MATHEMATICS OF FINANCE

CHAPTER 5

A BRIEF INTRODUCTION TO PORTFOLIO OPTIMIZATION PROBLEMS

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1. INTRODUCTION

Another classical problem of finance is optimal portfolio investments. In short, the problem here is to determine a self-financing trading strategy that maximizes the investor's utility given certain constrains such as an initial budget constraint or absence of short-selling.

There are roughly speaking two approaches: the dynamical programming (or HJB approach) and the martingale method. The first method leads to a characterization of the optimal solution in terms of a partial differential equation. It has the advantage of giving relatively explicit expressions of the optimal trading strategy; however, it can only be applied in Markov models. The martingale method does not require the later condition and provides with a close form for the optimal final wealth that can be attained. However, in order to find the optimal trading strategy, one would need to find the replicating strategy of the optimal final wealth, which itself can be a hard task in the non-markovian general case. We concentrate in the first approach, which can be applied in a more general *stochastic control framework*.

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2. THE STOCHASTIC CONTROL FRAMEWORK

Consider a stochastic process $\{X_t\}_{t\geq 0}$ which evolution can be influenced by a decision maker via some "controls". Concretely, the random measurement X_t, called the *state variable*, is assumed to follow the dynamics

$$dX_t = \mu(t, X_t, \hat{u}_t)dt + \sigma(t, X_t, \hat{u}_t)dW_t, \quad X_0 = x, \tag{1}$$

where $\mu(t, x, u)$ and $\sigma(t, x, u)$ are deterministic functions and $\{\hat{u}_t\}_{t\geq 0}$ is an $\mathbb{R}^m-$valued stochastic process which valued can be set by the decision maker. As usual, (1) should be understood in a matrix form in the sense that X can be an $\mathbb{R}^k$-valued process and W can be an $\mathbb{R}^d$-Wiener process, in which case $\mu : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ and $\mu : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^m \to M^{k\times d}$, the matrices of $k \times d$ real entries. Of course, $\hat{u}$ must be adapted to $\{W_t\}_{t\geq 0}$ since the decision maker must pick the value of the control $\hat{u}$ at time t based only upon the information available at that time. Strictly speaking, we have a family of processes $\{X_t^{\hat{u}}\}_{t\geq 0}$ indexed by each adapted *control process* $\{\hat{u}_t\}_{t\geq 0}$, which itself can be subjected to further restrictions.

Example 2.1. To fix some ideas, think of X as the wealth process of a self-financing trading strategy in a risky asset (stock) and a risk-free asset. The price process of the stock per share is denoted $\{S_t\}_{t\geq 0}$ and the risk-free asset has short rate of interest $\{r(t)\}_{t\geq 0}$. Clearly, the wealth process can be influenced by the trading strategy $\hat{u}$ of the investor. We set the control $\hat{u}$ to be the *percentage of wealth* invested in the risky asset. This wealth percentage, denoted hereafter as $\{\beta_t\}_{t\geq 0}$, together with the initial wealth $X_0 = x$ determine uniquely the wealth process (see Proposition 1.1). Concretely, we can specify the dynamics of X as follows. By definition, the amount of money invested in the risky and risk-free assets at time t are $\beta_t X_t$ and $\beta_t X_t$, respectively. Then, $(\beta_t X_t)/S_t$ is the number of shares of the risky asset held at time t. Hence, the change of the portfolio value during $[t, t + dt]$ is

$$dX_t = \left(\frac{\beta_t X_t}{S_t}\right) dS_t + ((1 - \beta_t)X_t)\, r(t)dt,$$

Assuming that X follows the dynamics

$$dS_t = S_t\left(\alpha_t dt + \sigma_t dW_t\right),$$

the dynamics of the wealth process $X = X^\beta$ is given by

$$dX_t = X_t\left\{\beta_t(\alpha_t - r(t)) + r(t)\right\} dt + X_t\beta_t\sigma_t dW_t, \quad X_0 = x.$$

With a little bit of extra effort, one can also consider trading strategies with consumption. Concretely, we say that the rate of consumption is $\{c_t\}_t$ if during a given time period $[s, t]$, the total consumption is $\int_s^t c_u du$. Equivalently, we can think that the consumption during a small time interval $[t, t + dt]$ is $c_t dt$. In this case, the dynamics of the wealth process $X = X^{\beta,c}$ with initial wealth x, risky asset proportion $\{\beta\}_{t\geq 0}$, and rate of consumption $\{\beta_t\}_{t\geq 0}$ is given by

$$dX_t = \left(X_t\left\{\beta_t(\alpha_t - r(t)) + r(t)\right\} - c_t\right) dt + X_t\beta_t\sigma_t dW_t, \quad X_0 = x. \tag{2}$$

In this case, the controls for the investor are β and c.

The optimal control problem consists of choosing the control so that to maximize certain measure of performance during a given time period $[0,T]$ of interest. The measure of performance is set of the general form

$$\mathcal{J}(\hat{u}) = \Phi(X_T^{\hat{u}}) + \int_0^T F(t, X_t^{\hat{u}}, \hat{u}_t)dt.$$

Besides the natural adaptability restriction, the control $\hat{u} := \{\hat{u}_t\}_{0\le t\le T}$ can be subjected to certain *admissibility conditions* such as restricting its value to certain domain $U \subset \mathbb{R}^m$. We can denote the set of all *admissible* controls $\hat{u}$ by $\mathcal{U}$. Hence, we have the optimization problem

$$V(0,x) = \sup_{\hat{u}\in\mathcal{U}} E\left(\Phi(X_T^{\hat{u}}) + \int_0^T F(t, X_t^{\hat{u}}, \hat{u}_t)dt\right). \tag{3}$$

We indexed the optimal value by the initial time $t = 0$ and the initial value x of the state variable.

Example 2.2. The optimal control problem associated with the portfolio wealth process in (2) typically takes the form:

$$V(0,x) = \sup_{(\beta,c)\in\mathcal{U}} E\left(\Phi(X_T^{\beta,c}) + \int_0^T F(t, X_t^{\beta,c}, c_t)dt\right). \tag{4}$$

In that case, x is interpreted as the initial portfolio wealth or endowment, $\Phi(X_T^{\beta,c})$ is the investor's utility coming from the final wealth, and $\int_0^T F(t, X_t^{\beta,c}, c_t)dt$ is the running investor's utility coming from consumption. As such, both $\Phi(x) : \mathbb{R}_+ \to \mathbb{R}$ and $F(t,x,c) : \mathbb{R}_+^3 \to \mathbb{R}$ should be increasing in wealth x and consumption c. Certain admissibility conditions must be imposed for the problem to be well-posed (to have a solution). The following are typical restrictions:

(i) $X_0^{\beta,c} \le x,$ or $X_0^{\beta,c} = x,$ **(Budget constrain)**

(ii) $X_t^{\beta,c} \ge 0, \quad \forall\, 0 \le t \le T$ **(Solvency condition)**

(iii) $\beta_t \ge 0, \quad \forall\, 0 \le t \le T$ **(No short-selling constrain)**

(iii) $\beta_t \le 1, \quad \forall\, 0 \le t \le T$ **(No borrowing constrain)**.

One can simplify the optimal control problem (3) by restricting the set of possible solution. As in the case of option pricing, one way to do so is to consider *Markov or feedback controls* . A feedback control is of the form:

$$\hat{u}_t = u(t, X_t), \tag{5}$$

where $u : \mathbb{R}_+ \times \mathbb{R}^k \to U \subset \mathbb{R}^m$ is a deterministic function. In this case, the state variable $X := X^u$ of (1) will be determined by the dynamics

$$dX_t = \mu(t, X_t, u(t,X_t))dt + \sigma(t, X_t, u(t,X_t))dW_t, \quad X_0 = x, \tag{6}$$

and (3) becomes

$$V(0,x) = \sup_{u \in \mathcal{U}} E\left(\Phi(X_T^u) + \int_0^T F(t, X_t^u, u(t, X_t^u))dt\right), \tag{7}$$

where the supremum is now over all deterministic function $u(t,x)$ satisfying certain admissibility conditions.

Once an optimal feedback control (or at least an approximation to the optimal) u^* has been determined, the decision maker can implement the control by discretizing time. Namely, at each time t_i of given control times $t_0 = 0 < t_1 < \cdots < t_n = T$, the decision maker will set the control value to

$$\hat{u}_{t_i} = u(t_i, X_{t_i}),$$

depending of the current state value X_{t_i}. For instance, in the case of the portfolio problem of Example 2.2, at the beginning of each trading day, the investor will rebalance the portfolio weights according to the previous day's final portfolio value.

3. The dynamical problem and the Hamilton-Jacobi-Bellman approach

The *Hamilton-Jacobi-Bellman approach* (HJB) immerse the optimal control problem in a dynamical setting, apparently making it more complex. Think of the what would be the control problem (7) at time t given that the state value X_t is x at that time. The decision maker will then be facing the following problem at that time t:

$$V(t,x) = \sup_{u \in \mathcal{U}_t} E\left(\Phi(X_T^{u,t,x}) + \int_0^T F(t, X_t^{u,t.x}, u(t, X_t^{u,t,x}))dt\right), \tag{8}$$

where the supremum is now over all deterministic function $u : [t,T] \times \mathbb{R}^k \to U \subset \mathbb{R}^m$, satisfying certain additional admissibility conditions $\mathcal{U}_t$, and $\{X_s^{u,t,x}\}_{t \le s \le T}$ is the state process during $[t,T]$ given that the control u was chosen, and the "initial" value is x. Concretely, $X = X^{u,t,x}$ follows the dynamics

$$\begin{aligned} dX_s &= \mu(s, X_s, u(s,X_s))ds + \sigma(s, X_s, u(s,X_s))dW_s, \quad t \le s \le T \\ X_t^{u,t,x} &= x. \end{aligned}$$

The notation is heavy, but the meaning is quite natural in terms of the problem faced by the decision maker. Also, notice that we are strongly using the Markov nature of the process X.

The crucial finding of the HJB method is that, under certain consistency and regularity conditions, one can find a PDE representation for the (8). The consistency condition is established in terms of the following principle called *Bellman principle*:

> "An optimal sequence of decisions in a multistage decision process problem has the property that whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions"

In relation with the problem (8), we can express the Bellman principle as follows. Suppose that $u^*_{t,x} : [s,T] \times \mathbb{R}^k \times \mathbb{R}^m$ is the optimal solution to the problem (8) and $\{X^{*,t,x}_s\}_{t \le s \le T}$ is the resulting state process:

$$X^{*,t,x}_s := X^{u^*_{t,x},t,x}_s, t \le s \le T.$$

Bellman principle could be mathematically be written as follows:

$$u^*_{t,x}(s) = u^*_{t', X^{*,t,x}_{t'}}(s),$$

for any $t \le t' \le s \le T$. Using this principle and assuming that $V(t,x)$ is smooth enough, the following result holds:

Theorem 3.1. *The function $V : [0,T] \times \mathbb{R}^k \to \mathbb{R}$ in (8) satisfies the* HJB equation*:*

$$\frac{\partial V}{\partial t} + \sup_{\bar{u} \in U} \{F(t,x,\bar{u}) + \mathcal{A}^{\bar{u}} V(t,x)\} = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^k, \tag{9}$$

$$V(T,x) = \Phi(x). \tag{10}$$

where

$$\mathcal{A}^{\bar{u}} V(t,x) := \sum_{i=1}^{k} \mu_i(t,x,\bar{u}) \frac{\partial V}{\partial x_i}(t,x) + \frac{1}{2} \sum_{i,j=1}^{k} C_{i,j}(t,x,\bar{u}) \frac{\partial^2 V}{\partial x_i \partial x_j}(t,x),$$

and $C_{i,j}$ is the (i,j) entry of the $d \times d$ matrix $\sigma\sigma^T$. Furthermore, for each (t,x) the supremum at (9) is attained at the $u^*(t,x)$ the optimal feedback control of the problem (6).

The previous theorem has a mainly heuristic value since it requires strong smoothness assumptions on the value function $V(t,x)$. But as it happened with the hedging problem, one can reverse the argument and show that the the conditions are actually sufficient. Concretely, the following so-called *verification theorem* is quite useful:

Theorem 3.2. *Under the notation of Theorem 3.1, suppose that*

(1) $H : [0,T] \times \mathbb{R}^k \to \mathbb{R}$ *satisfies the* HJB equation*:*

$$\frac{\partial H}{\partial t} + \sup_{\bar{u} \in U} \{F(t,x,\bar{u}) + \mathcal{A}^{\bar{u}} H(t,x)\} = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^k,$$

$$H(T,x) = \Phi(x).$$

(2) *For each (t,x), the maximization problem $\sup_{\bar{u} \in U} \{F(t,x,\bar{u}) + \mathcal{A}^{\bar{u}} H(t,x)\}$ attains its maximum value at $\bar{u} = g(t,x)$;*

(3) *The function $g : [0,T] \times \mathbb{R}^k \to \mathbb{R}^m$ satisfies the other admissibility conditions.*

Then, the optimal value function V of (8) is given by $V(t,x) = H(t,x)$ and there exists an optimal control u^* to the problem (7) given by $u^*(t,x) = g(t,x)$.