \geq \varepsilon} x M(dx, ds),$$

where $b_\varepsilon = b - \int_{\varepsilon < |x| \leq 1} x \nu(dx)$ and M is the jump-measure of X (a posterior a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_0^d$ with mean measure $\nu(dx)dt$). Consider the following pure jump Lévy process

$$X_\varepsilon(t) := X(t) - X_1^\varepsilon(t) = \int_0^t \int_{|x| < \varepsilon} x \left\{ M(dx, ds) - \nu(dx) ds \right\}.$$

Also, consider the jump-diffusion model

$$X_2^\varepsilon(t) := b_\varepsilon t + (\sigma^2 + \sigma^2(\varepsilon))^{1/2} W_t + \int_0^t \int_{|x| \geq \varepsilon} x N(dx, ds),$$

where $\sigma^2(\varepsilon) = \int_{|x| \leq \varepsilon} x^2 \nu(dx)$. Rosinski and Asmussen [4] establish the following results under the assumption that ν has no atoms in a neighborhood of the origin:

Theorem 4 *Suppose that ν has no atoms in a neighborhood of the origin. Then,*

(a) $\{\sigma^{-1}(\varepsilon) X_\varepsilon(t)\}_{t \geq 0}$ converges in distribution to a standard Brownian motion $\{B(t)\}_{t \geq 0}$ if and only if

$$\lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = \infty. \tag{26}$$

(b) Under (26), it holds that

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(X(t) \leq x) - \mathbb{P}(X_2^\varepsilon(t) \leq x)| \leq c \, \frac{\int_{|x| \leq \varepsilon} x^3 \nu(dx)}{\sigma^3(\varepsilon)} \leq c \, \frac{\varepsilon}{\sigma(\varepsilon)}.$$

The first part of the above theorem provides a way to approximate a the small jumps-component of X properly scaled by a Wiener process. Condition (26) can be interpreted as an assumption requiring that the size of the jumps of $\sigma^{-1}(\varepsilon) X_\varepsilon$ are asymptotically vanishing. Part (b) suggests that the distribution of certain Lévy processes (with infinite jump activity) can be approximated closely by the combination of a Wiener process with drift and a compound Poisson process.

4.3 Simulations based on series representations

Throughout, $X = \{X(t)\}_{t\in[0,1]}$ is a Lévy process on $\mathbb{R}^d$ with Lévy measure ν and without Brownian part (which can be separately simulated). Let M be the jump measure on $\mathcal{B}\left([0,1]\times\mathbb{R}_0^d\right)$ of the Lévy process X:

$$M(C) := \#\left\{t : (t, \Delta X_t) \in C\right\}.$$

Condition 1 *The following series representation holds:*

$$M(\cdot) = \sum_{i=1}^{\infty} \delta_{(U_i, H(\Gamma_i, V_i))}(\cdot), \quad a.s. \tag{27}$$

for a homogeneous Poisson process $\{\Gamma_i\}_{i=1}^{\infty}$ on $\mathbb{R}_+$ with unit intensity rate, an independent random sample $\{U_i\}_{i=1}^{\infty}$ uniformly distributed on $(0,1)$, and an independent random sample $\{V_i\}_{i=1}^{\infty}$ with common distribution F on a measurable space S. The response function $H : (0,\infty)\times S \to \mathbb{R}^d$ is from now a measurable function.

Remark 3 *Representation (27) can be obtained (in law) if the Lévy measure has the decomposition*

$$\nu(B) = \int_0^{\infty} \sigma(u; B)du, \tag{28}$$

*where $\sigma(u;B) = \mathbb{P}\left[H(u,\mathbf{V})\in B\right]$. It is not always easy to obtain (**??**). The following are typical methods: the inverse Lévy measure method, Bondensson's method, and Thinning method (see Rosinski [42] for more details).*

Define

$$A(s) = \int_0^s \int_S H(r,v) I\left(\|H(r,v)\| \le 1\right) F(dv)dr. \tag{29}$$

Condition 2

$$A(\Gamma_n) - A(n) \to 0, \quad a.s. \tag{30}$$

Lemma 1 *The limit in (30) holds true if either one of the following conditions is satisfied:*
i. *$b \equiv \lim_{s\to\infty} A(s)$ exists in $\mathbb{R}^d$;*
ii. *the mapping $r \to \|H(r,v)\|$ is nonincreasing for each $v \in S$.*

Proposition 1 *If the conditions 1 and 2 are satisfies then, a.s.*

$$X(t) = bt + \sum_{i=1}^{\infty} \left(H(\Gamma_i, V_i) I\left(U_i \le t\right) - tc_i\right), \tag{31}$$

for all $t\in[0,1]$, where $c_i \equiv A(i) - A(i-1)$.

Remark 4 *The series (31) simplifies further when $\int_{|x|\le 1} |x|\nu(d\mathbf{x}) < \infty$, namely, when X has paths of bounded variation. Concretely, a.s.*

$$X(t) = (b-a)\,t + \sum_{i=1}^{\infty} J_i \mathrm{I}\left(U_i \le t\right), \tag{32}$$

where $a = \int_{|x|\le 1} x\nu(dx)$. The vector $d \equiv b - a$ is called the drift *of the Lévy process.*

5 Density Transformation of Lévy processes

The following two results describes Girsanov's type theorems for Lévy processes. Concretely, the first result provides conditions for the existence of an equivalent probability measure under which X is still a Lévy process, while the second result provides the density process. These theorems have clear applications in mathematical finance as a devise to define risk-neutral probability measures. The proofs can be found in Section 33 of Sato [45].

Theorem 5 *Let $\{X_t\}_{t\leq T}$ be a real Lévy process with Lévy triple (σ^2, b, ν) under some probability measure $\mathbb{P}$. Then the following two statements are equivalent:*

(a) There exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\{X_t\}_{t\leq T}$ is a Lévy process with triplet (σ'^2, b', ν') under $\mathbb{Q}$.

(b) All the following conditions hold.

(i) $\nu'(dx) = k(x)\nu(dx)$, for some function $k : \mathbb{R} \to (0, \infty)$.

(ii) $b' = b + \int x 1_{|x|<1}(k(x) - 1)\nu(dx) + \sigma\beta$, for some $\beta \in \mathbb{R}$.

(iii) $\sigma' = \sigma$.

(iv) $\int \left(1 - \sqrt{k(x)}\right) \nu(dx) < \infty$.

Theorem 6 *Suppose that the equivalent conditions of the previous theorem are satisfied. Then, $\xi \equiv \frac{d\mathbb{Q}}{d\mathbb{P}}$, is given by the formula*

$$\begin{aligned}\xi \equiv \exp\Bigg(&\beta W_T - \frac{1}{2}\beta^2 T \\ &+ \lim_{\varepsilon \downarrow 0}\left(\int_0^T \int_{|x|>\varepsilon} \log k(x) M(ds, dx) - T\int_{|x|>\varepsilon} (k(x) - 1)\nu(dx)\right)\Bigg),\end{aligned}$$

with $\mathbb{E}_{\mathbb{P}}[\xi] \equiv 1$. The convergence on the righst-hand side of the formula above is uniform in t on any bounded interval.

6 Asset price modeling driven by Lévy processes

6.1 Generalities

- In general the market is incomplete: options are not superfluous assets whose payoff can be perfectly replicated in an ideal frictionless market. The option prices are themselves subject to modelling.

- There are infinitely many arbitrage-free prices for a given contingent claim: one for each risk-neutral martingale measure.

- Let us consider an option pricing model where the price of the option at time t is the expectation of the discounted payoff given the past with respect to a risk-neutral measure $\mathbb{Q}$,
$$C(t, S_t) = \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r(T-t)} H | S_u, u \le t \right\}.$$
Moreover, we assume that, under the risk-neutral measure $\mathbb{Q}$, S is given by an exponential Lévy model:
$$S_t = S_0 e^{rt+X_t},$$
where X is a Lévy process with triple (σ^2, b, ν) under Q. Notice that, bu definition, the discounted price process process $S_t^* = e^{-rt} S_t$ is a martingale under Q relative to the information generated by the prices process itself. The following is question arises naturally: Does C satisfy a partial differential equation (PDE) as in the Black-Scholes model?

6.2 Exponential Lévy model

Consider a claim which payoff is a function of the value of the stock at maturity:
$$H = f(S_T).$$
By the Markov Property, there is a measurable function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $C(t, S_t(\omega))$ is the value of the claim at time t; that is,
$$C(t, S_t) = e^{-r(T-t)} \mathbb{E}_Q \left[H | \mathcal{F}_t \right].$$
Concretely,
$$C(t, x) = e^{-r(T-t)} \mathbb{E}_Q \left[f\left(x e^{r(T-t)+X_{T-t}} \right) \right]. \tag{33}$$
Indeed,
$$\mathbb{E}_Q \left[f(S_T) | \mathcal{F}_t \right] = \mathbb{E}_Q \left[f\left(\frac{S_T}{S_t} S_t \right) | S_t \right] = c(S_t(\omega)),$$
where $c(x) = \mathbb{E}_Q \left[f(\frac{S_T}{S_t} x) | S_t = x \right]$. By homogeneity and independence of the increments of X, S_T/S_t is independent of S_t and has the same law as S_{T-t}/S_0. Therefore,
$$\mathbb{E}_Q \left[f(\frac{S_T}{S_t} x) \right] = \mathbb{E}_Q \left[f(\frac{S_{T-t}}{S_0} x) \right],$$
and (33) follows.

The following theorem provides an integro-partial differential equation (IPDE) for the option price function C of (33). The IPDE equation below is well-known in the literature (see e.g. [14] and [41]) and its proof can be found in [17, Proposition 12.1].

Proposition 5 *Suppose the following conditions:*

1. $\int_{|x| \ge 1} e^{2x} \nu(dx) < \infty$;

2. *Either* $\sigma > 0$ *or* $\liminf_{\varepsilon \searrow 0} \varepsilon^{-\beta} \int_{|x| \leq \varepsilon} |x|^2 \nu(dx) < \infty$.

3. $|f(x) - f(y)| \leq c|x - y|$, *for all* x, y *and some* $c > 0$.

Then, the function $C(t, x)$ *in (33) is continuous on* $[0, T] \times [0, \infty)$, $C^{1,2}$ *on* $(0, T) \times (0, \infty)$ *and verifies the integro-partial differential equation:*

$$\frac{\partial C(t,x)}{\partial t} + rx\frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(t,x) - rC(t,x)$$
$$+ \int_{-\infty}^{\infty} \left(C(t, xe^y) - C(t,x) - x(e^y - 1)\frac{\partial C}{\partial x}(t,x) \right) \nu(dy) = 0.$$

6.3 A general jump-diffusions model

Suppose that the risk-neutral asset price process is given by

$$S_t = S_0 e^{X_t},$$

where the driving process is the jump-diffusion

$$X_t = X_0 + \int_0^t \sigma(s, X_{s^-}) dW_s + \int \gamma(s, z, X_{s^-}) \widetilde{J}(ds, dz).$$

The price process of an European option with payoff $h(X_T)$ is given by

$$C_t = \mathbb{E}\{h(X_T) | \mathcal{F}_t\} = C(t, X_t),$$

where $C(t, x) = \mathbb{E}\{h(X_{T-t} + x)\}$. Assuming that C is $C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and applying Itô formula, we obtain that

$$\frac{\partial C(t,x)}{\partial t} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 C}{\partial x^2}(t,x)$$
$$+ \int_{-\infty}^{\infty} \left(C(t, x + \gamma(t,z,x)) - C(t,x) - \gamma(t,z,x)\frac{\partial C}{\partial x} \right) \nu(dy) = 0,$$

and the price process C_t admits the martingale representation:

$$C_t = C_0 + \int_0^t \frac{\partial C(s, X_s)}{\partial t} \sigma(s, X_s) dW_s +$$
$$+ \int_0^t \int \left(C(t, X_{s^-} + \gamma(s, z, X_{s^-})) - C(t, X_{s^-}) \right) \widetilde{J}(ds, dz).$$

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