

Advanced Probability and Options, with numerical
methods

Part I

Fixed-income securities and interest-rate models

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Instructor's class notes
Fall 2009

October 29, 2009

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Chapter 1

Introduction to fixed-income products

1.1 Introduction

The market for fixed-income securities, such as bonds and its corresponding interest-rate derivatives, is of primordial importance. For instance, according to the Office of Controller of the Currency (OCC), the notional value of derivatives held by U.S. commercial banks was about \$ 202.0 trillion during the first quarter of 2009, from which 84% corresponds to interest rate product, followed by foreign exchange (7.4%), credit (7.2%), and equity derivatives (1.1%) in that order. By comparison, U.S. GDP in 2006 was \$12.456 trillion. Besides commercial banks, many other industries rely heavily on this products such as insurance companies which often look to reduced their interest rate risk exposure via swaps.

Our objectives in this part is two-fold. On one hand, we will introduce some of the most important products of fixed-income markets and their associated terminology. Our second objective will be to go over several reasonable *models* for the bond market that allows *an arbitrage-free pricing* of the interest-rate derivatives as it was done with the equity derivatives. Recall that an arbitrage-free pricing theory typically identifies a primary set of traded assets, which are used to price a set of secondary traded assets. In the context of this part, the primary set of traded assets will consist of *zero-coupon bonds* and a *money market account*. Again, as it was done for equity derivatives, the basic pricing operation consists of constructing a portfolio of the primary assets, and dynamically rebalance it across time such that the cash flow and value of the portfolio replicate the cash flow and value of the secondary asset.

Conceptually, an arbitrage-free pricing model for the bond market is much harder than for equity derivatives for two main reasons. There are many outstanding bonds at any given time and hence, their price processes must be modeled jointly to avoid arbitrage. Second, potentially, the payoff of the interest-rate derivatives depends on certain quantities associated with the price processes of several bonds. However, these quantities, though are assumed to observable in time, do not represent the price of tradable assets and hence, the standard machinery of pricing based on replication does not apply.

Example 1. Consider two zero-coupon bonds with maturities $T_1 < T_2$. The prices at time t of these bonds are denoted by $p(t, T_1)$ and $p(t, T_2)$, respectively. The so-called “*LIBOR forward rate*” for the

period $[T_1, T_2]$ is defined by

$$L(t; T_1, T_2) = -\frac{p(t, T_2) - p(t, T_1)}{(T_2 - T_1)p(t, T_1)}.$$

Consider a derivative whose payoff at the expiration time T_1 is

$$(K - L(T_1; T_1, T_2))_+.$$

Such a derivative, called *caplet*, can be considered as a put option on the process $\{L(t; T_1, T_2)\}_{t \leq T_1}$. However, this process does not represent the price of any tradable asset.

In the previous example, one possible approach to price the caplet is to see this as a *basket option* on the two bonds $\{p(t, T)\}_{t \leq U}$ and $\{p(t, U)\}_{t \leq U}$. Lamentably, many of the vanilla fixed income products (such as a cap, which is just a sequence of caplets) consists of a sequence of cashflows at several time points T_1, \dots, T_n and hence, they will depend on a “high” number of bonds. The historical estimation of such a large covariance matrix is not statistically tractable (especially as there will be high correlation between the bond prices).

1.2 Bonds

Recall that a *bond* is a contract where, for a suitable price now, the buyer or holder gets a fixed payment L (called the *principal value*, *face value* or *par*), at a prespecified fixed time T , called the *maturity*. In addition to the principal, the bond could entail periodical payments of a fixed amount, called *coupon*. The coupon is quoted as a percentage of the principal. Hence, a 10-year bond with 10% coupon rate (per annum) paid semiannually means that in addition to the principal L , the bond pays its hold a coupon of $(.1/2) \times L$ every six months. If no coupon is paid, the bond is called a *zero-coupon* (or *pure discount*) *bond* or *zcb* for short. As these assets provide the investor with a fixed cash flow, they are also called *fixed income securities*.

One can realized any fixed income L at time T by buying L units of a zero-coupon bond with a face value of 1 dollar. For simplicity, we call this primary bond a *T-bond*. From now on, the time t price of a *T-bond* is denoted $P(t, T)$, which obviously must satisfy that

$$P(t, T) \leq 1, \quad \text{for all } t \leq T.$$

Also, for a given time t , the graph of the functions $T \rightarrow p(t, T)$ is called **the term structure of bond prices** (or **the bond price curve**) at t . The following are some expected features of the bond price process $\{P(t, T)\}_{t \geq 0}$:

1. $p(t, T) \nearrow p(T, T) = 1$, as $t \rightarrow T$. Why?
2. $t \rightarrow p(t, T)$ exhibit a very irregular trajectory (Brownian-like).
3. For each maturity T , $t \rightarrow p(t, T)$ can be considered the price process of a separate asset. Of course, all assets are correlated: the smaller $|T_1 - T_2|$, the stronger correlation.

Another natural assumption is that if there were bonds of all maturities for a fixed date t , the graph of $T \rightarrow p(t, T)$ (the **zero-bond price curve**) would be smooth in $T > t$. This presumed smoothness will allow us to infer a **zero-bond price curve** from the pure-discount and coupon bonds available in the market.

Most governments issue bonds; for instance, the US issues **treasury securities** of three types:

- **Bills:** zero-coupon bonds with maturities of less than one year. These are purchased in auctions at a discount of their face value.
- **Notes:** Bonds with semi-annual coupon payments and maturities between 2 and 10 years. Their prices are quoted in fractions of $1/32$.
- **Bonds:** Bonds with semi-annual coupon payments and maturities between 10 and 30 years.

In addition to bills, notes, and bonds, treasury securities called STRIPS are traded. These are zero-coupon bonds synthetically created from the coupons or principal amounts of treasury bonds.

1.2.1 The bond market

There are basically three markets to buy or sell a Treasury security:

- **The spot market:** Most of the US treasuries are traded in the over-the-counter (OTC) market. It is important to realize that the quotes of the traded bonds reflect not only today's interest rates (determined for instance by the prices of the most recent issued bonds), but also, current expectations of future interest rates.

Example 2. The following tables describes typical over-the-counter quotes taken place on Dec.1st 94 and reported in the Wall Street Journal of Dec. 2nd:

Rate	Maturity	Bid	Asked	Chg.	Ask yield	
$4^{1/4}$	Dec 95n	97 : 04	97 : 06	-2	7.02	
	Maturity	Days to Mat.	Bid	Asked	Chg.	Ask yield
	Dec 15' 94	10	4.70	4.60	-.13	4.67

A few important remarks are appropriate:

1. The first bond is a treasury note (hence, the letter n next to maturity) with a coupon rate of 4.25%. This is quoted for a face value of \$100 in fractions of $1/32$.
2. The quoted price is called the *clean price*, since the actual price to be paid when purchasing the bond (called the *dirty price*) must include the accrued interest since last coupon:

$$\text{dirty price at time } t = \text{clean price} + c \frac{t - T_{i-1}}{T_i - T_{i-1}},$$

where T_{i-1} and T_i are the times of the last and next coupon payments, respectively, and c is the coupon rate. Let us remark that the time intervals are measure using a day-count convention of "30/360" (see Example 3).

3. The third and fourth columns in the quote of the t-bill denote the bid and ask “banker’s discount yields”, respectively. These yields, together with the “ask yield” will be reviewed in Section 1.3.1.

- **The forward market**

A forward contract is an OTC agreement made in the present between two parties to exchange a commodity (say a bond) at a predetermined future date, but at a price agreed today, called the **forward price**. *There is no exchange of money at settlement date for entering in the contract.* The total profit/loss payoff of a long position (buyer side) on a forward contract on a zero-coupon bond will be

$$P(S, T) - F(t),$$

where t is the settlement date, T is the maturity of the bond, S is the exchange time, and $F(t)$ is the forward price.

- **The future market**

Similar to a forward contract, a future contract is an agreement to purchase a commodity at a prespecified date, called the delivery date, for a given price, called the **future price**. The buyer and seller negotiate indirectly, through an *exchange* (say CME, Chicago Mercantile Exchange). The price is paid via a sequence of installments over the contract’s life credited or charged to the investor’s account. These installments are exercised at the end of each intermediate trading date and equal to the change in the future price on that day; the future price is set according to an exchange. For instance, if the future price today at time t is $F(t)$, and the next day’s future price is $F(t + 1)$, then the amount of $F(t + 1) - F(t)$ is credited or charged to the investor’s account. At maturity S the future price is $p(S, T)$ and the total profit/payoff is

$$\begin{aligned} & (p(S, T) - F(S - 1)) + (F(S - 1) - F(S - 2)) + \dots \\ & + \dots (F(t + 1) - F(t)) = p(S, T) - F(t), \end{aligned}$$

same as forward contract. This procedure is called *making the market*.

Bonds are sometimes called risk-free assets, but this is far from being the case. There are many risk involved in buying or selling a bond. The most important risk is the *interest rate risk*. In other words, if we were to buy a bond that promise a rate of return of 10% per annum, then this will be indeed the return if one hold the bond until maturity. However, if at a given time, one decides to sell the bond, it is possible that the market rate of return (called the *yield* or interest rate of the bond) will be higher than 10% and one will end up loosing money. Other types of risks include *credit risk* due to a possible default of the creditor, *inflation risk* due to a possible decrease in the purchase power of the money received at maturity, and *liquidity risk* due to a reduction in the demand of the bond.

1.3 Interest rates

We have been talking about interest-rate derivatives. But, what are interest rates? In general terms, an interest rate is the amount of money that a borrower promises to pay to the lender per dollar borrowed. There are different ways to measure interest rates. We illustrate this point in the following example:

Example 3. A bank states that the interest rate on deposits is $r = 10\%$ per year. Suppose that an investor deposit $A = \$100$ dollars in the bank. After $t = 2$ years, the terminal value, say F , of the investment will depend on how the interest is applied.

(a) If the interest rate r is **simple**,

$$F = A(1 + rt) \implies 100(1 + (.1)(2)) = 120.$$

(b) If the interest rate is **compounded** $m = 2$ times per year,

$$F = A \left(1 + \frac{r}{m}\right)^{mt} \implies 100 \left(1 + \frac{.1}{2}\right)^4 = 121.55$$

(c) If the interest rate is **continuously compounded**,

$$F = Ae^{rt} \implies 100e^{(.1)(2)} = 122.14.$$

(This is the limit of multiple compounding when the compounding frequency, m , per year increases to infinity. *Why?*).

1.3.1 Bond yields

We can view a zero-coupon bond as a loan since the writer of the bond borrows money from the buyer of the bond. One can then talk about the interest rate of a bond. The simple (annualized) interest rate y associated with a pure-discount bond is called **the yield of the bond**. Concretely, this is the number y such that

$$1 = p(t, T) (1 + y(T - t)), \quad \text{or} \quad p(t, T) = \frac{1}{1 + y(T - t)}.$$

In the later formula, we say that the price of the zcb today, $p(t, T)$, is the present value of \$1 dollar in $T - t$ time units from now.

One can also think of the yield as the simple *rate of return* of the bond:

$$y = \frac{1}{\text{Elapsed time}} \times \frac{\text{Final value} - \text{Initial value}}{\text{Initial value}}.$$

There is a one-to-one relation between the yield and the bond: the higher the yield, the lower the price of bond. The **yield** is widely used in the US-treasury market to quote treasury bills.

Example 4 (Taken from Jarrow). The following quote of a treasury bill was obtained from the Wall Street Journal of Friday, December 2, 1994 (hence, the transaction took place on December 1st; see Example 2):

Maturity	Days to Mat.	Bid	Asked	Chg.	Ask yield
Dec 15' 94	10	4.70	4.60	-.13	4.67

The third and fourth columns denote the bid and ask "*banker's discount yields*" defined by

$$\text{Banker's yield} = \left(\frac{\text{Face Value} - \text{Price}}{\text{Face Value}} \right) \left(\frac{360}{\text{Number of days from settlement to maturity}} \right)$$

If, for instance, one wish to compute the “ask price” for the treasury bill, one will need to solve the following equation for p :

$$.0460 = \left(\frac{1 - \text{ask price}}{1} \right) \frac{10}{360}.$$

Thus, the asked price is $p = .998722$. Here, 360 represent the presumed number of days in a year according to the industry “30/360” convention for US bonds. Concretely, each month is assumed to have 30 days and each year 360 days.

The last column in the above quote represents the yield defined as before:

$$y = \frac{365}{\text{Number of days from settlement to maturity}} \times \frac{\text{Face value} - \text{ask price}}{\text{ask price}}.$$

Thus,

$$\text{ask yield} = \left(\frac{1 - 0.9998722}{0.998722} \right) \frac{365}{10}.$$

For a general bond with coupon rate c per annum (paid semiannually), the **yield** y is the (annualized) interest rate (compounded either semiannually or continuously) that, when used to discount all cash flows of the bond, gives a present value equal to the market bond price. For instance, for a coupon bond with maturity T (in years), par \$1, and coupon rate c , its **market price** P and **yield** y (semiannually compounded) are related as follows:

$$P = \sum_{i=1}^{2T-1} \frac{c/2}{(1 + y/2)^i} + \frac{1 + c/2}{(1 + y/2)^{2T}}.$$

Note that the quoted coupon rate c is annualized, even though it is typically paid semiannually as a simple interest of the principal. Similarly, the yield y_c of the bond (continuously compounded) will be such that

$$P = \sum_{i=1}^{2T} \frac{c}{2} e^{-y_c \frac{i}{2}} + \left(1 + \frac{c}{2}\right) e^{-y_c T}$$

Exercise 1. Verify that the relation above holds for the ask price and ask yield in the quote below:

Coupon Rate	Maturity	Bid	Asked	Chg.	Ask yield
4 ^{3/4}	Dec 94n	99 : 29	99 : 31	+1	4.97

1.4 Valuation of bonds with coupons

Consider a coupon bond specified as follows:

- $T = T_n$ is the maturity;
- L is the principal to be exchanged at maturity;
- c_i is a coupon payment at time T_i (assumed to be deterministic).

Suppose that there were zero-coupon bonds of all maturities in the market. What is the **time- t price $p(t)$** of the previous bond in terms of *zero-coupon bonds*? The following is the so-called *Bond Valuation Formula* that relates the price of a bond with coupon with discount bonds:

$$p(t) = Lp(t, T_n) + \sum_{i=1}^n c_i p(t, T_i), \quad (1.1)$$

In other words, one can see the coupon payment c_i at time T_i as the payment of c_i units of T_i -zcb bonds. Hence, the time- t value of such a payments must be $c_i p(t, T_i)$.

In most bond markets there are only a few traded discount bonds and most of the actively traded bonds bear coupons. In that case, the bond valuation formula (1.1) is typically used to infer zero-coupon bond prices implied by the traded coupon bonds. More specifically, one can use a traded coupon-bearing bond of certain maturity T and previous zcb in order to create synthetically a zcb of maturity T . The following example illustrates this procedure.

Example 5. Suppose the data below represents the prices of five treasury bonds *today*:

Principal	Time to mat. (years)	Coupon rate	Bond price
100	.25	0	97.5
100	.5	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

Recall that the US treasury coupon bonds pay semiannual coupons. The coupon are quoted as an annualized rate. Hence, the coupon rate of 8% for the 1.5–bond with \$100 par means that the bond will pay 4 dollars each six months.

1. Determine the (implied) zero-coupon bond prices for maturities $T=1.5$ and $T=2.0$.
2. Determine the (continuously compounding) **“treasury zero rates”** implied by the above quotes.
3. Plot the term structure of bond prices and of zero rates (**zero curve**). Is it upward or downward slopping?

The method outlined above is an instance of the so-called **Bootstrap Method** (c.f. Section 5.4 in *Hull* for more details and examples). This method involves deriving each new zero rate from previously determined rates. In practice we might not have exactly the maturities we need. The approach often used is to interpolate between the closest available maturities. Say, if we know that a 2.3-year bond with coupon 6% sells for 98 and a 2.7-year bond with coupon of 6.5 % sells for 99, it might be assumed that a 2.5-year bond with a coupon of 6.25 % sells by 98.5. We will talk in greater detail about the bootstrap method in a future section.

1.5 Forward contracts and forward LIBOR rates

1.5.1 Valuation of forward contracts

Consider a *forward contract*, written now, say at time t , to buy a T -zero-coupon bond at time $S \leq T$. Let $F(t; S, T)$ be the corresponding forward price (contracted at time t). Is it possible to choose

$F(t; S, T)$ “consistently” (i.e. in an arbitrage-free sense) with the bond market? The answer is affirmative. In fact, it is possible to devise a replicating trading strategy for the forward contract in terms of T - zcb’s and S - zcb’s. Consider the following strategy:

1. At time t ,

(1a) sell $p(t, T)/p(t, S)$ units of the S -bond;

(1b) Use the proceedings to buy 1 unit of T -bonds (worth $p(t, T)$ dollars);

Net cash flow at time t is 0.

2. At time S ,

(2a) pay the principal of the S -bonds;

Net cash flow at time S is $p(t, T)/p(t, S)$ dollars.

3. At time T ,

(3a) get 1 dollar from the T -bond.

Net cash flow at time T is 1 dollar.

Comparing the cash flow of the above strategy with that of a forward contract, one can deduce that both contracts must entail the same investment to avoid arbitrage. We conclude the following *valuation formula for forward contracts*:

$$F(t; S, T) = \frac{p(t, T)}{p(t, S)}. \quad (1.2)$$

In arbitrage terms, if there are two zcb bonds in the market with maturities S and T and a forward contract has a forward price different from (1.2), then there is an arbitrage opportunity in the market.

Exercise 2. Describe exactly the arbitrage strategy claimed in the previous paragraph if (i) $F(t; S, T) > p(t, T)/p(t, S)$ and if (ii) $F(t; S, T) < p(t, T)/p(t, S)$.

1.5.2 Forward LIBOR rates and instantaneous rates

It is more common to quote a forward contract in terms of its corresponding interest rates. For instance, the simple interest rate L of a forward contract is defined by the following equation:

$$1 = F(1 + L(T - S)),$$

where $F = F(t; S, T)$ is the time- t forward price on a zero-coupon bond with maturity T and exchange time S . Using the forward price (1.2), it follows that

$$L = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)}. \quad (1.3)$$

$L(t; S, T) := L$ above is called the (simple) *LIBOR forward rate for $[S, T]$ contracted at time t* , or for simplicity, the **LIBOR forward rate**. One can think of this rate as the prevailing rate for a \$1 dollar loan during $[S, T]$ as seen today.

When $S = t$, the rate is called the *spot rate* or the *LIBOR spot rate for $[t, T]$* , and it is denoted by $L(t, T)$:

$$L(t, T) = -\frac{p(t, T) - 1}{(T - t)p(t, T)}. \quad (1.4)$$

This is the simple rate today of a loan of duration T . Actually, note that $L(t, T)$ is just the **yield** of the discount bond with maturity T as introduced in Section 1.3.1.

Remark 1 (Terminology). The term LIBOR rate is widely used in industry. LIBOR stands for London Interbank Offer Rate. Formally, a LIBOR quote by a particular bank is the (simple) rate of interest at which a bank is prepared to make a large wholesale deposit with other banks. Large banks typically quote 1-month, 3-month, 6-month, and 12-month LIBORs in all currencies (*see Section 5.1. in Hull for more information*). Notice that a spot LIBOR rate for $[t, T]$ is equivalent to the existence of zero-coupon bond with maturity T .

Example 6. In principle, one can use forward rates combined with zero-coupon rates to determine other zero-coupon rates. For instance, suppose the data below represents forward rates:

Year (n)	Forward rate for the nth year
2	5 %
3	5.8%

Suppose that we know that the 1 year zero yield is 3%. Then, we can determine the (**implied**) “zero rates” for a maturity of 2 and 3 years. However, since forward contracts are traded over-the-counter, these instruments are not liquid enough, and thus, quotes are not widely available.

One can also talk about the continuously compounded rates. Thus, the *continuously compounded forward rate $R = R(t; S, T)$ for $[S, T]$ contracted at t* is defined by the following equation:

$$1 = Fe^{R(T-S)},$$

where $F = F(t; S, T)$ is again the forward price (1.2). The solution of this equation gives

$$R(t; S, T) := R = -\frac{\log p(t, T) - \log p(t, S)}{T - S}.$$

When $S = t$, this is called the continuously compounded **spot rate** and it is denoted by $R(t, T)$:

$$R(t, T) := -\frac{\log p(t, T)}{T - t}.$$

This is just the (continuously-compounding) rate today at time t of a loan of duration T .

For the mathematical formulas we will see later, it is convenient to consider *instantaneous forward rates*, which are obtained when one makes the duration of the loan “infinitesimally” small. Specifically, *the instantaneous forward rate at time S contracted at time t* is given by

$$f(t, S) := \lim_{T \searrow S} R(t; S, T) = -\frac{\partial \log p(t, S)}{\partial S}.$$

When $S = t$, this is then called the **short rate at time t** , hereafter denoted by $r(t)$. Therefore,

$$r(t) := \lim_{T \searrow t} -\frac{\log p(t, T)}{T - t} = -\left. \frac{\partial \log p(t, S)}{\partial S} \right|_{S=t}. \quad (1.5)$$

The short rate $r(t)$ is a convenient abstraction that represents the (log) rate of return at time t on a loan of “infinitesimal” maturity or the (continuously compounded) yield at time t of a zero-coupon bond of infinitesimal maturity. In reality, such bonds of infinitesimal maturity do not exist and one needs to use proxies for $r(t)$. Two common proxies are the yield of a liquid short-term bond, say a one-month T-bill (*Wilmott*) and also *LIBOR rates*, which is frequently used by derivatives traders (*Hull*).

1.5.3 Relationship between the rates

Note that for $t \leq s \leq T$, we have

$$p(t, T) = \exp \left\{ - \int_t^T f(t, s) ds \right\}.$$

Hence, the instantaneous forward rate determines uniquely the bond prices. This relation implies that once a model for the forward rates is chosen, the bond prices would be determined uniquely and *bis-versa*. Hence, there are two ways to build a model for the bond market:

- (1) By specifying the dynamics of all possible bonds $p(t, T)$;
- (2) By specifying the dynamics of all possible forward rates $f(t, T)$.

A natural question is whether the short-rate process $\{r(t)\}_{t \geq 0}$ determines uniquely the bond prices of all maturities. The existence of the short-rate process is equivalent to the existence of an asset with price process

$$B_t = e^{\int_0^t r(u) du},$$

the so-called risk-free asset. Thus, if we buy $1/B_T$ units of the risk-free asset today at time t , one will get 1 dollar at time T , the same payoff of buying one T -bond at time t . It is tempting to think that the value of a T -bond at time t should then be

$$p(t, T) = \frac{B_t}{B_T} = e^{-\int_t^T r(u) du}.$$

In particular, for any $t < S < T$,

$$p(t, T) = p(t, S)p(S, T) \implies p(S, T) = \frac{p(t, T)}{p(t, S)},$$

and one would be able to determine the price of any bond in the future from the information available today at time t . This is not a reasonable assumption. The main problem with the previous derivation of the bond price is that the value of the short-rate for $u > t$ is not available at time t . So, it is not possible to know how much $1/B_T$ is at time $t < T$.

Under certain conditions, we shall see that for the bond market to be arbitrage-free is necessary and sufficient that all bonds can be computed as expectations of their discounted payoffs under a suitable *risk-neutral probability measure* Q :

$$p(t, T) = E^Q \left\{ e^{-\int_t^T r(s)ds} \right\}.$$

This does not mean that the short-rate dynamics determine uniquely the bond prices because it will turn out that there are infinitely many risk-neutral martingale measures Q .

1.6 Interest rate swaps

An *interest rate swap* is an agreement to exchange a stream of fixed interest rate payments on a notional amount for a stream of floating interest rate payments on the same notional. The floating interest rate is reset at the beginning of each period typically according to a “**LIBOR rate**” (floating) on that period. There is no exchange of cash at the settlement date. Hence, the fixed rate R , called the **swap rate**, is chosen such that the initial fair value of this contract is 0.

Suppose that T_0 is the settlement time, $T_1 < T_2 < \dots < T_n$ are the payment dates, and R is the swap rate. Then, at each time T_i ($i = 1, \dots, n$), one party

1. will received $R(T_i - T_{i-1})K$, and
2. will pay $L(T_{i-1}, T_i)(T_i - T_{i-1})K$ to the counter party

Here, $L(T_{i-1}, T_i)$ stands for the LIBOR rate at time T_{i-1} for a loan on $[T_{i-1}, T_i]$.

1.6.1 Valuation using zero-coupon bonds

In terms of zero-coupon bond prices, the “fair” (arbitrage-free) value $\Pi(t)$ of the contract at time $t \leq T_0$ for the party receiving the floating-payment leg is given by

$$\Pi(t) = K \sum_{i=1}^n \{p(t, T_{i-1}) - (1 + \delta R)p(t, T_i)\}, \quad (1.6)$$

where we assumed a constant time span of δ between payments (called the **tenor** of the contract).

In particular, one can deduce the “fair” swap rate R , which, by definition, is the value R that makes the fair value of the contract at the settlement time T_0 equal to 0. Fixing $t = T_0$ and using the telescopic nature of the sum, the “fair” swap rate R is given by

$$R = \frac{1 - p(0, T_n)}{\delta \sum_{i=1}^n p(0, T_i)}. \quad (1.7)$$

Let us explain the idea behind formula (1.6). The swap contract consists of two components: the fixed interest leg and the floating interest leg. Without loss of generality, let us assume that the notional is $K = 1$ dollar. For the party making the fixed interest payments, the fixed payment leg is worth

$$-R \sum_{i=1}^n (T_i - T_{i-1})p(t, T_i) = -\delta R \sum_{i=1}^n p(t, T_i)$$

at time $t \leq T_0$. The floating payment leg is more tricky. First, note that the floating payment at time T_i is

$$L(T_{i-1}, T_i)(T_i - T_{i-1}) = -1 + \frac{1}{p(T_{i-1}, T_i)}, \quad (1.8)$$

where we used the formula (1.4). The cash-flow of -1 dollar is worth $-p(t, T_i)$ at time T . In order to determine how much is worth is $1/p(T_{i-1}, T_i)$, we use a replicating trading strategy. Indeed, suppose that

1. At time t , we buy a T_{i-1} -zero bond. The net cash flow will be $p(t, T_{i-1})$;
2. At time T_{i-1} , we receive \$1 dollar and buy T_i -zero bonds. One can buy up to $\frac{1}{p(T_{i-1}, T_i)}$ units of bonds. The net cash flow here is 0.
3. At time T_i , obtains $\frac{1}{p(T_{i-1}, T_i)}$ dollars.

We conclude that receiving $1/p(T_{i-1}, T_i)$ at time T_i is worth $p(t, T_{i-1})$ at time t , and hence, receiving (1.8) at time T_i is worth

$$p(t, T_{i-1}) - p(t, T_i),$$

at time t . In that case, the floating interest leg is worth

$$\sum_{i=1}^n \{p(t, T_{i-1}) - p(t, T_i)\}.$$

Now, it is clear that the net cash flow of the swap will be worth (1.6), the sum of the fair value of both legs.

1.6.2 Application to compute zero-rates

Let us draw some conclusions of the previous formulas. First, note that (1.7) give us a relationship between the swap rate of a swap contract with payment dates $T_1 < \dots < T_n$ and the bond prices with those maturities. In particular,

$$p(0, T_n) = \frac{1 - R\delta \sum_{i=1}^{n-1} p(0, T_i)}{1 + \delta R}. \quad (1.9)$$

Equivalently,

$$R\delta \sum_{i=1}^{n-1} p(0, T_i) + (1 + \delta R)p(0, T_n) = 1. \quad (1.10)$$

Note that if we interpret R in the previous equation as a coupon rate, the previous formula is saying that the swap rate is the coupon rate at which the value of the bond is the same as the principal or "par". Such a rate is called the *par yield* of the bond. Hence, we can write

$$\text{Swap-rate} = \text{"Par yield"}.$$

The formula (1.9) is commonly used to compute a bond price $p(0, T_n)$ that is consistent with previous bond prices and the swap rate R of the swap contract with maturity T_n . The following example illustrated this procedure.

Example 7 (Hull, Section 7.3). The table below shows the quotes of plain US dollar swaps¹ as posted by a market-maker. The bid and offer rates are for the fix payment leg. Therefore, the bid quote of 6.03% means that market-maker is willing to pay a fixed interest of 6.03% in exchange for a floating LIBOR interest, while the offer quote of 6.06% means that the market-maker is willing to pay a floating LIBOR rate in exchange for receiving a fixed payments interest of 6.06%.

Maturity (years)	Bid	Offer	Swap rate
2	6.03	6.06	6.045
3	6.21	6.24	6.225
4	6.35	6.39	6.370
5	6.47	6.51	6.490
7	6.65	6.68	6.665
10	6.83	6.87	6.850

Suppose that the zero rates for 6-month, 1-year, and 1.5-year are respectively 5.5%, 5.75%, and 5.9% per annum with continuous compounding. Compute a 2-year and a 2.5-year zero-rates that are consistent with the previous swap rates.

Swaps are the most important interest-rate derivatives in the over-the-counter market due to their high liquidity. They are also quite important to strip the current zero-rates. Typically, the short end of the LIBOR curve (zero-rate curve or term structure of yields) is constructed from market LIBOR rates (typically, 1, 3, 6, and 12 month maturities). The longer term zero-rates are inferred from the market prices of other instruments, such as swaps. It is believe that the curve inferred from swaps is the most relevant for pricing other (exotic) interest-rate options, as it is the curve at which the bank can directly trade swaps and hence, hedge the option (see *Yoshi*)

Exercise 3. Let $t \leq T_0 < T_1 < \dots < T_n = T$ and consider a bond with principal $K = 1$ that pays the floating coupon $c_i = (T_i - T_{i-1})L(T_{i-1}, T_i)$ at time T_i (for $i = 1, \dots, n$). Devise a replication strategy for the previous bond with initial endowment $p(t, T_0)$. Your trading strategy can only make transactions in the bond market at times t, T_0, \dots, T_{n-1} .

Exercise 4. Suppose that, at time $t < T_0$, there are available in the market $n - 1$ zero-coupon bonds of maturities $T_0 < T_1 < \dots < T_{n-1}$ and a swap contract with tenor $T_0 < \dots < T_n$. Is it possible to synthetically create a zero-coupon bond of maturity T_n using the available bonds and the swap? You can use a replication strategy that trades (only) at the times t, T_0, \dots, T_{n-1} .

Exercise 5. Suppose now that at time $t < T_0$, there are available in the market $n - 1$ zero-coupon bonds of maturities $T_0 < T_1 < \dots < T_{n-2} < T_n$ and a swap contract with tenor $T_0 < \dots < T_n$. Is it possible to synthetically create a zero-coupon bond of maturity T_{n-1} using the available bonds and the swap? You can use a replication strategy that trades (only) at the times t, T_0, \dots, T_{n-2} .

Exercise 6. (i) Deduce the fair value of a swap contract, assuming that the fixed interest payments are every 6 months, but the floating interest payments are every 3 months. (ii) Repeat the problem, assuming that there is no interest payment at the end of the first 3 months. (iii) In the case of (ii), what is the fair value of the swap rate in terms of zero-coupon bond prices.

¹The standard swap in the United States is one where fixed payments made every 6 months are exchange for floating LIBOR payments every 3 months. Also, there is no interest payment at the end of the first 3-month period.

1.7 Duration and bond hedging

Recall that the most important risk when trading a bond is the *interest rate risk*; that is, the risk due to an increase or decrease in the future interest rates. For instance, suppose that an investor buys a zero-coupon bond at time t with maturity T and yield y (say, continuously compounded). The price of the bond is

$$P = e^{-y(T-t)}.$$

It is natural to measure the sensitivity of the bond price to changes in the yield in terms of the derivative:

$$\frac{dP}{dy} = -(T-t)P.$$

Hence, the percent change in the bond price per unit change in the yield is approximately equal to the negative of the duration of the bond:

$$\frac{1}{P} \frac{dP}{dy} = -(T-t)$$

In that case, the duration of the bond

$$D = T - t,$$

arises naturally as a measure of the risk-exposure of the bond to changes in the interest rate of the bond.

In the case of a coupon bond the situation is more complicated since the bond will be exposed to changes in the "whole" term structure of yields. Concretely, consider a fixed-income security paying the amount c_i at time T_i , for $i = 1, \dots, n$. At time $T_0 = 0$, the price of the security is given by

$$p = \sum_{i=1}^n c_i p(0, T_i) = \sum_{i=1}^n c_i e^{-y_i T_i}, \quad (1.11)$$

where y_i is the yield of bond maturing at time T_i . Hence, the bond is exposed to changes in all of these yields. Suppose that there is a small parallel shift in the terms structure of size Δy . Then, the change in the price of the bond is

$$\Delta p = \sum_{i=1}^n c_i (e^{-(y_i + \Delta y)(T_i - t)} - e^{-y_i(T_i - t)}) \approx -\Delta y \sum_{i=1}^n T_i c_i e^{-y_i T_i}.$$

Hence, the percent change of the bond price as a result of a 1% parallel increase in the market interest rates can be approximated as follows:

$$\frac{1}{p} \frac{\Delta p}{\Delta y} \approx -\tilde{D}, \quad (1.12)$$

where \tilde{D} is the quantity:

$$\tilde{D} = \sum_{i=1}^n T_i \tilde{w}_i, \quad \text{with} \quad \tilde{w}_i = \frac{1}{p} c_i e^{-y_i T_i}. \quad (1.13)$$

Note that $\sum_{i=1}^n \tilde{w}_i = 1$ and hence, \tilde{D} is a weighted average of the security's payment dates with weights proportional to the present value of the corresponding payment amounts. \tilde{D} is a measure

of the real duration of security, which gives greater weight to those payment dates that are more valuable as seen today. As suggested by (1.12), the duration (1.13) serves also as a measure of the risk-exposure of the security to (parallel) changes in the market interest rates.

In the literature of fixed-income securities, the above analysis is simplified by taking the yield of the security as a summary measure of the terms structure of yields. Recall that the yield y of the security is defined as the discounting rate (continuously compounded) that makes the total present value of the security's payments equal to its market value:

$$p = \sum_{i=1}^n c_i e^{-yT_i}.$$

In that case, the change in the bond price as a result of a small change Δy in the yield of the bond can be approximated as follows:

$$\Delta p \approx -\Delta y \sum_{i=1}^n T_i c_i e^{-yT_i}.$$

We can then identify the following duration measure:

$$D = \sum_{i=1}^n T_i w_i, \quad \text{with} \quad w_i = \frac{1}{p} c_i e^{-yT_i}. \quad (1.14)$$

As before, this measurement is a weighted average of the payment dates T_i with weights proportional to the present value of the corresponding payment amounts, but this time taking the yield of the bond as discounting rate.

The duration (1.14) has still the interpretation of a measure of risk-exposure to interest rate changes, but not to parallel changes in the term structure but rather to changes in the yield of the bond:

$$\frac{1}{p} \frac{\Delta p}{\Delta y} \approx -D.$$

Note that $D = \tilde{D}$, if the terms structure of interest rates ($T_i \rightarrow y_i$) is flat (namely, constant).

One can similarly construct a measure of risk exposure to changes in the discretely compounded yield y of the bond. Suppose that the bond pays the amount c_i at time $i/2$ for $i = 1, \dots, 2T$ (here, T is the duration of the bond in years). In terms of the bond yield y (semiannually compounded), the price of the bond is given by

$$p = \sum_{i=1}^{2T} \frac{c_i}{(1 + y/2)^i}$$

The sensitivity of the bond price to changes in the yield y will be

$$\frac{dp}{dy} = - \sum_{i=1}^{2T} \frac{i}{2} \cdot \frac{c_i}{(1 + y/2)^{i+1}} = - \frac{1}{1 + y/2} \cdot \sum_{i=1}^{2T} \frac{i}{2} \frac{c_i}{(1 + y/2)^i}.$$

Hence, the percentage change in bond price per unit change in the yield can be approximated as follows:

$$\frac{1}{p} \frac{\Delta p}{\Delta y} \approx - \frac{1}{1 + y/2} D,$$

where we define

$$D := \sum_{i=1}^{2T} \frac{i}{2} w_i, \quad \text{with} \quad w_i = \frac{1}{p} \frac{c_i}{(1 + y/2)^i}.$$

Once again, the duration D above is a weighted average of the bond payment dates with weights proportional to the present value of the corresponding payment amounts.

Duration plays a crucial role in hedging the interest-rate risk of a portfolio of bonds. Suppose that we hold a portfolio of n bonds with corresponding positions β_1, \dots, β_n bond units (which can be either positive or negative depending on whether we buy or sell the bond). Thus, if p_i denotes the price of one unit of the i^{th} bond ($i = 1, \dots, n$), then the value of the portfolio is given by

$$p = \sum_{i=1}^n \beta_i p_i.$$

In that case, one can show that, when the term structure of yields is flat, the duration D of the portfolio is given by

$$D = \sum_{i=1}^n \beta_i D_i \frac{p_i}{p}, \tag{1.15}$$

where D_i is the duration of the i^{th} bond. Note that in terms of the invested wealth percentages

$$w_i = \frac{\beta_i p_i}{p}, \quad i = 1, \dots, n,$$

we have that

$$D = \sum_{i=1}^n w_i D_i, \quad \text{with} \quad w_1 + \dots + w_n = 1.$$

Exercise 7. Show (1.15) in the case that the term structure is flat.

As it was done with delta-gamma immunization, it is natural to hedge the interest-rate risk of a portfolio of bonds by acquiring other liquid fixed-income securities in such proportions that the duration of the overall portfolio is zero. Such a procedure is called *immunization*. The following simple example illustrates this point.

Example 8 (Cvitanic and Zapatero, pp. 10.4). Suppose that the term structure is flat and the annual interest rate is 5%. An investor holds a liability with face value \$100 and maturity 2 years. His initial wealth is

$$100e^{-.05(2)} = 90.4837$$

In the market there are only two liquid zero-coupon bonds with maturities 1 and 4 years, respectively. With a face value of \$100 dollars, the prices of these bonds are

$$100e^{-.05(1)} = 95.1229, \quad 100e^{-.05(4)} = 81.8730,$$

respectively. The only way for the investor to guarantee that he will be able to meet the payment of the liability in two years is by buying a bond with a 2 year maturity. However, such a bond does not exist in the market (or maybe it exists and it is too expensive). Instead, he invest his initial cash in

the other two bonds to match the duration of the liability. To this end, we need to find w_1 and w_2 such that

$$w_1 + w_2 = 1, \quad 2 = (1)w_1 + (4)w_2.$$

Thus, $w_1 = 2/3$ and $w_2 = 1/3$. This means that he should buy

$$\beta_1 = \frac{2}{3} \frac{95.1229}{90.4837} \quad \text{and} \quad \beta_2 = \frac{1}{3} \frac{81.8730}{90.4837},$$

units of 1-year bond and 4-year bond, respectively.

Exercise 8. In the context of the previous example,

- (ii) analyze the changes in the value of the liability and the portfolio if the interest rates drop 4% immediately after forming the portfolio.
- (iii) analyze the changes in the value of the liability and the portfolio if the interest rates go up 6% immediately after forming the portfolio.

1.8 Bootstrapping the yield curve

See Chapter 4 of Filippo's notes on fixed-income securities.

Chapter 2

Short rate models

2.1 Introduction to short-rate modeling

Recall that one of our main goals is to build an arbitrage-free model for the *bond market* and related contingent claims. The simplest possible model for bond prices is to assume a constant deterministic rate of return r for all bonds so that the time- t price of the T -zero bond is given by

$$p(t, T) = e^{-r(T-t)}.$$

This of course is not consistent with market bond prices where e.g. the yield of one bond changes from day to day. The next natural step is to assume that the rate of return changes with time, say $\{r(t)\}_{t \geq 0}$, in a deterministic fashion so that

$$p(t, T) = e^{-\int_t^T r(u)du}. \quad (2.1)$$

Note that $\{r(t)\}_{t \geq 0}$ coincides with the short-rate defined in (1.5). There are several issues with this valuation formula. For instance, (2.1) would imply that

$$p(S, T) = \frac{p(t, T)}{p(t, S)},$$

for any $t < S < T$ and hence, future bond prices can be completely determined from current bond prices. This is far from being the case in reality.

A natural idea is to assume that $\{r(t)\}_{t \geq 0}$ is stochastic. The right-hand side on (2.1) will then depend on the future spot rates. Hence, it is natural to take the expected value of this unpredictable quantity given the information available at time t , say \mathcal{F}_t :

$$p(t, T) = E \left\{ e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t \right\}. \quad (2.2)$$

Note that $r(t)$ in (2.2) can still be interpreted as the short rate at time t :

$$r(t) = - \left. \frac{\partial \log p(t, T)}{\partial T} \right|_{T=t}. \quad (2.3)$$

Another approach for bond pricing is to assume the existence of a money market account with instantaneous rate of return $\{r(t)\}_{t \geq 0}$ or what amounts to the same, to assume the existence of a risk-free asset with price process:

$$B_t = e^{\int_0^t r(u) du}. \quad (2.4)$$

By buying (resp. short-selling) this asset at time t and then selling (resp. buying back) the asset at time $t + \delta t$, one can lend (resp. borrow) money at a rate of $r(t)dt$. Note that, in this approach, $\{r(t)\}_{t \geq 0}$ is not seen as the short-rate of return (2.3) but rather as the rate of return of the risk-free asset (2.4). We can, however, justify heuristically that these two will coincide. Indeed, suppose that at any time t , there exist bond prices of all maturities $T > t$ and also suppose that

$$r(t) < - \left. \frac{\partial \log p(t, T)}{\partial T} \right|_{T=t} = - \left. \frac{\partial p(t, T)}{\partial T} \right|_{T=t}.$$

Then, there exists a short-enough time δ such that

$$\frac{1 - p(t, t + \delta)}{\delta p(t, t + \delta)} > r(t),$$

which produces an arbitrage opportunity. Indeed, one can short-sell a discount bond with maturity $t + \delta$ and put the money in the money market account. At time $t + \delta$, one will get $p(t, t + \delta)(1 + r(t)dt)$, which is more than the principal of the bond. We would be able to lock the profit $1 - p(t, t + \delta)(1 + r(t)dt)$ without risk.

As it was explained in Section 1.5.3, if r in (2.4) is assumed deterministic, the arbitrage-free bond prices will have to be given by (2.1), which is again not realistic. This suggests to take the rate of return $\{r(t)\}_{t \geq 0}$ to be stochastic. This is precisely the approach we adopt in this part. Concretely, we assume that

- (A) At any time t , there exists a frictionless market for zero coupon bonds of all maturities $T \geq t$;
- (B) There exists a money market account with stochastic rate of return $\{r(t)\}_{t \geq 0}$.

In this chapter, we will consider in some detail the following stochastic model for r :

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t, \quad t \geq 0, \quad r(0) = r_0, \quad (2.5)$$

where $\mu(t, r)$ and $\sigma(t, r)$ are deterministic functions and W is a one-dimensional Wiener process (in which case, the model is said to have one-factor). Intuitively, this model postulates that the change of the short-rate during a small time interval $[t, t + \Delta t]$ is approximately Gaussian with mean $\mu(t, r(t))\Delta t$ and variance $\sigma^2(t, r(t))\Delta t$:

$$r(t + \Delta t) - r(t) \approx \mu(t, r_t)\Delta t + \sigma(t, r(t))\sqrt{\Delta t} \varepsilon, \quad \varepsilon \sim N(0, 1).$$

The solution to (2.5), which existence is assumed from now on, is a Markov process; namely, for any nice function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $0 < t \leq T$,

$$E[f(T, r(T)) | W_u, u \leq t] = E[f(T, r(T)) | r_u, u \leq t] = E[f(T, r(T)) | r(t)].$$

We warn again that the short rate $r(t)$ is just a convenient abstraction or a latent unobservable quantity. The short-rate r can be seen as the glue that will allow us to price all the bonds and related

interest-rates in a consistent manner. Two key questions are of course what kind of model is suitable for the short-rate r and how to determine the parameters of this model if r is not even observable. There are two approaches to deal with these points. First, one can use a "proxy" for $r(t)$; that is, an observable quantity that is believed to behave like r . Two common proxies are the one-month T-bill and short-term *LIBOR rates*. Another common approach is to adjust the parameters of the model for r so that the theoretical market prices of the bonds (and interest-rate derivatives) are close to the corresponding market prices. We of course need to show first how to obtain the theoretical arbitrage-free prices of bonds and interest-rate derivatives.

2.2 A one-factor bond market model

In order to develop an arbitrage-free pricing model for the bond market (A)-(B) above, we start by adopting certain standing assumptions about the bond prices and the short-rate. We can then analyze under what additional conditions this pricing setting will be consistent with the absence of arbitrage. The following are our standing assumptions about the market (A)-(B) described in the introduction:

- (1) The instantaneous or short rate $r(t)$ of the money market account follows the dynamics

$$dr(t) = \mu_t dt + \sigma_t dW(t), \quad t \geq 0,$$

where $\{W(t)\}_{t \geq 0}$ is a Wiener process under the historical or objective probability measure P , and μ and σ are adapted processes.

- (2) All bond prices are driven by the Wiener process $\{W(t)\}_{t \geq 0}$ such that

$$dp(t, T) = p(t, T) \{ \alpha_T(t) dt + \sigma_T(t) dW(t) \}, \quad t \leq T, \quad (2.6)$$

where $\alpha_T(t)$, the time- t mean-rate of return of the T -bond, and $\sigma_T(t)$, the time- t volatility of the T -bond, are adapted processes.

The previous general framework is called a one-factor Itô model since it is assumed that there is only one source of randomness in the market represented by the Wiener process W . For this market model, we shall address the two fundamental problems of mathematical finance: arbitrage-free valuation and replication (or hedging). We can summarize our finding in this section as follows:

- In order to avoid arbitrage opportunities, at any time t , all outstanding bonds must have the same **excess return $\lambda(t)$ per unit of risk**, regardless of maturity.
- Under the absence of arbitrage, the bond prices will satisfy the following risk-neutral valuation formula:

$$p(t, T) = \mathbb{E}^Q \left\{ e^{-\int_t^T r(u) du} \middle| \mathcal{F}_t^W \right\}, \quad (2.7)$$

where, under Q , the short-rate $\{r(t)\}_{t \geq 0}$ has dynamics:

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma(t)dW^\lambda(t). \quad (2.8)$$

- Given a particular "benchmark" bond, the price of any other bond, with maturity prior to the benchmark, can be replicated perfectly by a portfolio of the benchmark bond and the money market account, dynamically rebalanced across time.

2.2.1 Arbitrage-free pricing of bonds

As it was previously mentioned, under the pricing setting (1)-(2) above, the bonds must satisfy a consistency constraint in order to avoid arbitrage opportunities. Loosely speaking, all bonds, regardless of their maturities, must share the same (instantaneous) **excess rate of return, with respect to the short rate, per unit of risk** or in other words, the same **risk premium per unit of volatility**.

Theorem 1. *Under the absence of arbitrage, there exists a random process $\lambda(t)$ such that*

$$\lambda(t) = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}, \quad (2.9)$$

for any $0 < t \leq T$, regardless of the maturity $T > 0$. Such a process is called the **market price of risk**.

Remark 2 ()*. Let us briefly outline the heuristics behind (2.9). As in the Black-Scholes case, the idea is to create a portfolio that is “locally risk-free”. Consider a portfolio at time t consisting of one T -bond and $\Delta(t)$ -units of, say, S -bonds. The value of the portfolio is

$$V_t := p(t, T) + \Delta(t)p(t, S).$$

By self-financibility, during a small time interval $[t, t + dt]$, the change in the value of the portfolio is

$$\begin{aligned} dV_t &= dp(t, T) + \Delta_t dp(t, S) \\ &= \{\alpha_T p(t, T) + \Delta \alpha_S p(t, S)\} dt + \{\sigma_T p(t, T) + \Delta \sigma_S p(t, S)\} dW_t. \end{aligned}$$

where we used (2.6). Then, it follows that by taking $\Delta(t) = -(\sigma_T p(t, T))/(\sigma_S p(t, S))$, one can make a (locally) risk-free return since

$$dV_t = \{\alpha_T p(t, T) + \Delta \alpha_S p(t, S)\} dt.$$

In that case, the instantaneous risk-free rate of return of the portfolio at time t will be

$$\frac{\alpha_T p(t, T) + \Delta \alpha_S p(t, S)}{V_t},$$

which must coincide with the instantaneous rate of return of the money market account $r(t)$. We deduce that

$$\frac{\alpha_T p(t, T) + \Delta \alpha_S p(t, S)}{p(t, T) + \Delta p(t, S)} = r(t).$$

Plugging $\Delta(t) = -(\sigma_T p(t, T))/(\sigma_S p(t, S))$ in and simplifying, we get

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \frac{\alpha_S(t) - r(t)}{\sigma_S(t)},$$

and thus, this ration is independent of the maturity.

Note that Theorem 1 states a necessary condition for the absence of arbitrage; namely, if two bonds did not share the excess rate of return per unit risk at certain time t , it would be possible to devise an arbitrage opportunity. A related question is whether the existence of the market price of risk (2.9) implies that the bond market is arbitrage-free. To answer this question, we invoke *Girsanov's theorem*, which, given an adapted process $\{\theta_t\}_{t \leq T}$, states the existence of a probability measure $Q^{(\theta)}$, equivalent to P^1 , such that the process:

$$W^{(\theta)}(t) = W(t) + \int_0^t \theta(s) ds, \quad t \leq T, \quad (2.10)$$

is a Wiener process under $Q^{(\theta)}$. Girsanov's theorem holds true if, for instance, the following so-called Novikov's condition is satisfied:

$$E \left(e^{\frac{1}{2} \int_0^T \theta_s^2 ds} \right) < \infty. \quad (2.11)$$

Let us assume that there exists a process λ such that (2.9) holds for any maturity $T > 0$ and any $t \leq T$, and also, assume that Girsanov's theorem holds true with $\theta = \lambda$. Then, by simple manipulation of the dynamics (2.6):

$$dp(t, T) = p(t, T) \{r(t)dt + \sigma_T(t) dW^{(\lambda)}(t)\}. \quad (2.12)$$

By Itô's formula, the discounted bond price $\tilde{p}(t, T) := B_t^{-1}p(t, T)$ is such that

$$d\tilde{p}(t, T) = \tilde{p}(t, T)\sigma_T(t) dW^{(\lambda)}(t),$$

and, since $W^{(\lambda)}$ is a martingale under $Q := Q^{(\lambda)}$, \tilde{p} is a martingale under Q . In particular, the following natural risk-neutral valuation formula is satisfied:

$$p(t, T) = \mathbb{E}^Q \left\{ e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t^W \right\}, \quad (2.13)$$

In plain words, the previous formula says that the bond price at any time is the risk-neutral expected discounted payoff of the bond given the information available at that time. Note that in order to compute (2.13) all what is needed is the dynamics of $\{r(t)\}_{t \geq 0}$ under Q , which can be seen to be

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma(t)dW^{(\lambda)}(t). \quad (2.14)$$

The fact that all discounted bond prices are martingales under a martingale measure will suffice to preclude any arbitrage-opportunity. Concretely, we say that there is an arbitrage-opportunity if there exists a self-financing portfolio $\{V_t\}_{t \leq T}$ consisting of finitely-many bonds such that all three conditions below are satisfied:

$$(i) V_0 = 0, \quad (ii) V_T \geq 0, \quad (iii) P(V_T > 0) > 0.$$

We left the details to the reader to show that any self-financing portfolio V will be such that $\tilde{V}_t := B_t^{-1}V_t$ is a martingale under Q . Then,

$$E^Q \{V_T\} = V_0 = 0,$$

¹Namely, $Q(A) = 0$ if and only if $P(A) = 0$.

and thus, since $V_T \geq 0$, $Q(V_T > 0) = 0$, which contradicts that $P(V_T > 0) > 0$.

Note that by an argument similar to that in the previous paragraph, if we were to price a contingent claim with payoff $\mathcal{X} = \Phi(r(T))$ and maturity T by the formula

$$\Pi(t) := \Pi(t; \mathcal{X}, T) := \mathbb{E}^Q \left\{ e^{-\int_t^T r(u) du} \Phi(r(T)) \middle| \mathcal{F}_t^W \right\}, \quad (2.15)$$

then the market consisting of the money market account, the bonds, and the claim will remain arbitrage-free. Thus, (2.15) provides an arbitrage-free pricing procedure for contingent claims in relation to the bond market.

We summarize the previous results in the following theorem:

Theorem 2. *If there exists a process λ , satisfying Novikov's condition (2.11), and such that (2.9) holds for any maturity $T > 0$ and any $t \leq T$, then the following statements hold true:*

- (i) *The bond market is arbitrage-free;*
- (ii) *There exists a probability measure Q such that*
 - (a) *the discounted bond prices $\tilde{p}(t, T) = B_t^{-1}p(t, T)$ are martingales under Q ; and*
 - (b) *the bond prices satisfy the risk-neutral valuation formula (2.13).*
- (iii) *The price process $\Pi(t)$ given by (2.15) is consistent with the absence of arbitrage relative to bond market.*

2.2.2 Some calibration and computational issues

The previous part gives necessary and sufficient conditions for a consistent pricing of the bond market and interest-rate derivatives in terms of a process λ . This process must be calibrated or estimated if we were to apply this theory. There are different approaches for this end. One method is to extract the market price of risk λ from the dynamics of a liquid bond, say $p(t, S)$, as follows:

$$\lambda(t) = \frac{\alpha_S(t) - r(t)}{\sigma_S(t)}. \quad (2.16)$$

The S -bond works as a "benchmark", by which the market price of risk λ is set. Typically, one would need to parameterize further α_S and σ_S for them to be computable from historical observations $p(t_0, S), \dots, p(t_n, S)$ of the benchmark bond. For instance, if we assume σ_S and α_S to be constant, one can estimate them from the historical volatility and mean of the log returns

$$\frac{1}{\sqrt{\Delta}} \log \frac{p(\Delta, S)}{p(0, S)} \dots, \frac{1}{\sqrt{\Delta}} \log \frac{p(n\Delta, S)}{p((n-1)\Delta, S)}.$$

Once the benchmark is set and the market price of risk λ is computed, any other bond price (or derivative) can be priced by the valuation formula (2.13) (or (2.15)), where r has dynamics

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma(t)dW^{(\lambda)}(t). \quad (2.17)$$

A second approach consists of parameterizing the dynamics of the short-rate $\{r(t)\}_t$ under the risk-neutral measure Q (which essentially accounts to parameterize the drift $\mu - \lambda\sigma$ and the volatility

σ). The parameters of the model will be then chosen in such a way that the resulting theoretical prices of the bonds and possibly other derivatives are close to the corresponding market prices of the bonds and derivatives. This second approach is sometimes called the *martingale method* and the market-price-matching procedure is called *calibration*. See Section 2.5 for more details.

Once the risk-neutral dynamics of the short-rate has been calibrated, the prices of bonds and interest-derivatives can be computed via the risk-neutral valuation formulas (2.13, 2.15). In such a general setting, Monte Carlo is the method of choice for computations. Suppose that we want to compute the time- t price of a T -claim with payoff $\mathcal{X} = \Phi(r(T))$. For simplicity, let us assume that the *risk-neutral* model for the short-rate satisfies the Markov formulation (2.5) with $\mu^Q(t, r)$ replacing $\mu(t, r)$. A typical implementation of the Monte Carlo method will be as follows:

1. *Simulation of paths of r* . This can be done by the so-called Euler's method. Concretely, given a number of time steps N and a number of simulations M (both chosen by the user), we set $\tilde{r}_0^{(i)} = r(t)$ (the current spot rate), and compute recursively

$$\tilde{r}_j^{(i)} = \tilde{r}_{j-1}^{(i)} + \mu^Q(u_{j-1}, \tilde{r}_{j-1}^{(i)})\Delta + \sigma(u_{j-1}, \tilde{r}_{j-1}^{(i)})\sqrt{\Delta} \cdot \varepsilon_j^{(i)}, \quad j = 1, \dots, N,$$

for each $i = 1, \dots, M$, where $\Delta := (T - t)/N$ is the time step and $\varepsilon_j^{(i)}$ are independent standard normal variables. Here, $\tilde{r}_j^{(i)}$ is a simulated approximation of $r(j\Delta + t)$.

2. *Evaluation of the expected values*. The time- t price of the T -derivative with payoff $\mathcal{X} = \Phi(r(T))$ can be approximated by

$$\tilde{\Pi}_t = \frac{1}{M} \sum_{i=1}^M \Phi(r_N^{(i)}) \exp \left\{ \Delta \sum_{j=0}^{N-1} \tilde{r}_j^{(i)} \right\}.$$

Monte Carlo methods can be implemented easily for a general payoff and general models. However, they are known to be slow.

2.2.3 Replication of bond prices and other derivatives

Given that there are infinitely many bonds and no tradable underlying, one can expect that it is not possible to replicate the payoff of every bond using only the money market account. However, the following result shows that it is possible to replicate the payoff of any bond with a portfolio of another bond and the money market account:

Theorem 3. *Fix a "benchmark" bond with maturity S and suppose that the absence of arbitrage condition of Theorem 1 is satisfied. Then, the value of any other bond (with maturity prior to the benchmark) can be replicated via a self-financing trading strategy on the S -bond and the money market account; in other words, if $T < S$,*

$$dp(t, T) = p(t, T) \left\{ \beta^B(t) \frac{dB(t)}{B(t)} + \beta^S(t) \frac{dp(t, S)}{p(t, S)} \right\},$$

for certain β^B and β^S (representing the proportions of total wealth invested in the money market account and the S -bond, respectively).

Exercise 9. Find the replicating portfolio of the above theorem.

2.2.4 Multiple sources of randomness

A natural extension is to consider multiple sources of randomness. Concretely, let us assume that W in (2.6) is a k -dimensional Wiener process and σ_T is a \mathbb{R}^k -valued process. Thus, (2.6) takes the form:

$$dp(t, T) = p(t, T) \left\{ \alpha_T(t) dt + \sum_{i=1}^k \sigma_T^i(t) dW^i(t) \right\}. \quad (2.18)$$

It turns out that our results in this section will hold true for this more general setting. Let us address in detail the issue of arbitrage-freeness in this context. It is intuitive that (2.9) will then take the following form:

$$\sigma_T \lambda(t) = \alpha_T(t) - r(t). \quad (2.19)$$

The idea behind (2.19) is as before. We consider a portfolio of k -bonds with maturities S_1, \dots, S_k and the T -bond:

$$V_t = p(t, T) + \sum_{i=1}^k \Delta^i(t) p(t, S_i)$$

It will turn out that it is possible to choose $\Delta^1(t), \dots, \Delta^k(t)$ at time t so that the portfolio is locally risk-free with a rate of return higher than the short-rate $r(t)$ unless the following matrix is singular:

$$H = \begin{bmatrix} \alpha_{S_1} - r & \sigma_{S_1} \\ \vdots & \vdots \\ \alpha_{S_k} - r & \sigma_{S_k} \\ \alpha_T - r & \sigma_T \end{bmatrix}.$$

In particular, the first column is a linear combination of the remainder columns and thus, there exists a vector $\lambda(t) = [\lambda^1(t), \dots, \lambda^k(t)]$ such that

$$\sigma_{S_1} \lambda(t) = \alpha_{S_1} - r, \quad \text{and} \quad \sigma_T \lambda(t) = \alpha_T - r.$$

This shows the existence of λ in (2.19) and that this will be independent of the maturity T .

One can again interpret $\lambda^i(t)$ as the market price of the i^{th} source of randomness, W^i . The larger λ^i is, the larger is the risk premium of the bond per one unit of the volatility σ_T^i . As before, it is expected that the market can be completed with the help of k benchmark bonds.

2.3 A markov pricing model and the term structure equation

One drawback of the general one-factor market model of the previous section is that the resulting risk-neutral valuation formulas are computationally expensive. It is desirable to have a more explicit computational tool. Under additional pricing constraints, we can obtain a type of Black-Scholes Partial Differential Equation for the bond prices, called the *term structure equation* or the *bond price equation*. Our pricing setting is as follows:

(1') The short rate $r(t)$ of the money market account follows the Markov dynamics

$$dr(t) = \mu(t, r(t)) dt + \sigma(t, r(t)) dW_t, \quad t \geq 0, \quad r(0) = r_0, \quad (2.20)$$

where $\mu(t, r)$ and $\sigma(t, r)$ are deterministic functions and W is a one-dimensional Wiener process.

(2') For each maturity T , the time- t price of the T -bond is a smooth function of t and of the spot short rate $r(t)$; that is,

$$p(t, T) = F_T(t, r(t)), \quad (2.21)$$

for a smooth function $F_T(t, r)$.

Throughout this section, we assume that (2.20) admits a solution which takes values on $\mathbb{R}_+ := (0, \infty)$. The second assumption (2') means that the bond price is "Markov" in the sense that its value depends only on the current short-rate spot price and not on the past history of the short-rate.

Remark 3. Note that if one assumes (1') above and (2) from the previous section, then, assuming that the market is arbitrage-free and that the market price of risk λ satisfies the Novikov's condition, the bond prices will satisfy the risk-neutral valuation formula (2.14). It will follow from the Markov properties of $\{r(t)\}_{t \geq 0}$ that (2.21) is satisfied for a function F_T . In particular models, one can further show that such a function is smooth and hence, the PDF (2.24) below will hold true.

The key towards a PDE formulation for F_T is to observe that the pricing setting (1')-(2') is a particular case of (1)-(2) in the previous section. Indeed, by Itô's formula,

$$dp(t, T) = dF_T(t, r(t)) = \frac{\partial F_T}{\partial t} dt + \frac{\partial F_T}{\partial r} dr(t) + \frac{1}{2} \frac{\partial^2 F_T}{\partial r^2} (dr(t))^2,$$

which can be written as

$$dp(t, T) = p(t, T) \{ \alpha_T(t) dt + \sigma_T(t) dW(t) \},$$

with

$$\alpha_T(t) := \frac{\partial_t F_T + \mu \partial_r F_T + \frac{1}{2} \sigma^2 \partial_{rr} F_T}{F_T} \Big|_{(t, r(t))}, \quad \sigma_T(t) := \frac{\sigma \partial_r F_T}{F_T} \Big|_{(t, r(t))}. \quad (2.22)$$

Here, $\partial_t F_T = \partial F_T / \partial t$ (with similar notations holding for ∂_r and ∂_{rr}) and $|_{(t, r(t))}$ means that all the functions on the right hand-side of the equations are evaluated at $r = r(t)$.

Since the above pricing model is a particular case of the one-factor market model of the previous section, the arbitrage-free condition of Theorem 1 will apply. Hence, under absence of arbitrage, there exists a process $\lambda(t)$ such that

$$\lambda(t) = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \frac{\partial_t F_T + \mu \partial_r F_T + \frac{1}{2} \sigma^2 \partial_{rr} F_T - r F_T}{\sigma \partial_r F_T} \Big|_{(t, r(t))}, \quad (2.23)$$

regardless of the maturity $T > 0$. Equivalently, the function $\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$\lambda(t, r) := \frac{\partial_t F_T(t, r) + \mu(t, r) \partial_r F_T(t, r) + \frac{1}{2} \sigma^2(t, r) \partial_{rr} F_T(t, r) - r F_T(t, r)}{\sigma(t, r) \partial_r F_T(t, r)},$$

does not depend on the maturity T . Then, we have the following result:

Theorem 4. *Under the above pricing assumptions (1')-(2') and absence of arbitrage opportunities, there exists a function $\lambda(t, r) : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that F_T satisfies the PDE:*

$$\frac{\partial F_T}{\partial t} + (\mu(t, r) - \lambda(t, r) \sigma(t, r)) \frac{\partial F_T}{\partial r} + \frac{1}{2} \sigma^2(t, r) \frac{\partial^2 F_T}{\partial r^2} - r F_T = 0,$$

for $0 < t < T$ and $r > 0$ with boundary condition $F_T(T, r) = 1$.

Let us now consider a general European contingent claim with maturity T and payoff $\mathcal{X} = \Phi(r(T))$ and let $\Pi(t)$ denotes the time- t price of the claim. One can analogously show that the only "Markov" pricing process of the form

$$\Pi(t) = F(t, r(t)),$$

for a smooth function $F(t, r)$, that is consistent with the absence of arbitrage, is when F solves the PDE:

$$\frac{\partial F}{\partial t} + (\mu(t, r) - \lambda(t, r)\sigma(t, r)) \frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2(t, r) \frac{\partial^2 F}{\partial r^2} - r F_r = 0, \quad (2.24)$$

with boundary condition

$$F(T, r) = \Phi(r). \quad (2.25)$$

Remark 4 ()*. The equation (2.24) will follow from the same arguments as we did with the bond prices. In general terms, the first step will be to write the dynamics of Π_t

$$d\Pi(t) = \Pi(t) \{ \alpha_{\mathcal{X}}(t) dt + \sigma_{\mathcal{X}}(t) dW_t \},$$

in terms of the partial derivatives of F . Secondly, we devise a (locally) risk-free portfolio of a T -bond and the claim \mathcal{X} (as in Section 2.2.1), which will be compared to the money market account dynamics. This will imply that a necessary condition for the market consisting of the bond, the money market account, and the claim to be arbitrage-free is that

$$\lambda(t) = \frac{\alpha_{\mathcal{X}}(t) - r(t)}{\sigma_{\mathcal{X}}(t)}.$$

We finally plug in the rate of return $\alpha_{\mathcal{X}}$ and volatility $\sigma_{\mathcal{X}}$ in terms of F and rearrange terms.

The PDE (2.24-2.25) admits the following stochastic (Feymann-Kac) representation:

$$F(t, r) = \mathbb{E}^Q \left\{ e^{-\int_t^T r(u) du} \Phi(r(T)) \middle| r(t) = r \right\}, \quad (2.26)$$

where under Q , the dynamics of r is given by

$$dr(s) = \{ \mu - \lambda \sigma \} ds + \sigma dW^Q(s), \quad (2.27)$$

with $\{W_t^Q\}_{0 \leq t \leq T}$ denoting a Q -Wiener process. The existence of such a probability measure Q follows from Girsanov's theorem (under certain technical assumptions on λ). More concretely, Q is defined such that

$$W^Q(t) := W(t) + \int_0^t \lambda(s, r(s)) ds,$$

is a Q -Wiener process.

Remark 5 ()*. Let us briefly recall the general steps towards the Feymann-Kac representation:

- (i) First, find the dynamics of $F(s, r(s))$ using Itô formula;
- (ii) By Itô's multiplication formula, find the dynamics of $e^{-\int_t^s r(u) du} F(s, r(s))$ with respect to s ;
- (iii) Integrate from $s = t$ to $s = T$ and take conditional expectations give \mathcal{F}_t^W .
- (iv) Use the Martingale property of Wiener integrals ($E \left\{ \int_t^T f_s dW(s) \middle| \mathcal{F}_t^W \right\} = 0$) and the Markov's property of r .

2.4 Some popular one-factor Markov models

For future reference we now describe a few popular one-factor models of the form (2.20). These models can serve to model the dynamics of the short-rate under both the objective (statistical) measure P and the martingale (risk-neutral) measure Q .

Vasicek model

The dynamics of the model is given by

$$dr(t) = \alpha(\theta - r(t))dt + \sigma dW(t), \quad (2.28)$$

where α , θ , and σ are positive constants. Using Itô's formula, one can easily show that the solution of (2.29) is given by

$$r(t) = r(0)e^{-\alpha t} + \theta(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-u)} dW(u).$$

Hence, $r(t)$ conditional to $r(0)$ is Normally distributed with mean and variance

$$\begin{aligned} E \{ r(t) | r(0) \} &= r(0)e^{-\alpha t} + \theta(1 - e^{-\alpha t}), \\ \text{Var} \{ r(t) | r(0) \} &= \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \end{aligned}$$

We can describe the dynamics of the process in terms of their parameters:

- (i) The value of σ measures the volatility of the process.
- (ii) The value of θ determines the "long-term" mean value of the process. Intuitively, at any time, the short rate $r(t)$ will experience a drift towards the level θ .
- (iii) The strength of the drift is proportional to the value of α , and thus, α determines the speed of mean reversion. From the expression of the conditional mean, one can also see that, starting at 0, the process takes on "average" $t = \log 2/\alpha$ to reach the half-mean value $\theta/2$.

One can easily show that $\{r(t)\}_{t \geq 0}$ has a Normal stationary distribution with mean θ and variance $\sigma^2/2\alpha$. That is, the following two properties hold:

- $r(t)$ in the long run behaves like a $\mathcal{N}(\theta, \sigma^2/2\alpha)$;
- If $r(0)$ is $\mathcal{N}(\theta, \sigma^2/2\alpha)$, then $r(t)$ has the same distribution for any t .

One major drawback of Vasicek's model is that with positive probability r can take negative values.

Cox-Ingersoll-Ross (CIR)

This model exhibits similar dynamical properties to Vasicek's model but overcomes the negativity issue described above. The dynamics of the model is given by

$$dr(t) = \alpha(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \quad (2.29)$$

where α , θ , and σ are positive constants. Under the condition

$$2\alpha\theta > \sigma^2,$$

their solutions exists and remain positives. As with the Vasicek model, σ determines the volatility of the short-rate, θ is its mean "long term" value, and α measure the speed of mean reversion. Lamentably, there is no closed formula for its solution. Its conditional mean and variance given $r(0)$ are determined by

$$\begin{aligned} \mathbb{E}[r(t)|r(0)] &= r(0)e^{-\alpha t} + \theta(1 - e^{-\alpha t}), \\ \text{Var}(r(t)|r(0)) &= r(0)\frac{\sigma^2}{\alpha}(e^{-\alpha t} - e^{-2\alpha t}) + \theta\frac{\sigma^2}{2\alpha}(1 - e^{-\alpha t})^2. \end{aligned}$$

The equation has a $\Gamma(\frac{2\theta\alpha}{\sigma^2}, \frac{\sigma^2}{2\alpha})$ stationary distribution.

Hull-White model or generalized Vasicek

As we will see later on, none of the previous two model can actually recover a desired initial term structure of bond prices $T \rightarrow p^*(0, T)$. In order to be able to overcome this point, Hull-White proposes to take a functional mean parameter in the Vasicek model. Concretely, the dynamics of the short-rate is given by

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW(t), \quad (2.30)$$

where a , θ , and σ are positive constants. Its solution is given by

$$r(t) = r(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-u)}\theta(u)du + \sigma \int_0^t e^{-\alpha(t-u)}dW(u).$$

Hence, $r(t)$ conditional to $r(0)$ is Normally distributed with mean and variance

$$\begin{aligned} E\{r(t)|r(0)\} &= r(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-u)}\theta(u)du, \\ \text{Var}\{r(t)|r(0)\} &= \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \end{aligned}$$

Here, σ determines the volatility of the short-rates, $\theta(t)/a$ is its mean value, and a is the speed of mean reversion.

2.5 The martingale method, calibration, and yield curve fitting

The previous two sections gave conditions for a consistent pricing of bonds and interest rate derivatives. Concretely, under the one-factor model of Section 2.2, a necessary condition for the market to be arbitrage-free is the existence of the market price of risk process (2.9). Subsequently, the risk-neutral pricing procedure (2.15) will generate interest-rate derivative prices that are arbitrage-free provided that Q is chosen so that $\{r(t)\}_{t \geq 0}$ has the dynamics (2.14) under Q . In the case that r follows a Markov model of the form (2.20), the risk-neutral prices (2.15) can be computed further via the PDE equation (2.24-2.25). Both computation procedures depend on λ , μ , and σ only through $\mu^Q := \mu - \lambda\sigma$ and σ , the drift and volatility of the Q -dynamics of the short-rate r , respectively. Thus, if our main goal is option pricing, we only need to model the dynamics of the short-rate under the martingale measure Q . For each risk-neutral drift μ^Q of the short-rate dynamics $r(t)$, there is an associated arbitrage-free price model for the bond market and related interest-rate derivatives given by (2.14)-(2.15).

The martingale method exploits the previous simple observation and works directly with the Q -dynamics of the short-rate. Typically, the martingale dynamics is defined exogenously in terms of certain parameter(s) Θ , which in turn are “calibrated” to match the market prices of bond and interest-rate derivatives. For instance, suppose that there are k interest-rate derivatives with respective payoffs $\Phi_i(r(T_i))$ and maturities T_i , $i = 1, \dots, k$ (these derivatives might include some bonds where $\Phi_i(r(T_i)) \equiv 1$). Then, given their initial market prices $F_i^*(0)$, one can choose the values of the parameter(s) Θ as follows

$$\Theta^* := \arg \inf_{\Theta} \sum_{i=1}^k \left(\frac{\mathbb{E}^{\Theta} \left\{ e^{-\int_0^{T_i} r(u) du} \Phi_i(r(T_i)) \right\} - F_i^*(0)}{F_i^*(0)} \right)^2.$$

We can think of Θ^* as market implied parameters (making an analogy with implied volatility) and the procedure to adjust the parameters to observable market prices is called *calibration*. Typical derivatives are of course the bond prices themselves and other interest-rate derivatives such as caps, captions, and swaptions, which will be introduced later on.

A second more ambitious approach consists of choosing the parameter(s) Θ so that one is able to recover a given initial term structure of bond prices $T \rightarrow p^*(0; T)$. This approach is called *inversion of the yield curve* or *yield curve fitting*. Concretely, if we had already estimated the market term structure $T \rightarrow p^*(0; T)$, we wish to choose the parameters Θ such that

$$p^{(\Theta^*)}(0; T) \equiv p^*(0; T), \quad \text{for all } T > 0, \quad (2.31)$$

where $p^{(\Theta)}(0; T)$ represent the theoretical bond price of the T -zcb corresponding to the value parameter Θ . Note that the initial terms structure $T \rightarrow p^*(0; T)$ will be typically constructed, before the calibration procedure, from bonds, swaps, and/or futures available in the market via the bootstrap method and smoothness assumptions.

In practice, one combines the two previous calibration approaches. For instance, in the Hull-White model

$$dr = (\theta(t) - ar)dt + \sigma dW^Q(t), \quad (2.32)$$

one can use derivatives to estimate a and σ , and then used zero bond prices to calibrate $\Theta(t)$. Given a and σ , we will see that (2.31) is indeed achievable for certain value $\theta(t)$. As a preview, we state this point in the following theorem.

Theorem 5. Let $\{p^*(0, T)\}_{T>0}$ be a desirable initial terms structure subject only to the condition of being twice differentiable. Suppose that the martingale short-rate dynamics is given by the Hull-White model (2.32) with

$$\theta(t) = \frac{\partial f^*(0, t)}{\partial T} + a f^*(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at})$$

where $f^*(0, T) := -\partial \log p^*(0, T) / \partial T$ is the initial forward rate curve. Then, the theoretical initial bond prices

$$p(0, T) = E^Q \left\{ e^{-\int_0^T r(u) du} \right\},$$

are such that

$$p(0, T) = p^*(0, T), \quad \text{for all } T > 0.$$

The previous result can be shown using the term-structure equation. Rather, we shall introduce a whole new framework (the so-called HJM approach) that will allow us, in particular, to construct short-rate model that are able to fit the initial terms structure automatically. The previous result will then appear as a particular instance of this methodology.

Chapter 3

Forward rate models

TBP

Chapter 4

Change of numeraire and option pricing

4.1 Introduction

A numéraire is the unit of account in which other assets are denominated. In this part, we will introduce a method to compute option prices based on the idea of changing the numeraire. This procedure will turn out to be very useful in the case of interest-rate derivatives, where we will use a T-bond as numeraire.

Intuitively, the choice of numeraire does not introduce any arbitrage opportunity in the market. In other words, if the assets under consideration are priced so that there is no arbitrage opportunity, this property will remain true regardless of whether the assets are priced in dollars or in any other currency. This idea can be formalized if we introduce some basic definitions and axioms about how our market operates (e.g. what is a dynamical trading strategy, what is a self-financing portfolio, etc). To this end, suppose that $\{S_1(t)\}_t, \dots, \{S_n(t)\}_t$ represent the price processes of n assets. We can define a self-financing trading strategy $\beta = (\beta_1, \dots, \beta_n)$ as an adapted process such that the value process

$$V_t = \beta_1(t)S_1(t) + \dots + \beta_n(t)S_n(t),$$

satisfies the following condition:

$$V_t = V_0 + \int_0^t \beta_1(u)dS_1(u) + \dots + \int_0^t \beta_n(u)dS_n(u),$$

which can equivalently be written in differential form as follows:

$$dV_t = \beta_1(t)dS_1(t) + \dots + \beta_n(t)dS_n(t),$$

The previous definition formalized the idea of a self-financing strategy as one where there is no infusion or withdrawal of money into the portfolio of assets.

Now, suppose that $\{S_0(t)\}_t$ represents the price process of one unit of certain asset that will be used as numeraire. At time t , the i^{th} asset is worth

$$\bar{S}_i(t) := \frac{S_i(t)}{S_0(t)},$$

in units of the asset S_0 . The only condition about this numeraire is that its price process remains strictly positive at all times and does not pay dividends. The following result, called the *Invariance Lemma*, is key to introduce the method of change of numeraire:

Lemma 1. A trading strategy β in the market $[S_1, \dots, S_n]$ is self-finance if and only if it is self-financing in the market $[\bar{S}_1, \dots, \bar{S}_n]$. Concretely,

$$V_t := \beta_1(t)S_1(t) + \dots + \beta_n(t)S_n(t) = V_0 + \sum_{i=1}^n \int_0^t \beta_i(u) dS_i(u), \quad \text{for all } t > 0,$$

if and only if

$$\bar{V}_t := \beta_1(t)\bar{S}_1(t) + \dots + \beta_n(t)\bar{S}_n(t) = \bar{V}_0 + \sum_{i=1}^n \int_0^t \beta_i(u) d\bar{S}_i(u), \quad \text{for all } t > 0.$$

Again, the previous lemma is quite natural, since injecting (or withdrawing) external money into (resp. from) a portfolio is a property that should not depend on the currency of choice. One can expect that the previous Lemma holds in a very general setting. The following exercise shows its validity in the case of an Itô market:

Exercise 10. Suppose that the price processes of the assets have the following dynamics

$$d S_i(t) = S_i(t) \{b_i dt + \sigma_i(t) dW(t)\}, \quad i = 0, \dots, n, \quad (4.1)$$

where W is d -dimensional Wiener process. Use Itô's formula to compute the differentials $d\bar{S}_i$ and conclude the validity of the invariance lemma.

The invariance lemma has several consequences. For instance, recall that an arbitrage opportunity is a self-financing strategy β which corresponding portfolio value V_t^β satisfies

$$V_0^\beta = 0, \quad V_T^\beta \geq 0, \quad \text{and} \quad P(V_T^\beta > 0) > 0.$$

Then, a trading strategy β is an arbitrage opportunity in the market $[S_1, \dots, S_n]$ if and only if this is an arbitrage opportunity in the market $[\bar{S}_1, \dots, \bar{S}_n]$. Thus, we can say that a market is arbitrage-free regardless of the numeraire.

The second key result towards the method of change of numeraire is the *Fundamental Theorem of Finance* (FTF), which should be by now a quite natural result given our experience with the Binomial market model and Black-Scholes market mode. FTF states that a market is arbitrage-free if and only if there exists a martingale measure Q (equivalent to the objective measure P) such that all assets have the same expected rate of return under Q . The fundamental theorem of finance holds true in a very general setting. In the case of the Itô model (4.10), this can be shown using *Girsanov's theorem* following argument similar to those in Section 2.2.1, but taking the normalized asset $\bar{S}_0(t) = 1$ as the money market account.

Proposed originally by Nicole El Karoui, the method of change of numeraire combines the fundamental theorem of finance with the invariance of arbitrage-freeness under a change of numeraire. Concretely, suppose that $[S_0, \dots, S_n]$ is arbitrage-free and suppose that we take as numeraire the asset S_0 . The normalized price process $[\bar{S}_0, \dots, \bar{S}_n] = [1, \bar{S}_2, \dots, \bar{S}_n]$ will be still arbitrage-free and, by the FTF, there will be a martingale measure, that we denote Q_0 , so that

$$E^{Q_0} \left\{ \frac{\bar{S}_0(T) - \bar{S}_0(t)}{\bar{S}_0(t)} \middle| \mathcal{F}_t \right\} = E^{Q_0} \left\{ \frac{\bar{S}_i(T) - \bar{S}_i(t)}{\bar{S}_i(t)} \middle| \mathcal{F}_t \right\}, \quad (4.2)$$

for each $i = 1, \dots, n$. But, the first expected value is identically 0. Hence, we can write the previous condition as follows:

$$E^{Q_0} \{ \bar{S}_i(T) | \mathcal{F}_t \} = \bar{S}_i(t) \iff E^{Q_0} \left\{ \frac{S_i(T)}{S_0(T)} \middle| \mathcal{F}_t \right\} = \frac{S_i(t)}{S_0(t)}, \quad (4.3)$$

for all $0 \leq t \leq T$ and any $i = 1, \dots, n$. In words, under Q_0 , the price process of any *tradable* asset denominated in S_0 units is a martingale.

Given that some of the assets S_i could be taken to be derivatives of other primary assets, this result applies also to derivatives. Concretely, suppose that $S_n = \Pi(t)$ represents the price process of a contingent claim with final payoff \mathcal{X} (that is, $S_n(T) = \mathcal{X}$). Then, the contingent claim \mathcal{X} is priced consistently (relative to the other assets $[S_0, \dots, S_{n-1}]$) if and only if its price process $\{\Pi(t)\}_{0 \leq t \leq T}$ satisfies the following relationship

$$\Pi(t) = S_0(t) E^{Q_0} \left\{ \frac{\mathcal{X}}{S_0(T)} \middle| \mathcal{F}_t \right\}, \quad (4.4)$$

where Q_0 is the martingale measure that makes the market $[\bar{S}_0 = 1, \bar{S}_1, \dots, \bar{S}_{n-1}]$ arbitrage-free. Formula (4.4) states a risk-neutral valuation formula for the claim \mathcal{X} taking the asset S_0 as the numeraire (or *deflator*).

For future reference, we state the previous findings in the following result:

Theorem 6. *Suppose that S_0, \dots, S_n represent the price processes of $n + 1$ assets defined on a probability space (Ω, P) and adapted to an information process $\{\mathcal{F}_t\}_{t \geq 0}$. Suppose that the market is arbitrage-free and that S_0 is strictly positive. Then, the following statements hold true:*

1. *There exists a probability measure Q_0 equivalent to P such that the normalized price processes*

$$\bar{S}_0 = 1, \bar{S}_1 = \frac{S_1}{S_0}, \dots, \bar{S}_n = \frac{S_n}{S_0},$$

are martingales under Q_0 .

2. *If $\Pi(t) := S_i(t)$, for some $i \in \{1, \dots, n\}$, represents the price process of a European claim with maturity T and payoff \mathcal{X} , then*

$$\Pi(t) = S_0(t) \mathbb{E}^{Q_0} \left\{ \frac{\mathcal{X}}{S_0(T)} \middle| \mathcal{F}_t \right\}, \quad (4.5)$$

any $0 \leq t \leq T$.

3. *Given a claim with payoff \mathcal{X} and maturity T , the process defined by*

$$\Pi(t) := S_0(t) \mathbb{E}^{Q_0} \left\{ \frac{\mathcal{X}}{S_0(T)} \middle| \mathcal{F}_t \right\},$$

is a valid arbitrage-free price process for the claim \mathcal{X} . That is, Π is such that the market $[S_0, \dots, S_n, \Pi_t]$ is arbitrage-free and $\Pi(T) = \mathcal{X}$.

The first part of the previous theorems states the existence of an equivalent probability measure Q_0 such that the price process of any tradable asset in S_0 units is a martingale under Q_0 . Such a measure Q_0 will be called hereafter the *martingale measure associated with the numeraire S_0* . The second part of the above theorem says that the only price processes of contingent claims that are consistent with the absence of arbitrage are necessarily of the form (4.5), which can be interpreted as a *Risk-Neutral Valuation Formula* with general numeraire. The last statement says that pricing a claim according to (4.5) will suffice to avoid arbitrage opportunities.

Example 9. Suppose that we take as numeraire a (locally) risk-free asset with instantaneous rate of return $\{r(t)\}_{t \geq 0}$. Specifically, the numeraire will be

$$B(t) = e^{\int_0^t r(u) du}.$$

Then, the martingale measure associated with B is the standard martingale measure Q under which the discounted prices of the primary assets

$$\bar{S}_i(t) = e^{-\int_0^t r(u) du} S_i(t), \quad i = 0, \dots, n,$$

are martingales. In that case,

$$\Pi(t) := \mathbb{E}^Q \left\{ e^{-\int_t^T r(u) du} \mathcal{X} \middle| \mathcal{F}_t \right\},$$

is an arbitrage-free price process for the claim with payoff \mathcal{X} and maturity T .

Exercise 11. Assume the setting of the previous example and additionally assume the existence of another asset S_0 with strictly positive price process. Let Q_0 be the martingale measure associated with the numeraire S_0 . Show that Q_0 is given by the formula

$$Q_0(A) = \frac{1}{S_0(0)} \mathbb{E}^Q \left\{ e^{-\int_0^T r(u) du} S_0(T) \mathbf{1}_A \right\}.$$

Moreover, shows that the following formula is satisfied for any $\mathcal{Y} \in \mathcal{F}_T$:

$$\mathbb{E}^{Q_0} \{ \mathcal{Y} | \mathcal{F}_t \} = \frac{1}{S_0(t)} \mathbb{E}^Q \left\{ e^{-\int_t^T r(u) du} S_0(T) \mathcal{Y} \middle| \mathcal{F}_t \right\}.$$

It is useful to write the last formula in the previous exercise as follows:

$$\mathbb{E}^Q \left\{ e^{-\int_t^T r(u) du} \mathcal{X} \middle| \mathcal{F}_t \right\} = S_0(t) \mathbb{E}^{Q_0} \left\{ \frac{\mathcal{X}}{S_0(T)} \middle| \mathcal{F}_t \right\}. \quad (4.6)$$

Formula (4.6) shows explicitly the connection between the two Risk-Neutral Valuation Formulas.

4.2 Exchange options

In this section we give a classical example that illustrates the power of changing numeraire. Consider a "basket" option that gives the holder the right (but not the obligation) to exchange one unit of a

certain asset S_1 for one unit of another asset S_2 at a given time T . This option is equivalent to a contingent claim with maturity T and payoff:

$$\mathcal{X} = (S_2(T) - S_1(T))_+.$$

Let us assume that the asset prices follow a Black-Scholes type model of the form

$$dS_i = S_i(t) \{b_i dt + \sigma_i(t) dW(t)\}, \quad i = 1, 2,$$

where $W = [W_1, \dots, W_d]^T$ is a d -dimensional Wiener process, $\sigma_i : [0, \infty) \rightarrow \mathbb{R}^d$ is a deterministic volatility process, and b_i is an adapted \mathbb{R} -valued process such that the market is arbitrage-free.

Suppose that we want to price the previous option. The classical approach will price the claim by the RNVF:

$$\Pi(t) = E^Q \left\{ e^{-\int_t^T r(u) du} (S_2(T) - S_1(T))_+ \middle| \mathcal{F}_t \right\}, \quad (4.7)$$

where r is the instantaneous rate of return of the locally risk-free asset (whose existence is assumed), and Q is such that the discounted price process are martingales under Q . The previous formula is not easy to compute, especially if r is assumed stochastic driven by W .

We now give another method to compute (4.7). Consider the martingale measure Q_1 associated with the numeraire S_1 , which is assumed to be strictly positive. We know that the price process (4.7) is arbitrage-free relative to $[S_1, S_2]$ (namely, $[S_1, S_2, \Pi]$ are arbitrage-free) and by the second part of Theorem 4.7, this price will be given as well by the following formula:

$$\Pi(t) = S_1(t) E^{Q_1} \left\{ \frac{1}{S_1(T)} (S_2(T) - S_1(T))_+ \middle| \mathcal{F}_t \right\},$$

which can be written as

$$\Pi(t) = S_1(t) \mathbb{E}^{Q_1} \left\{ (\bar{S}_2(T) - 1)_+ \middle| \mathcal{F}_t \right\},$$

where $\bar{S}_2(t) := S_2(t)/S_1(t)$ is the normalized asset price process.

Surprisingly, one can derive precisely the distribution of \bar{S}_2 under Q_1 . Indeed, recall that, by definition of Q_1 , \bar{S}_2 is a martingale. Hence, the dynamics will be of the form:

$$d\bar{S}_2(t) = \bar{S}_2(t) \bar{\sigma}_2(t) d\bar{W}(t),$$

for certain $\bar{\sigma}_2$, where \bar{W} is a Wiener process under Q_1 . But also, we know that a change of probability measure does not modify the volatility of the process. Thus, it suffices to determine the volatility of the process under any probability measure. Under P , one can compute this volatility by applying Itô's formula. Indeed, defining $f(s_1, s_2) = s_2/s_1$,

$$d\bar{S}_2(t) = df(S_1, S_2) = \frac{\partial f}{\partial s_1} dS_1 + \frac{\partial f}{\partial s_2} dS_2 + \text{etc.} = \frac{-S_2}{S_1^2} S_1 \sigma_1 dW(t) + \frac{1}{S_1} S_2 \sigma_2 dW(t) + \text{etc.}$$

The terms left unwritten are not relevant to determine the volatility of the process. One can then check that the volatility of the process is $\bar{\sigma}_2 = \sigma_2 - \sigma_1$. We conclude that

$$d\bar{S}_2(t) = \bar{S}_2(t) (\sigma_2(t) - \sigma_1(t)) d\bar{W}(t). \quad (4.8)$$

Furthermore, $\bar{S}_2(t)$ can be written as

$$d\bar{S}_2(t) = \bar{S}_2(t)\tilde{\sigma}(t)d\bar{W}(t),$$

where $\tilde{\sigma}(t) := \|\sigma_2(t) - \sigma_1(t)\|$ and $\bar{W}(t) = \int_0^t (\sigma_2(u) - \sigma_1(u))/\tilde{\sigma}(u)d\bar{W}(u)$ is a 1-dimensional Wiener process under Q_2 .

Finally, since $\tilde{\sigma}$ is deterministic, one can show that $E^{Q_1} \left\{ (\bar{S}_2(T) - 1)_+ \middle| \mathcal{F}_t \right\}$ corresponds to the time- t price of a call option with strike $K = 1$ and maturity T on a asset following a vanilla Black-Scholes model with rate of return $r = 0$, underlying spot price $s = S_2(t)/S_1(t)$, and volatility σ given according to

$$\sigma^2 = \frac{1}{T-t} \int_t^T \tilde{\sigma}^2(u)du.$$

Recalling the Black-Scholes formula for call options, we obtain that

$$\Pi(t) = S_1(t) (sN(d_1) - e^{-r(T-t)}KN(d_2)) = S_2(t)N(d_1) - S_1(t)KN(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln \left(\frac{S_2(t)}{S_1(t)} \right) + \frac{\sigma^2}{2}(T-t) \right\}, \quad d_2 = d_1 - \sigma\sqrt{T-t}. \quad (4.9)$$

Using the same tricks leading to (4.8), one can show the following result:

Exercise 12. Suppose that the price processes of 2 assets are given by the dynamics

$$d S_i(t) = S_i(t) \{b_t dt + \sigma_i(t)d W(t)\}, \quad i = 1, 2, \quad (4.10)$$

where W is d -dimensional Wiener process, and the market is arbitrage-free.

- Let Q_1 be the martingale measure associated with the numeraire S_1 . Find the dynamics of the standardized assets $\bar{S}_2(t) := S_2(t)/S_1(t)$ under the martingale measure Q_1 associated with the numeraire S_1 (your dynamics must be in terms of the Wiener process W^{S_1} under Q_1).
- Let $B_t = \exp\{\int_0^t r(u)du\}$ be a (locally) risk-free asset and suppose that the market $[B, S_1, S_2]$ is arbitrage-free. Let W^Q (resp. W^{S_1}) be the martingale measure associated with B (resp. S_1). Show that

$$W^{S_1}(t) = W^Q(t) + \int_0^t \sigma_1^T(u)du.$$

Hint: As before consider $\bar{S}_2 = S_2/S_1$. Determine $d\bar{S}_2$ under Q and compare this with the dynamics that $d\bar{S}_2$ should have under Q^{S_1} .

4.3 The forward neutral measure

The martingale measure associated with the T -zero-coupon bond is called the T -forward measure, denoted throughout this part by Q_T . Recall that, by definition, Q_T is the probability measure equivalent to the objective probability measure P for which the price process $\Pi(t)$ of any tradable asset is such that its normalized process

$$\bar{\Pi}(t) := \frac{\Pi(t)}{p(t, T)}, \quad (4.11)$$

is a martingale under Q_T . Again, $\bar{\Pi}(t)$ represents the value of the asset in T -bonds units. But also, $\bar{\Pi}(t)$ can be seen to be the *forward price* of a forward contract on the asset Π with settlement time t and exchange time T . We call $\bar{\Pi} = \Pi(t)/p(t, T)$ the T -forward price. Since Q_T has the property of making all T -forward prices of tradable assets martingales, Q_T is also called the *T -forward neutral measure*.

The T -forward measure is quite useful to price derivatives under the presence of stochastic interest rates. To understand why this is the case, consider an interest-rate derivative $\mathcal{X} = \Phi(r(T))$ with maturity T . Under absence of arbitrage, the arbitrage-free price of the claim is given by

$$\Pi(t) = E^Q \left\{ e^{-\int_t^T r(u)du} \Phi(r(T)) \middle| \mathcal{F}_t \right\}, \quad (4.12)$$

where Q is a martingale measure for the bond market (that is, under Q , all discounted bond prices $e^{-\int_0^t r(u)du} p(t, T)$ are martingales). On the other hand, using the T -forward measure, we have that $\bar{\Pi}(t) = E^{Q_T} \{ \bar{\Pi}(T) | \mathcal{F}_t \}$ and hence,

$$\Pi(t) = p(t, T) E^{Q_T} \{ \Phi(r(T)) | \mathcal{F}_t \} \quad (4.13)$$

At time $t = 0$, this becomes

$$\Pi(0) = p(0, T) E^{Q_T} \{ \Phi(r(T)) \}. \quad (4.14)$$

Formula (4.14) often is easier to apply than (4.12) since $p(0, T)$ is easily observable in the market and the computation of $E^{Q_T} \{ \Phi(r(T)) \}$ requires only the distribution of one variable $\Phi(r(T))$, while the computation of $E^Q \left\{ e^{-\int_0^T r(u)du} \Phi(r(T)) \right\}$ requires the joint distribution of $\int_0^T r(u)du$ and $\Phi(r(T))$.

The conclusion in the above paragraph holds for many other claims. For instance, if $[B, S_1, \dots, S_n]$ is an arbitrage-free market and \mathcal{X} is the payoff of a European claim which payoff depends on the underlying assets $[B, S_1, \dots, S_n]$, then

$$E^Q \left\{ e^{-\int_t^T r(u)du} \mathcal{X} \middle| \mathcal{F}_t \right\} = p(t, T) E^{Q_T} \{ \mathcal{X} | \mathcal{F}_t \}. \quad (4.15)$$

The left-hand side will involve to determine the joint distributions of $\int_0^T r(u)du$ and \mathcal{X} , while the right-hand side will involve only the distribution of \mathcal{X} under Q_T .

The following exercise show another interesting property of T -forward measures:

Exercise 13.

1. Show that, under the T -forward measure Q_T (namely, the martingale measure associated with the numeraire $p(t, T)$), the instantaneous forward rate today, $f(0, T)$, is an "unbiased estimator" for the short $r(T)$ at the future time T . That is,

$$f(0, T) = E^{Q_T} \{ r(T) \}.$$

2. Is it true that the forward rate for $[S, T]$ contracted now, $L(0; S, T)$, is an unbiased estimator for the (simple) LIBOR spot rate for $[S, T]$, $L(S, T)$? That is,

$$L(0; S, T) = E^{Q_T} \{ L(S, T) \}?$$

(see Section 1.5.2 for the notation).

Hint: Consider the price of the S -bond in the currency $p(t, T)$.

4.4 Valuation of a call option via change of numeraires

In this part, we give an alternative formula for a call option on an asset with strictly positive price process $\{S(t)\}_{t \geq 0}$. This formula can be considered the analog of the Black-Scholes formula for stochastic interest rates in terms of its simplicity and applicability.

Recall that the key assumption in the B-S model was that both the volatility σ of the underlying and the risk-free rate of return r are constants. In that case, the value of a call option takes the form

$$\Pi(t) = sN(d_1) - e^{-rT}KN(d_2), \quad (4.16)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left\{ \ln \left(\frac{s}{Ke^{-rT}} \right) + \frac{1}{2}\sigma^2T \right\}, \quad d_2 = d_1 - \sigma\sqrt{T}. \quad (4.17)$$

Above, T is the maturity, s is the spot price of the underlying at time $t = 0$, and r is the short-rate. With little effort, it is possible to generalize the formula (4.16) to the case of deterministic time-varying volatility $\sigma(t)$ and risk-free rate of return $r(t)$ by replacing σ and r in (4.16-4.17) by their "mean values" $\bar{\sigma}$ and \bar{r} defined by

$$\bar{\sigma} = \sqrt{\frac{1}{T} \int_0^T \sigma^2(u) du}, \quad \bar{r} = \frac{1}{T} \int_0^T r(u) du.$$

Under a stochastic short-rate model, we can also derive a simple formula for the price of call options. However, in this context what is taken to be deterministic is the volatility of the " T -forward price" of the underlying $\{\bar{S}(t)\}_{t \geq 0}$ defined by

$$\bar{S}(t) := S(t)/p(t, T).$$

Concretely, we assume that the market $[B, p(\cdot, T), S(\cdot)]$ is arbitrage-free and under the objective measure P ,

$$d\bar{S}(t) = \bar{S}(t) \{ \bar{b}(t)dt + \bar{\sigma}(t)dW(t) \}, \quad (4.18)$$

where $\bar{\sigma} : [0, T] \rightarrow \mathbb{R}_+$ is a *deterministic function* and \bar{b} an arbitrary adapted process. Note that, given that \bar{S} is a martingale under the T -forward measure Q_T , the dynamics of \bar{S} under Q_T is

$$d\bar{S}(t) = \bar{S}(t)\bar{\sigma}(t)d\bar{W}(t),$$

where \bar{W} is the Wiener process under Q_T . Furthermore,

$$\bar{S}(t) = \bar{S}(0) \exp \left\{ \int_0^t \bar{\sigma}(u)d\bar{W}(u) - \frac{1}{2} \int_0^t \|\bar{\sigma}(u)\|^2 du \right\},$$

and since $\bar{\sigma}$ is deterministic, we have that the distribution of $\ln \bar{S}(T)$ under Q_T is Normal with the following mean and variances:

$$\ln \bar{S}(T) \sim \mathcal{N} \left(\ln \bar{S}(0) - \frac{1}{2} \int_0^T \|\bar{\sigma}(t)\|^2 dt, \int_0^T \|\bar{\sigma}(t)\|^2 dt \right). \quad (4.19)$$

We are now ready to give the formula for the call option price:

Theorem 7 (Geman-El Karoui-Rochet). *Under an arbitrage-free (stochastic) interest rate model and under the deterministic (forward) volatility assumption (4.18), the price of the call option on the asset S with strike K and maturity T is given by*

$$\Pi(0) = S(0)N(d_1) - K p(0, T)N(d_2),$$

where $N(z)$ is the distribution function of a standard normal and

$$d_1 = \frac{\ln \{S(0)/Kp(0, T)\} + \frac{1}{2} \int_0^T \|\bar{\sigma}(t)\|^2 dt}{\sqrt{\int_0^T \|\bar{\sigma}(t)\|^2 dt}}, \quad d_2 = d_1 - \sqrt{\int_0^T \|\bar{\sigma}(t)\|^2 dt}.$$

Interestingly, the option value above depends only on the volatility of the T -forward price of the underlying, and the current market prices of the asset and the T -bond. Albeit of its complicated appearance, the previous formula is quite natural in light of the analog Black-Scholes formula (4.16-4.17) for time-varying deterministic volatility and short rate. The only difference is to replace e^{-rT} (which is in fact the bond price under a constant short-rate r assumption) with the market bond price $p(0, T)$ at time $t = 0$ and replace σ with the mean value $\sqrt{\int_0^T \|\bar{\sigma}(t)\|^2 dt}$ of $\bar{\sigma}$. Hence, its validity is not a surprise. The most interesting contribution is to formalize the required assumption for its validity.

Remark 6. We outline the general steps behind the previous formula:

1. The arbitrage-free price of the call option is given by

$$\Pi(0) = E^Q \left\{ e^{-\int_0^T r(u)du} (S(T) - K)_+ \right\},$$

which can be written as

$$\begin{aligned} \Pi(0) &= E^Q \left\{ e^{-\int_0^T r(u)du} (S(T) - K) \mathbf{1}_{\{S(T) \geq K\}} \right\} \\ &= E^Q \left\{ e^{-\int_0^T r(u)du} S(T) \mathbf{1}_{\{S(T) \geq K\}} \right\} - K E^Q \left\{ e^{-\int_0^T r(u)du} \mathbf{1}_{\{S(T) \geq K\}} \right\}. \end{aligned}$$

2. In terms of the martingale measure Q_S associated with the numeraire S and the T -forward martingale measure Q_T ,

$$\Pi(0) = S(0)Q_S(S(T) \geq K) - KQ_T(S(T) \geq K), \quad (4.20)$$

where we used the formulas (4.6) and (4.15).

3. To evaluate the second probability in (4.20), we note that

$$Q_T(S(T) \geq K) = Q_T(\ln \bar{S}(T) \geq \ln K),$$

which can be evaluated using (4.19).

4. To evaluate the first probability in (4.20), we use the following trick:

$$Q_S(S(T) \geq K) = Q_S\left(\frac{1}{K} \geq \frac{1}{\bar{S}_T}\right) = Q_S\left(\frac{1}{K} \geq Y_T\right),$$

where we set $Y(t) := 1/\bar{S}_t$. Given that $Y(t) := p(t, T)/S(t)$, this process is a martingale under Q_S . Its volatility can be determined using the dynamics (4.18) and Itô's formula. Indeed, taking $f(\bar{s}) = 1/\bar{s}$:

$$dY(t) = df(\bar{S}(t)) = \frac{\partial f}{\partial \bar{s}} d\bar{S} + \text{etc.} = -\frac{1}{\bar{S}^2} \bar{S} \bar{\sigma} dW(t) + \text{etc.}$$

Hence, under Q_S ,

$$dY(t) = -Y(t) \bar{\sigma}(t) dW^S(t),$$

where W^S is the Wiener process under Q_S . Hence, under Q_T ,

$$\ln \bar{Y}(T) \sim \mathcal{N}\left(\ln \bar{Y}(0) - \frac{1}{2} \int_0^T \|\bar{\sigma}(t)\|^2 dt, \int_0^T \|\bar{\sigma}(t)\|^2 dt\right), \quad (4.21)$$

and we can evaluate the first probability in (4.20) as we did with $S(T)$.

4.4.1 Price of a call option on a zero-bond in a Hull-White model

Under the Hull-White model for the short rate,

$$dr = \{\theta(t) - ar\}dt + \sigma dW_t, \quad (4.22)$$

it can be proved that a zero-bond with maturity $T' > T$ admits a deterministic volatility under the forward measure. Concretely, $\bar{S}_T(t) = \frac{p(t, T')}{p(t, T)}$ satisfies

$$d\bar{S}_T(t) = \bar{S}_T(t) \{(\text{stuff})dt + \bar{\sigma}(t)dW_t\},$$

with

$$\bar{\sigma}(t) = \frac{\sigma}{a} \left\{ e^{-a(T'-t)} - e^{-a(T-t)} \right\}.$$

Therefore, we can apply the Theorem 7 to price a call option on a T' -bond with strike K and maturity $T < T'$. The option will have payoff $\mathcal{X} = (p(T, T') - K)_+$ at time T and its price will be

$$\Pi(0) = p(0, T')N(d_1) - K p(0, T)N(d_2), \quad (4.23)$$

where

$$d_1 = \frac{\ln \{p(0, T')/Kp(0, T)\} + \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}, \quad d_2 = d_1 - \hat{\sigma}, \quad (4.24)$$

$$\hat{\sigma} = \frac{\sigma^2}{2a^3} \{1 - e^{-2aT}\} \left\{1 - e^{-a(T'-T)}\right\}^2. \quad (4.25)$$

Using the following put-call parity relationship

$$P(0; T, T', K) = C(0; T, T', K) + Kp(0, T) - p(0, T'),$$

one can easily obtain the price of a put option on a T' -bond with strike K and maturity T as follows:

$$\Pi(0) = Kp(0, T)N(-d_2) - p(0, T')N(-d_1),$$

where d_1 and d_2 are as above.

It is important to note that the call option prices does not depend on θ . An important application of this fact for calibration purposes. Indeed, if we were able to determine market prices of call or put options on bonds, then we can attempt to adjust the parameters σ and a of (4.22) so that the theoretical arbitrage-free prices (4.23) are close to the corresponding market prices. Once the parameters σ and a are calibrated, one can fit the parameter θ to the initial terms structure as in Theorem 5. Lamentably, quotes of call or put options on bonds are not widely available. However, we shall see next that these call and put option prices can be "extracted" from other extremely liquid interest-rate derives: Caps and Floors.

4.5 Caps and Floors

4.5.1 Generalities and usage

A *cap* is a financial contract specified by a sequence of *reset times* $T_0 = 0 < T_1 < \dots < T_n$ (also called *tenor*), a so-called *cap rate* R , and a *notional or principal* value M . Under this specifications, the cap will consist of a sequence of payments at times T_1, \dots, T_n in the respective amounts

$$\mathcal{X}_i = M (T_i - T_{i-1}) (L(T_{i-1}, T_i) - R)_+, \quad i = 1, \dots, n,$$

where $L(T_{i-1}, T_i)$ is the spot LIBOR rate on $[T_{i-1}, T_i]$:

$$L(T_{i-1}, T_i) = \frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)}; \quad (4.26)$$

Hence, one can see a cap as portfolio of n contingent claims $\mathcal{X}_1, \dots, \mathcal{X}_n$ with respective maturities T_1, \dots, T_n . Each of these claims is called *caplet*.

A *cap* works as a financial insurance that protect a borrower from paying more than a prespecified interest rate, *the cap rate* R . This is how this insurance procedure works. Suppose that an agent borrows the amount M at time $T_0 = 0$ and will pay a floating interest rate at times T_1, \dots, T_n , determined by the prevailing LIBOR rates in each period. Thus, the borrower will pay

$$M(T_i - T_{i-1})L(T_{i-1}, T_i),$$

at each time T_i , $i = 1, \dots, n$. It is natural for the agent to try to hedge away his risk against the floating interest rate and he wishes to lock his interest payment to be at most a certain cap value R . In order to do this, the agent invest in a cap contract with tenor T_0, \dots, T_n , notional M , and cap rate R . Indeed, if at any given time T_i the LIBOR forward rate $L(T_{i-1}, T_i)$ exceeds the cap rate R , the cap contract will cover the difference between his desired interest payment $R(T_i - T_{i-1})K$ and the interest payment $M(T_i - T_{i-1})L(T_{i-1}, T_i)$ he has to made.

A *floor* is defined similarly but the payoff at time T_i (called *floorlet*) is given by

$$\mathcal{X}_i = M (T_i - T_{i-1}) (R - L(T_{i-1}, T_i))_+.$$

The idea of a floor instrument is to guarantee that the interest rate received by a lender will never go below some predetermined floor rate R .

Exercise 14. Consider a cap with tenor $T_0 < \dots < T_n$ with cap rate R and also a swap contract with the same tenor that exchange a fixed rate R_{swap} by a LIBOR rate. What will be the time $t = 0$ price of a cap with cap rate $R = R_{swap}$? This is called *at-the-money* cap.

4.5.2 Arbitrage-free pricing in terms of bond options

It turns out that there is close relation between the price of a cap and put options on bonds. To see this connection, consider the payoff of the caplet with maturity T_i :

$$\mathcal{X}_i = \delta (L(T_{i-1}, T_i) - R)_+,$$

where, for simplicity, we fixed the notional M to 1 and assumed a fixed span δ between reset times ($\delta = T_i - T_{i-1}$, for each i). Now, plugging (4.26) in, we can write this payoff as

$$\mathcal{X}_i = \frac{R^*}{p(T_{i-1}, T_i)} \left(\frac{1}{R^*} - p(T_{i-1}, T_i) \right)_+$$

where $R^* = 1 + \delta R$. Now, it is important to realized that the payment of the above caplet is at time T_i . However, this will be “equivalent” to a payoff of

$$\tilde{\mathcal{X}}_i = R^* \left(\frac{1}{R^*} - p(T_{i-1}, T_i) \right)_+$$

at time T_{i-1} . More precisely, the initial price of the claim \mathcal{X}_i with maturity T_i is the same as that of the claim $\tilde{\mathcal{X}}_i$ with maturity T_{i-1} . Indeed, the price $\Pi(0; \mathcal{X}_i)$ of the caplet with maturity T_i is such that

$$\begin{aligned} \Pi(0; \mathcal{X}_i) &= R^* E^Q \left\{ e^{-\int_0^{T_i} r(u) du} \frac{\tilde{\mathcal{X}}_i}{p(T_{i-1}, T_i)} \right\} \\ &= R^* E^Q \left\{ E^Q \left\{ e^{-\int_0^{T_i} r(u) du} \frac{\tilde{\mathcal{X}}_i}{p(T_{i-1}, T_i)} \middle| \mathcal{F}_{T_{i-1}} \right\} \right\} \\ &= R^* E^Q \left\{ e^{-\int_0^{T_{i-1}} r(u) du} \mathcal{X}_i \right\}, \end{aligned}$$

since $\tilde{\mathcal{X}}_i \in \mathcal{F}_{T_{i-1}}$ and $p(T_{i-1}, T_i) = E^Q \left\{ \exp\left\{-\int_{T_{i-1}}^{T_i} r(u) du\right\} \middle| \mathcal{F}_{T_{i-1}} \right\}$. However, $\tilde{\mathcal{X}}_i$ is precisely the payoff of R^* units of a put option on the T_i -bond with maturity T_{i-1} and strike $1/R^*$. Hence, we show the following result:

Proposition 1. Let $P(0; T, T', K)$ (resp. $C(0; T, T', K)$) be the time-0 price of a put option (resp. call option) on a T' -bond with maturity $T < T'$ and strike K . Then, the price of a cap contract with tenor $T_0 = 0 < \dots < T_n$, cap rate R , and notional M is given by either of the following expressions:

$$MR^* \sum_{i=1}^n P(0; T_{i-1}, T_i, \frac{1}{R^*}) = MR^* \sum_{i=1}^n \left(C(0; T_{i-1}, T_i, \frac{1}{R^*}) + \frac{1}{R^*} p(0, T_{i-1}) - p(0, T_i) \right).$$

where $R^* = 1 + \delta R$.

Note that above we use the following put-call parity relationship:

$$P(0; T, T', K) = C(0; T, T', K) + Kp(0, T) - p(0, T').$$

Example 10 (Clewlow and Strickland, Section 7.3.4). Consider the Hull-White model $dr = (\theta - ar)dt + \sigma dW$ with $\sigma = .0116$ and $a = .10$. Using the formula in Section 4.4.1, verify that the price of a caplet with reset time $T_0 = .16$ and time of payment $T_1 = .41$, principal of 1, and cap rate of 7% is .000227 if $p(0, T_0) = .9898$ and $p(0, T_1) = .9733$.

Being a very liquid product, caps are typically used for calibration purposes. For instance, in the Hull-White model, the parameters a and σ are typically calibrated to approximately match market prices of caplets. Concretely, supposed that $T_0 < \dots < T_n$ and R are the tenor and cap rate of a cap contract and let Π_i^* denotes the price of the corresponding caplet with maturity T_i and let $\Pi_i(a, \sigma)$ denotes its theoretical arbitrage-free price under the Hull-White model. Then, under this calibration approach, a and σ are chosen as the values where the following minimization problem is reached:

$$\min_{a, \sigma} \sum_{i=1}^n \left(\frac{\Pi_i^* - \Pi_i(a, \sigma)}{\Pi_i^*} \right)^2.$$

Once the parameters a and σ have been fixed, one can subsequently calibrate $\theta(t)$ to recover the initial term structure as it was described at the end of Section 2.5. See [Clewlow and Strickland, Section 7.3.4] for an example of this calibration procedure.

4.5.3 Market practices for caps

The market typically quotes the prices of caps and floors in terms of Black volatilities, which themselves are obtained from the Black formula for cap prices. In this part, we introduce this pricing formula. Let us recall that the caplet with maturity T_i and cap rate R is a derivative of the bond market with payoff

$$\mathcal{X}_i = (L(T_{i-1}, T_i) - R)_+. \quad (4.27)$$

Note that the payoff can be written as

$$\mathcal{X}_i = (L(T_{i-1}; T_{i-1}, T_i) - R)_+,$$

where

$$L(t; T_{i-1}, T_i) = \frac{p(t, T_{i-1}) - p(t, T_i)}{(T_i - T_{i-1})p(t, T_i)}, \quad t \leq T_{i-1},$$

is the LIBOR forward rate for $[T_{i-1}, T_i]$ contracted at time t . The Black-76 formula for the caplet (4.27) is given by

$$Capl_i^B(t; \sigma_i) = \delta_i p(t, T_i) \{L(t; T_{i-1}, T_i)N(d_1) - RN(d_2)\}, \quad (4.28)$$

$$d_1 = \frac{1}{\sigma_i \sqrt{T_i - t}} \left\{ \ln \left(\frac{L(t; T_{i-1}, T_i)}{R} \right) + \frac{1}{2} \sigma_i^2 (T_i - t) \right\}, \quad d_2 = d_1 - \sigma_i \sqrt{T_i - t}, \quad (4.29)$$

where σ_i is certain parameter called the *Black volatility* of the caplet. Note that it is not clear whether the resulting Black cap prices are consistent with an arbitrage-free bond market model. The answer to this question is the main motivation behind the so-called *LIBOR market model* introduced by Brace, Gatarek, Musiela. We postpone the analysis of this issue to the following section.

Note that the volatility σ_i determines the price of the caplet with maturity T_i . We call such a volatility the *spot or forward Black volatility*. Given spot Black volatilities $\sigma_1, \dots, \sigma_n$, the Black price of the cap with tenor T_0, \dots, T_i is given by

$$Cap_i(t) = \sum_{k=1}^i Capl_k^B(t; \sigma_k), \quad i = 1, \dots, N.$$

There is another procedure to compute the price of the caps. Instead, we can associate one volatility to each cap. Concretely, given volatilities $\sigma_1, \dots, \sigma_n$, the price of the cap with tenor T_0, \dots, T_i is given by

$$Cap_i(t) = \sum_{k=1}^i Capl_k^B(t; \sigma_i), \quad i = 1, \dots, N.$$

In that case, $\sigma_1, \dots, \sigma_n$ are called *flat Black volatilities*.

Example 11. Suppose that, at the present time $T_0 = 21/Jan/95$, we have two US Dollar cap volatility quotes of $\sigma_1 = 15.25\%$ and $\sigma_2 = 17.25\%$ corresponding to the maturities $T_1 = 21/Mar/95$ and $T_2 = 21/Jun/95$. The cap rate is 7% and principal is $K = 1$. If σ_1 and σ_2 are spot volatilities, the price of the cap with maturity T_2 is given by

$$Capl_1^B(0; \sigma_1) + Capl_2^B(t; \sigma_2).$$

However, if the volatilities are flat, the price of the cap will be

$$Capl_1^B(0; \sigma_2) + Capl_2^B(t; \sigma_2).$$

As with the case of vanilla options which are priced in terms of Black-Scholes implied volatilities, cap prices are typically priced in terms of Black volatilities. Also, in the same way as the Black-Scholes implied volatilities exhibit a persistent smile-like shape, the Black volatilities of cap prices exhibit a hump-like shape, which in general is hard to incorporate by one-factor interest rate model.