Math516/Stat541 Fall 2008
Homework Assignment 3

1. 21.1. *Note: In this problem, you can assume that either* $\Pi(t) = F(t, r(t))$ *for a smooth function* F *or that* $\Pi(t)$ *is given the risk-neutral valuation formula.*

2. Given a "benchmark" bond (say a S-bond), verify that the value of any other bond (with maturity prior to the benchmark) can be replicated via a trading strategy on the S-bond and the money market account; in other words, if $T < S$, show that

$$dp(t,T) = p(t,T)\left\{\beta^B(t)\frac{dB(t)}{B(t)} + \beta^S(t)\frac{dp(t,S)}{p(t,S)}\right\},$$

for certain β^B and β^T (representing the proportions of total wealth invested in the money market account and the S-bond, respectively) that you need to specify explicitly.

3. Exercise 21.3.

4. Consider a one-factor arbitrage-free bond market where under the risk-neutral probability measure Q, the short rate follows the Vasicek model:

$$dr(t) = (b - ar(t))dt + \sigma dW_t^Q.$$

Prove that the risk-neutral bond prices,

$$p(t,T) = E^Q\left\{ e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t^W \right\},$$

has the so-called *Affine Term Structure*,

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)},$$

for deterministic functions A and B.

5. Prove the first part of Proposition 20.5 and fill out the details of the proof of the second part.

6. This is similar to Exercise 23.1 in Björk: Show that an arbitrage-free forward HJM model of the form:

$$df(t,T) = \alpha(t,T)dt + \sigma e^{a(T-t)}dW_t^Q,$$

where a and σ are constants is *equivalent* to an arbitrage-free Hull-White model of the form

$$dr = (\Theta(t) - ar)dt + \sigma dW_t^Q,$$

that is being fitted to the initial term structure.

7. Use the HJM framework, to show that the Ho-Lee model

$$dr(t) = \Theta(t)dt + \sigma dW_t,$$

can be fitted exactly to today's observed bond prices if one pick

$$\Theta(t) = \frac{\partial f^*(0,t)}{\partial T} + \sigma^2 t,$$

where $f^*(t,T)$ is the observed forward rates.

8. Exercise 23.2.
9. Exercise 23.3.

   Note: A few facts from Chapter 17 (currency derivatives) are needed for this problem. Concretely, it can be argued that if X is the exchange rate from foreign to domestic currency, then under the domestic martingale measure $dX_t = (r_d - r_f)dt + \sigma_X dW_t$, where r_d and r_f are the riskfree rate of returns in the domestic and foreign market. See equation (17.6) in Björk for more details.
10. Exercise 23.4.
11. Exercise 24.2.