Math515/Stat540 Spring 2009
Computational Project II Figueroa-Lopez

Description

- This project has the following objectives:
  1. Emphasizing the importance of Feyman-Kac representations
  2. PDE formulations of option prices based on the absence of arbitrage at the Black-Scholes style
  3. Euler method for the simulation of SDEs
  4. Basic Monte Carlo methods for option pricing
- The subparts marked with $\star$ must be done individually and turn in their solutions separately (these are a total of 7 subparts).
- The rest of the parts can and should be done in groups.
- Write a team report describing briefly the methods, the computational experiments, the results, and conclusions (include your codes *properly documented*).

Problems:

I.* **Feynman-Kac representation:** Björk 5.10 and 5.12.

II. **Euler method.**
Consider an SDE in differential form as follows:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x, \tag{1}$$

where $\mu(t, x)$ and $\sigma(t, x)$ are deterministic functions. The *Euler method* is an iterative method to simulate an (approximate) trajectory of the process X during a time horizon $[0, T]$. Given a desired number of steps n and a horizon T, the following are the steps:

(1) **Initialization:** $dt = T/n$, $t_i = idt$, and $X_{t_0} = x$;

(2) **Random numbers generation:** Generate independent identically Normally distributed random variables $Z_0, \ldots, Z_n$.

(3) **Iterative step:**

for $i = 0..n$

$$X_{t_{i+1}} = X_{t_i} + \mu(t_i, X_{t_i})dt + \sigma(t_i, X_{t_i}) \cdot \sqrt{dt} \cdot Z_i \tag{2}$$

end for

The intuitive idea is to see dt in (1) as a given time step and dX_t and dW_t as the change of X and W during the time period $[t, t+dt]$, respectively. Thus,

$$X_{t+dt} = X_t + \mu(t, X_t)dt + \sigma(t, X_t)(W_{t+dt} - W_t).$$

Then, (2) follows from using the fact that $W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ will be distributed as $\sqrt{dt} \cdot Z_1, \ldots, \sqrt{dt} \cdot Z_n$.

Do the following parts:

(a) ⋆ Justify the Euler's method in terms of Riemann-Stieltjes approximations to the integral form of the SDE (1).

(b) ⋆ Give a pseudo code similar to (1)-(3) above to simulate two SDEs of the form:

$$X_t = x + \int_0^t \mu_1(s, X_s, Y_s)ds + \int_0^t \sigma_1(s, X_s, Y_s)dB_1(s), \tag{3}$$

$$Y_t = y + \int_0^t \mu_2(s, X_s, Y_s)ds + \int_0^t \sigma_2(s, X_s, Y_s)dB_2(s), \tag{4}$$

using Euler method. Here, B_1 and B_2 are correlated Wiener processes such that $dB_1 \cdot dB_2 = \rho(t)dt$ and $\rho(t), \mu_1(t, x, y), \mu_2(t, x, y), \sigma_1(t, x, y), \sigma_2(t, x, y)$ are deterministic functions.

(c) (i) Implement the Euler method in the case that X and Y are stock prices following Black-Scholes models with constant correlation ρ, respective constant mean rate of returns μ_1 and μ_2, and respective constant volatilities σ_1 and σ_2. (ii) Use your program to verify the theoretical effects (seing in class) that the parameters μ_i, σ_i, and ρ have in the price dynamics of the stock prices (you need to illustrate your claims by showing simulated trajectories of the stock prices for different parameters setting).

III. **Monte-Carlo evaluation of claims**.

Suppose that the price of the claim has the following risk-neutral valuation formula:

$$\Pi_0 = e^{-rT} E^Q \left[\Phi\left(S_T\right)\right], \tag{5}$$

where Φ is the payoff of the option, r is the short-rate of interest (assumed to be constant), $\{S_t\}_{t \geq 0}$ is the price process of the underlying, and Q is the risk-neutral probability measure. The Monte Carlo method is a very popular and easy to use method. It consists of simulating independent copies of S_T and use a sample mean to approximate the expectation in (5). Concretely, if $s_T^1, \ldots, s_T^m$ are the values of m independent copies of S_T, then

$$\Pi_0 \approx e^{-rT} \cdot \frac{1}{m} \sum_{j=1}^{m} \Phi(s_T^j).$$

Implement the Monte-Carlo method to value a European call option in the Black-Scholes model. Verify your algorithm by comparing it to that of a Black-Scholes calculator, widely available on the web.

IV. In the *Heston stochastic volatility model*, the stock price $\{S_t\}_{t\geq 0}$ follows the dynamics:

$$dS_t = S_t\mu dt + S_t\sqrt{V_t}dB_1(t), \tag{6}$$

where *volatility* $\{V_t\}_{t\geq 0}$ is determined by the dynamics

$$dV_t = \delta(m - V(t))dt + \nu\sqrt{2\delta}\sqrt{V(t)}dB_2(t). \tag{7}$$

Here, μ, δ, m, ν are positive constants such that $m/\nu^2 > 1$[1], and B_1 and B_2 are correlated Wiener processes such that $dB_1 \cdot dB_2 = \rho dt$.

(a) Implement the Euler method for simulating a path of $\{S_t\}_{t\leq T}$ under the Heston model (6)-(7).

(b) Use simulations to illustrate why the parameter δ is called the mean-reversion parameter of the volatility. What do you think is the interpretations of ν and m? Illustrate your claims using simulations.

(c) Let $\{\Pi_t\}_{t\leq T}$ be the process

$$\Pi_t = e^{-r(T-t)}E^Q\left[\Phi(S_T)\middle|\,\mathcal{F}_t^B\right], \tag{8}$$

where Φ is a given payoff function, and under Q, $\{S_t\}_{t\geq 0}$ has dynamics as in (6)-(7) with μ replaced by r, the short rate of interest. In that case, if a claim is priced according to the price process $\{\Pi_t\}_{t\leq T}$, the market consisting of the claim, the stock, and money market account is arbitrage-free. Write a program to evaluate the price of a European call option at time $t = 0$ using the Monte Carlo method combined with the Euler's method.

(d) * Because of Markov's property it follows that $\{\Pi_t\}_{t\leq T}$ in (8) is of the form

$$\Pi_t = F(t, S_t, V_t),$$

for a function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$. Assuming that $F(t, s, v)$ is a smooth function, show that $e^{-rt}F(t, S_t, V_t)$ is a martingale under Q and use this fact to obtain the following PDE:

$$\frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \delta(m - v)\frac{\partial F}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 F}{\partial s^2} + \rho\sigma sv\frac{\partial^2 F}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F}{\partial v^2} - rF = 0,$$
$$F(T, s, v) = \Phi(s),$$

where $\sigma := \nu\sqrt{2\delta}$.

VI. **Asian type options**

Consider the extended Black-Scholes model

$$d\,S(t) = S(t)\alpha(t, S(t))dt + S(t)\sigma(t, S(t))dW(t). \tag{9}$$

[1]This condition will guarantee that (7) admits a positive solution.

Consider European contingent claim with payoff $\mathcal{X} = \Phi(S(T), Z(t))$ at maturity T where

$$Z(t) = \int_0^t g(u, S(u))du.$$

Through all this problem, assume that the price process of the above claim is of the form

$$\Pi(t) = F(t, S(t), Z(t)),$$

for a smooth function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$.

(a) * Using the same ideas as for the derivation of the Black-Scholes equation, show that for $\Pi(t)$ to be arbitrage-free, $F(t, s, z)$ must satisfy the PDE

$$\begin{cases} \frac{\partial F}{\partial t} + sr\frac{\partial F}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2} + g \cdot \frac{\partial F}{\partial z} - rF & = 0 \\ F(T, s, z) = \Phi(s, z). \end{cases} \tag{10}$$

(b) * Show that the self-financing strategy that takes the position

$$\Delta(t) = \frac{\partial F}{\partial s}(t, S(t), Z(t))$$

in the stock at time t, and that borrow (or lends) the necessary (or exceeding) money from (or to) the money market account is a *hedging strategy* for the claim with payoff $\mathcal{X} = \Phi(S(T), Z(t))$.

(c) * Show that F admits the risk-neutral representation

$$F(t, s, z) = e^{-r(T-t)}E^Q_{t,s,z}\left\{\Phi(S(T), Z(T)\right\},$$

where under Q, the processes $(S(t), Z(t))$ has dynamics

$$\begin{cases} d\,S(u) & = S(u)rdu + S(u)\sigma(u, S(u))dW^Q(u), \quad u \geq t \quad S(t) = s, \\ d\,Z(u) & = g(u, S(u))du, \qquad u \geq t \quad Z(t) = z. \end{cases}$$

(d) Write a program that gives the price of European claim with payoff

$$\mathcal{X} := \left(\frac{1}{T}\int_0^T S_u du - K\right)_+.$$

Hint: Use Riemann sums to approximate the integral $\int_0^T S_u du$.