Math516/Stat541 Fall 2009

Solutions homework assignment 3

1. *Solution to 21.1 (a)*: Assume that the value of the claim at time t, denoted by Π_t, is a smooth function of t and $r(t)$, say $F(t, r(t))$. By Itô Formula and the dynamics of r (namely, $dr = \mu dt + \sigma d\bar{W}(t)$),

$$\begin{aligned} d\Pi(t) &= F_t dt + F_r dr + \frac{1}{2}\sigma^2 F_{rr} dt \\ &= F\left\{\left(\frac{F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr}}{F}\right) dt + \sigma \frac{F_r}{F} d\bar{W}(t)\right\} \\ &= F\left\{\left(\frac{F_t + (\mu - \lambda\sigma) F_r + \frac{1}{2}\sigma^2 F_{rr}}{F}\right) dt + \sigma \frac{F_r}{F} dW(t)\right\} \\ &= F\left\{r(t)dt + \sigma \frac{F_r}{F} dW(t)\right\}, \end{aligned}$$

where $W(t) = \bar{W}_t + \int_0^t \lambda_s ds$. In the last step we used the term structure equation. ■

2. Let σ_S and σ_T be the volatilities of the S- and T-bonds , respectively. To replicate the value of the S-bond, we need to invest a proportion $\frac{\sigma_S}{\sigma_T}$ of wealth in the $T-$bond and the remainder in the money market account. Then, the number of T-bond shares held at time t are

$$\frac{\frac{\partial F^S}{\partial r}}{\frac{\partial F^T}{\partial r}},$$

which we should contrast with the delta hedging strategy of the standard Black-Scholes model. ■

3. Exercise 21.3. (Interest rate swap: fixed$\longleftrightarrow$ short)

Solution prepared by Professor Viens.

This is an interesting exercise, because you have to take the point of view of B to find the swap rate. From the point of view of A, the price of the contract is always = 0, which I now show, for the sake of curiosity!

⊛ Cash flow for A:

t=0	t=T
$-K$	$Ke^{\int_0^T r(s)ds}$

where the total cash flow at T consists of $K_A = Ke^{RT}$, surplus to B: $-(Ke^{RT} - K)$ and surplus from B: $K(e^{\int_0^T r(s)ds} - 1)$.

Price of cash flow contract:

$$-K \cdot 1 + K \cdot \underbrace{\texttt{price of } e^{\int_0^T r(s)ds}}_{\Pi^*(0)}$$

$*$ The easy way to calculate $\Pi^*(0)$ is via replication: If you start with \$ 1 and you invest it all in the bank, you end up with $e^{\int_0^T r(s)ds}$, so $\Pi^*(0) = 1$.

$*$ The hard way: use the pricing formula, which we extend to non-simple claims: $\Pi^*(0) = E^Q_{0,r(0)}\left[e^{-\int_0^T r(s)ds}e^{\int_0^T r(s)ds}\right] = 1$.

Price of contract for $A: -K + K = 0$.

⊛ Cash flow for B:

t=0	t=T
$-K$	$K + (Ke^{RT} - K) = Ke^{RT}$

Price of contract for this cash flow:

$$-K \cdot 1 + Ke^{RT}p(0,T) = K\left(e^{RT}E^Q_{0,r(0)}\left[e^{-\int_0^T r(s)ds}\right] - 1\right)$$

⊛ So far a total price of 0, we get:

$$R = -\ln E^Q_{0,r(0)}\left[e^{-\int_0^T r(s)ds}\right]/T$$

■

6. Exercise 23.1.

Solution: *Steps of the HJM algorithm:*

(a) *First calculate α by HJM drift condition (see (23.9)):*

$$\begin{aligned}\alpha(t,T) &= \sigma e^{-a(T-t)}\int_t^T \sigma e^{-a(s-t)}ds = e^{-aT}e^{2at}\sigma^2\frac{1}{a}(e^{-at} - e^{-aT}) \\ &= \frac{\sigma^2}{a}(e^{-a(T-t)} - e^{-2a(T-t)}).\end{aligned}$$

(b) *"Integrate" f using the "initial condition" $f(0,T) = f^*(0,T)$:*

$$df(t,T) = \frac{\sigma^2}{a}(e^{-a(T-t)} - e^{-2a(T-t)})dt + \sigma e^{-a(T-t)}dW(t)$$

$$f(t,T) = f^*(0,T) + \frac{\sigma^2}{a^2}\left(e^{-a(T-t)} - e^{-aT} - \frac{1}{2}(e^{-2a(T-t)} - e^{-2aT})\right) + \sigma e^{-aT}\int_0^t e^{as}dW_s$$

(c) *Get $r(t) = f(t,t)$:*

$$r(t) = f^*(0,t) + \frac{\sigma^2}{a^2}\left(1 - e^{-at} - \frac{1}{2}(1 - e^{-2at})\right) + \sigma e^{-at}\int_0^t e^{as}dW_s \qquad (1)$$

(d) *Finally,* ***find the dynamics of r*** *(namely we want to find dr). It is more convenient to consider $R(t) = e^{at}r(t)$. Then,*

$$dR(t) = e^{at}[dr(t) + ar(t)dt] \quad \Longrightarrow \quad dr(t) = e^{-at}dR(t) - ar(t)dt.$$

On the other hand,

$$\begin{aligned} dR(t) &= e^{at}\Big[\Big(\frac{\partial f^*}{\partial T}(0,t) + \frac{\sigma^2}{a}(e^{-at} - e^{-2at})\Big)dt \\ &\quad + \Big(af^*(0,t) + \frac{\sigma^2}{a}(\frac{1}{2} - e^{-at} + \frac{1}{2}e^{-2at})\Big)dt + \sigma e^{at}dW_t\Big] \\ &= e^{at}\Big[\frac{\partial f^*}{\partial T}(0,t) + af^*(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})\Big]dt + \sigma e^{at}dW(t) \end{aligned}$$

Plugging in the equation for $dr(t)$, we get

$$dr(t) = \Big[\frac{\partial f^*}{\partial T}(0,t) + af^*(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) - ar(t)\Big]dt + \sigma e^{at}dW(t).$$

■

8. Exercise 23.2. Take as given an HJM model (under the risk neutral measure Q) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW, \tag{2}$$

where the volatility $\sigma(t,T)$ is deterministic.

(a) Show that all forward rates, as well as the short rate, are normally distributed.

Solution:
Let $df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$. Since σ is deterministic, by HJM drift condition, $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds$ is also deterministic. Therefore

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s)$$

is a Gaussian random variable with mean $f^*(0,T) + \int_0^t \alpha(s,T)ds$ and variance $\int_0^t \sigma(s,T)^2 ds$. Since $r(t) = f(t,t), r(t)$ is also Gaussian. ■

(b) Show that the log prices are log-normally

Solution:
The bond prices are: $p(t,T) = \exp\{-\int_t^T f(t,s)ds\}$. However, we can also show that $-\int_t^T f(t,s)ds$ is Gaussian. Indeed, using Funibi's theorem,

$$\begin{aligned} -\int_t^T f(t,s)ds &= -\int_t^T f^*(0,s)ds - \int_t^T \int_0^t \alpha(u,s)du\,ds - \int_t^T \int_0^t \sigma(u,s)dW_u\,ds \\ &= -\int_t^T f^*(0,s)ds - \int_t^T \int_0^t \alpha(u,s)du\,ds - \int_0^t \int_t^T \sigma(u,s)ds\,dW_u. \end{aligned}$$

Since $\int_t^T \sigma(u,s)ds$ is deterministic, the exponent of $p(t,T)$ is Gaussian with mean

$$-\int_t^T f^*(0,s)ds - \int_t^T \int_0^t \alpha(u,s)du\, ds.$$

and variance

$$\int_0^t \left\{ \int_t^T \sigma(u,s)ds \right\}^2 du.$$

■

9. Exercise 23.3.

Note: For this problem, we need a few facts from Chapter 17 (currency derivatives).

Solution:

Consider the foreign bond price in domestic currency. The price process of this instrument will be $p_f^d(t,T) = X(t)p_f(t,T)$. We think of this as an investment in the domestic market which price process

$$dp_f^d(t,T) = \alpha_f^d(t,T)dt + \sigma_f^d(t,T)dW,$$

must satisfy the absence of arbitrage condition:

$$\alpha_f^d(t,T) - r_d(t) = \sigma_f^d(t,T)\lambda,$$

where λ is the market price of risk of the bond market and r_d is domestic riskfree rate of return. Also, note that because we are working with the Q-dynamics, the market price of risk λ, under Q, is identically 0 (Why?). Then,

$$\alpha_f^d(t,T) - r_d(t) = 0.$$

We now proceed to obtain the rate of return α_f^d.

By chapter 17 (see equation (17.6)), we know that under the domestic martingale measure $dX/X = (r_d - r_f)dt + \sigma_X dW$. Also, in light of Proposition 20.5[1], under the domestic martingale measure,

$$dp_f(t,T) = p_f(t,T)\Big[(r_f + A_f + \frac{1}{2}||S_f||^2)dt + S_f dW(t)\Big]$$

where $A_f(t,T) = \int_t^T \alpha_f(t,s)ds$ and $S_f(t,T) = -\int_t^T \sigma_f(t,s)ds$.

Therefore, by Itô Formula,

$$\begin{aligned} p_f^d(dt,T) &= dXp_f + dp_f X + d < X, p_f > \\ &= p_f^d\Big[(r_d - r_f)dt + \sigma_X dW + (r_f + A_f + \frac{1}{2}||S_f||^2)dt + S_f dW + S_f \sigma_X dt\Big] \end{aligned}$$

We conclude that

$$\alpha_f^d(t) = r_d - r_f + r_f + A_f + \frac{1}{2}||S_f||^2 + S_f \sigma_X.$$

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[1] The relations among $r(t), f(t,T)$ and $p(t,T)$

Thus, under absence of arbitrage, which yields

$$r_d = r_d - r_f + r_f + A_f + \frac{1}{2}||S_f||^2 + S_f \sigma_X$$

Taking $\frac{\partial}{\partial T}$ of this equation yields the result. ■

10. Exercise 23.4. First, we note that

$$df_f(t,T) = dg(t,T) + df_d(t,T) = \{\alpha_g(t,T) + \alpha_d(t,T)\} + \{\sigma_g(t,T) + \sigma_d(t,T)\}\, dW(t).$$

Then, we use the conclusion of Exercise 23.3 that will give $\alpha_f := \alpha_g(t,T) + \alpha_d(t,T)$ in terms of $\sigma_f = \sigma_g(t,T) + \sigma_d(t,T)$ and σ_X. We use the HJM drift condition on α_d and solve for α_g.