Homework Assignment 5.

1. (a) In class we saw several ways to approximate the derivative of \( f'(x_0) \) of a smooth function by finite-differences. There are other type of approximations. Show that

\[
\Delta_h^{(3)} f(x_0) := \frac{1}{6h} [2f(x_0 + h) + 3f(x_0) - 6f(x_0 - h) + f(x_0 - 2h)],
\]  

(1)

can approximate \( f'(x_0) \) up to an error \( O(h^2) \).

(b) Consider four values \( x_1 < x_2 < x_3 < x_4 \). Find an approximation of \( f''(x_2) \) based on the four values \( y_1 = f(x_1) \), \( y_2 = f(x_2) \), \( y_3 = f(x_3) \), \( y_4 = f(x_4) \). What is the order of the error approximation in terms of the \( h_1 = x_2 - x_1 \), \( h_2 = x_3 - x_2 \), and \( h_3 = x_4 - x_3 \).

2. We saw in class that the explicit method is stable when applied to the heat equation

\[
\begin{align*}
\partial_\tau u(\tau, x) - \partial_{xx} u(\tau, x) &= 0, \quad \tau > 0; \\
u(0, x) &= \Phi(x).
\end{align*}
\]  

(2)

if \( \alpha := \frac{\delta \tau}{(\delta x)^2} \leq 1/2 \). Using the same type of arguments, find an analog stability condition for the explicit method when applied to the more general system

\[
\begin{align*}
\partial_t u(t, x) + \mu(t, x) \partial_x u(t, x) + a(t, x) \partial_{xx} u(t, x) - r u(t, x) &= 0, \quad t < T, x > 0; \\
u(T, x) &= \Phi(x).
\end{align*}
\]  

(3)

Assuming that \( \mu, a, \) and \( r \) are constant with \( a \) being positive. What can it be said if the coefficients therein are functions satisfying certain boundedness conditions?

3. Consider the Black-Scholes PDE for the value of an option \( V(t, s) \) with terminal condition \( V(T, s) = (K - s)_+ \). Write a program to find a numerical approximation for \( V \) using the explicit method and the following boundary conditions:

\[
\lim_{s \to 0} \frac{\partial^2 V}{\partial s^2} = 0, \quad \lim_{s \to \infty} \frac{\partial^2 V}{\partial s^2} = 0.
\]

Compare the performance of your program with the values obtained from the Black-Scholes formula using the following pricing parameters: \( S_0 = 100, K \in \{90, 100, 110\} \), \( r = 2\% \), \( T = 1 \) year, and \( \sigma \in \{15\%, 30\%, 50\%\} \).

4. Repeat the previous exercise using this time the fully implicit method.