Math516/Stat541 Fall 2009
Solutions to selected problems of HW 1

Problem 1. Exercise 20.1

(a) Find price of the contract yielding the cash flow:

at time S: $-K$

at time T: $Ke^{R^*(T-S)}$

Price at time $t = \Pi(t) = -Kp(t,S) + Ke^{R^*(T-S)}p(t,T)$ by definiton.

(b) To get $\Pi(0) = 0$, we must have

$$0 = -Kp(0,S) + Ke^{R^*(T-S)}p(0,T)$$

or

$$R^* = -\frac{\log p(0,T) - \log p(0,S)}{T-S}$$

So we see that R^* is the rate, as seen from time 0, which yields a bank account yielding a riskless rate of continuously compounded return over the future interval $[S,T]$. Thus the interpretation of R^* as a forward rate is justified, since nothing has to be paid today (t=0) for the contract. ■

Problem 2. Exercise 20.3

Times	T_0	T_1	$\cdots$	T_i	$\cdots$	T_n
Payment Stream		$Kr(T_1 - T_0)$	$\cdots$	$Kr(T_i - T_{i-1})$	$\cdots$	$Kr(T_n - T_{n-1}) + K$

Price of contract at time $t (t \le T_0)$:

$$p(t) = \sum_{i=1}^{n} Krp(t,T_i)(T_i - T_{i-1}) + Kp(t,T_n)$$

Want $p(T_0) = K \Longrightarrow$

$$K = K[\sum_{i=1}^{n} rp(T_0,T_i)(T_i - T_{i-1}) + p(T_0,T_n)]$$

Thus,

$$r = \frac{1 - p(T_0,T_n)}{\sum_{i=1}^{n} p(T_0,T_i)(T_i - T_{i-1})}$$

■

Problem 5.

(a) vs. (b)

Let $p^c(0,5)$ be the price of a bond with maturity 5 years, par value \$1, and coupon rate c. The yield y^c of the coupon bond is, by definition, such that

$$p^c(0,5) = \frac{c}{2}\, e^{-.5y^c} + \cdots + \frac{c}{2}\, e^{-4.5y^c} + \frac{c}{2}\, e^{-5y^c} + e^{-5y^c}.$$

Let $y(T)$ denote the yield (with continuous compounding) of the T-zero bond; that is, $y(T)$ satisfies

$$p(0,T) = e^{-y(T)T}.$$

Then, under absence of arbitrage,

$$\begin{aligned} p^c(0,5) &= \frac{c}{2}\, p(0,.5) + \cdots + \frac{c}{2}\, p(0,4.5) + (c/2+1)\, p(0,5) \\ &= \frac{c}{2}\, e^{-.5y(.5)} + \cdots + \frac{c}{2}\, e^{-4.5y(4.5)}\, (c/2+1)\; e^{-5y(5)}. \end{aligned}$$

By the upward-slopping assumption, $y(.5) \le \cdots \le y(5)$; therefore, $y(5) > y^c$. *Conclusion: The five-year zero rate is greater than the yield on a five-year coupon bearing bond* (under upward term structure).

(a) vs. (c)

Let us take R as the forward rate (with continuous compounding). Then,

$$\begin{aligned} R(0;5,5.25) &= -\frac{\log p(0,5.25) - \log p(0,5)}{.25} \\ &= (y(5.25) - y(5))\, \frac{5.25}{.25} + y(5). \end{aligned}$$

Since $y(5.25) > y(5)$, $R(0;5,5.25) > y(5)$. *Conclusion: The forward rate on* $[5,T]$ *contracted today (time* 0*) is greater than the five-year zero rate* (under upward term structure). ■