**Math516/Stat541** Fall 2009
**Homework assignment 4**

1. Suppose that the price processes of $n$ assets are given by the dynamics

$$d\,S_i(t) = S_i(t)\left\{b_i(t)dt + \sigma_i(t)dW(t)\right\}, \quad i = 0, \dots, n, \tag{1}$$

where $W$ is $d$-dimensional Wiener process.

(a) Use Itô's formula to compute the differentials $d\bar{S}_i$ and conclude the validity of the *invariance lemma.*

(b) Suppose that the market is arbitrage-free and let $Q_0$ be the martingale measure associated with the numeraire $S_0$. Find the dynamics of the standardized assets $\bar{S}_i(t) := S_i(t)/S_0(t)$, for $i = 1, \dots, n$, under $Q_0$ (your dynamics must be in terms of the Wiener process $W^{S_0}$ under $Q_0$).

(c) Let us assume for simplicity that $d = 1$ and let $W^Q$ (resp. $W^{S_0}$) be the martingale measure associated with the risk-free asset $B_t = e^{\int_0^t r(u)du}$ (resp. $S_0$). Show that

$$W^{S_0}(t) = W^Q(t) + \int_0^t \sigma_0(u)du.$$

*Hint: Determine $d\bar{S}_i$ under $Q$ and compare this with the dynamics that $d\bar{S}_i$ should have under $Q^{S_0}$.*

2. Let $[B, S_0, \dots, S_n]$ be an arbitrage-free market, where $B_t = e^{\int_0^t r(u)du}$ is the (locally) risk-free asset and $S_0$ is a strictly positive asset. Let $Q_0$ be the martingale measure associated with the numeraire $S_0$ and let $Q$ be the standard martingale measure (associated with $B$). Explain why the following formula holds for any contingent claim $\mathcal{Y} \in \mathcal{F}_T$:

$$E^Q\left\{ e^{-\int_t^T r(u)du}\mathcal{Y} \,\middle|\, \mathcal{F}_t \right\} = S_0(t)E^{Q_0}\left\{ \frac{\mathcal{Y}}{S_0(T)} \,\middle|\, \mathcal{F}_t \right\}.$$

In particular, show that $Q_0$ is given by the below formula in terms of $Q$:

$$Q_0(A) = \frac{1}{S_0(0)} E^Q\left\{ e^{-\int_0^T r(u)du} S_0(T)\mathbf{1}_A \right\}.$$

3. Consider the Hull-White model for bond prices, where

$$d\,r(t) = (\Theta(t) - a\,r) + \sigma d\,W(t),$$

under the risk-neutral measure $Q$. Show that, under $Q$, the normalized bond price,

$$\bar{S}_t := \frac{p(t, T_2)}{p(t, T_1)}, \quad (t \le T_1 \le T_2)$$

has the representation

$$\bar{S}_t = \bar{S}_t\left\{(\text{stuff})\,dt + \bar{\sigma}(t)\,d\,W(t)\right\},$$

with a deterministic, time dependent volatility $\bar{\sigma}(t)$.

*Hint: Recall and use that in this model, the bond price accept an Affine Term Structure (AFT):*

$$p(t,T) = e^{A(t,T)-r(t)B(t,T)},$$

*where* $B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$. *Use Itô's formula to find* $d\bar{S}_t$.

4. Exercise 24.2.

5. Exercise 24.3.

6. (a) Show that, under the $T$-forward measure $Q_T$ (namely, the martingale measure associated with the numeraire $p(t,T)$), the instantaneous forward rate today, $f(0,T)$, is an "unbiased estimator" for the short $r(T)$ at the future time $T$. That is,

$$f(0,T) = E^{Q_T}\left\{r(T)\right\}.$$

   (b) Is it true that the forward rate for $[S,T]$ contracted now, $L(0;S,T)$, is an unbiased estimator for the (simple) LIBOR spot rate for $[S,T]$, $L(S,T)$? That is,

$$L(0;S,T) = E^{Q_T}\left\{L(S,T)\right\}?$$

   (see Definition 20.2 for the notation).
   *Hint: Consider the price of the* $S-$*bond in the currency* $p(t,T)$.