MATHEMATICS OF FINANCE

CHAPTER 4

OPTION PRICING IN CONTINUOUS-TIME

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1. THE MARKET MODEL

1.1. **Description of the assets.**

We consider a market consisting of a risk free type of investment (called the *money market account*) and a risky asset. We assume the following conditions:

(1) The money market account allows investors to borrow and lend money. If an agent borrows (resp. lends) x dollars at time s, he shall pay (resp. receive) the amount $xe^{\int_s^u r(v)dv}$ at time u, where $\{r(t)\}_{t\geq 0}$ is a *non-random* known function, called the *short-rate of interest*.

(2) The risky asset has a price process $\{S_t\}_{t\geq 0}$ following the dynamics:

$$dS_t = S_t\left\{\alpha_t dt + \sigma_t dW_t\right\}, \quad S_0 = s_0, \tag{1}$$

where W is a Wiener process and α and σ are processes adapted to the information $\{\mathcal{F}_t^W\}_{t\geq 0}$.

One can think of the time-t short-rate $r(t)$ as the interest (per unit dollar and per unit time) that will prevail during a short time period $[t, t+dt]$. Hence, if we invest x dollars at time t, we shall receive the interest $xr(t)dt$ at time $t+dt$. The amount $xe^{\int_s^u r(t)dt}$ can be realized by investing x dollars at time s and rolling over the interest every dt until time u, where dt is a short time period (see Section 3 in Chapter 3 for more details). We remark that the assumption (1) above is equivalent to assuming the existence of a risk-free asset with (instantaneous) rate of return is $r(t)$:

$$d\,B_t = r(t)B_t dt, \quad B_0 = 1. \tag{2}$$

The value of B_t^{-1} can also be interpreted as the present value of 1\$ at time t. Notice that, in terms of the short-rate r, the time t price of a zero-coupon bond with maturity T and principal 1 dollar is

$$P(t,T) := e^{-\int_t^T r(u)du}.$$

This formula is valid only when r is deterministic.

We represent the space of all possible states of nature or all possible market conditions by Ω, equipped with a probability measure P. The actual form of Ω and P are not important. What is important is that the real stock price process is consistent with the probabilistic model (3) where W is a Wiener process under P. This model is called the *generalized Black-Scholes model* and P is called the *objective* or *statistical* probability P. Notice that (by Itô formula)

$$S_t = S_0 e^{\int_0^t \mu_s ds + \int_0^t \sigma_s dW_s}, \tag{3}$$

where

$$\mu_t = \alpha_t - \frac{1}{2}\sigma_t^2.$$

The values of α_t and μ_t determine, respectively, the instantaneous rate of simple return and log return at time t given the information at that time. In other words, under some regularity conditions,

$$\alpha_t = \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \frac{S_{t+\Delta} - S_t}{S_t} \middle| \mathcal{F}_t^W \right\} \tag{4}$$

$$\mu_t = \lim_{\Delta\to 0} \frac{1}{\Delta} E\left\{ \log\left(\frac{S_{t+\Delta}}{S_t}\right) \middle| \mathcal{F}_t^W \right\}. \tag{5}$$

The value of σ_t is called the instantaneous volatility of the asset at time t; a measure of the riskiness of the asset the time t. More formally, we have that σ_t^2 measures the variability of log return at time t given the information available at that time:

$$\sigma_t = \lim_{\Delta\to 0} \frac{1}{\Delta} \mathrm{Var}\left(\log \frac{S_{t+\Delta}}{S_t} \middle| \mathcal{F}_t^W \right). \tag{6}$$

If during a given time period $[s, u]$, the parameters μ and σ are constant, then the expressions in (5) and (6) will hold without limit for any $t, t+\Delta \in [s, u]$. Hence, one can statistically estimate μ and σ^2 using sample means and variances of log returns based on equally space price observations $S_{t_0}, \dots, S_{t_n}$, with $t_i = i\Delta$ and $\Delta := (u-s)/n$. Concretely, if $R_i = \log(S_{t_{i+1}}/S_{t_i})$ is the log return during $[t_i, t_{i+1}]$, then

$$\hat{\mu} := \frac{1}{\Delta}\bar{R}, \quad \hat{\sigma} := \sqrt{\frac{1}{\Delta} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} \left(R_i - \bar{R}\right)^2}, \tag{7}$$

where $\bar{R} := \sum_{i=1}^n R_i/n$. Notice that even if α and σ are constants, the value of α can be estimated using the sample mean of simple returns

$$\hat{\alpha} := \frac{1}{\Delta n} \sum_{i=1}^{n} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}},$$

only if Δ is small.

1.2. **Portfolio dynamics and self-financiabiliy.**

We assume that transactions take place under the following *frictionless conditions*:

(1) No transaction costs, taxes, and bid/ask spread.
(2) One can sell and buy any amount of the risky asset at any time (hence, there is no short-selling restrictions and no illiquidity issues in the market)

The information is generated by a process Z and we denote $\mathcal{F} := \{\mathcal{F}_t\}_{t\geq 0}$ the information process. We assume that the Wiener process W driving the stock price is adapted to $\mathcal{F}$ and thus, the stock price value S_t is known at time t.

We can represent a *continuous-time trading strategy or portfolio* (also called *dynamical portfolio*) by a pair of adapted processes $h := \{h_t := (x_t, y_t) : t \leq T\}$, where y_t determines the number of shares of the stock held at time t and x_t determines the money invested in the money market account at time t. The value of the portfolio at time t is

$$V_t^h = x_t + y_t S_t. \tag{8}$$

Adaptiveness of h just means that the position of the investor at time t are determined based only on the information available at that time. We said that h is *self-financing* if

$$V_t^h = V_0^h + \int_0^t x_u r(u)du + \int_0^t y_u dS_u, \tag{9}$$

for all $t \leq T$. Mathematically, the two integrals in the previous equation are understood as limits of Riemann-Stieljes sums:

$$(i) \quad \int_0^t x_u r(u)du := \lim \sum_{t_j \leq s} x_{t_{j-1}} r(t_{j-1})(t_j - t_{j-1}),$$

$$(ii) \quad \int_0^t y_u dS_u := \lim \sum_{t_j \leq t} y_{t_{j-1}} (S_{t_j} - S_{t_{j-1}})$$

when the mesh $\bar{\pi} := \max_j (t_j - t_{j-1})$ of the discrete-times $\pi : 0 = t_0 < t_1 < \cdots < t_n = t$ converges to 0. The financial interpretation is that $\int_0^t x_u r(u)du$ and $\int_0^t y_u dS_u$ are the respective *cumulative gains* in money market account and stock during the time period $[0, t]$. The self-financiability condition (9) means that at all times, the value of the portfolio is a result only of the gains and losses in the money market account and stock, and not due to infusion or withdrawal of money from the portfolio. Furthermore, in differential form, the self-finacibility conditions can be written as follows:

$$dV_t^h = x_t r(t)dt + y_t dS_t,$$

which has again the heuristic interpretation that the change of the portfolio value during a small time interval $[t, t + dt)$ equals to the interest earned (or paid) in the money market plus the gain/loss in the stock during the same time interval $[t, t + dt)$.

Equipped with the concept of self-financibility , one can easily generalizes other related concepts of mathematical finance.

Definition 1.1. A self-financing trading strategy or portfolio is an arbitrage opportunity if the portfolio value V_t^h is such that

$$(i)\ V_0^h = 0, \quad (ii)\ V_T^h \geq 0, \quad (iii)\ P(V_T^h > 0) > 0.$$

Definition 1.2. A European contingent claim $\mathcal{X} : \Omega \to \mathbb{R}$ is said to be *reachable* or *replicable* if there exists self-financing portfolio h such that

$$V_T^h(\omega) = \mathcal{X}(\omega),$$

for any “market condition" ω.

1.3. **A multi-asset model.**

All the definitions of this section have obvious generalizations when dealing with multiple risky assets. Indeed, suppose that, in addition to the money market account, there are N risky assets with respective price processes:

$$dS_t^i = S_t^i \left\{ \alpha_t^i dt + \sigma_t^i dW_t \right\}, \quad S_0^i = s_0^i, \quad i = 1, \dots, N, \tag{10}$$

where $W = (W^1, \dots, W^d)^T$ is a d-dimensional Wiener process, and $(\alpha^1, \dots, \sigma^N)$ and $\sigma = (\sigma^{i,j})$, for $i = 1, \dots, N, j = 1, \dots, d$, are processes adapted to the information $\{\mathcal{F}_t^W\}_{t \geq 0}$ generated by the Wiener process (see Section 6 for more details about this model).

A portfolio $h := \{(x_t, y_t^1, \dots, y_t^N)\}_{t \geq 0}$ is said to be self-financing if

(1) the process x and each process y^i are adapted;
(2) the value process $V_t^h := x_t + \sum_{i=1}^N y_t^i S_t^i$ is such that

$$V_t^h = V_0^h + \int_0^t x_u r(u) du + \sum_{i=1}^N \int_0^t y_u^i dS_u^i, \tag{11}$$

for all t.

Again, the differential form of (12), namely

$$dV_t^h = x_t r(t) dt + \sum_{i=1}^N y_t^i \, dS_t^i, \tag{12}$$

has a very natural interpretation: the change of the portfolio during a small time period $[t, t + dt]$ is a result (only) of the interest gained plus the gain/losses in the assets during the same period $[t, t + dt]$.

The following result gives an equivalent characterization of self-financing portfolios in terms of the asset price processes discounted to present value (*discounted prices process* for short):

$$\tilde{S}_t^i := \frac{S_t^i}{B_t} = e^{-\int_0^t r(u) du} S_t^i.$$

Proposition 1.1. *The portfolio h is self-financing if and only if the discounted value process $\tilde{V}_t^h = V_t / B_t$ is such that*

$$\tilde{V}_t^h = V_0^h + \sum_{i=1}^N \int_0^t y_u^i d\tilde{S}_u^i, \quad \textit{for all } t \geq 0.$$

In particular, the portfolio is uniquely determined by the initial wealth V_0^h and positions $(y^1, \dots, y^N)$ in the risky assets.

The previous result is a consequence of the product formula for Itô processes and it is left as an exercise.

Exercise 1. Show the two statements of Proposition 1.1.

Hint: Write the dynamics of $d\left(e^{-\int_0^t r(u) du} V_t^h\right)$ in terms of the dS_t^i's and express each dS_t^i in terms of $d\tilde{S}_t^i$.

2. Arbitrage-free pricing - The Black-Scholes PDE approach

Throughout this part, we assume the model (1)-(2) of Section 1.1. Furthermore, the price of the risky asset is assumed to be given by the following dynamics

$$dS_t = S_t \{\alpha(t, S_t)dt + \sigma(t, S_t)dW_t\}, \quad S_0 = s_0, \tag{13}$$

where $\alpha : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are deterministic functions, and W is a Wiener process under the objective probability measure P. Sometimes, the model (13) is also called a *local stochastic volatility model*. For future reference we also write the dynamics of the risk-free asset associated with the money market account:

$$d\,B_t = r(t)B_t dt, \quad B_0 = 1. \tag{14}$$

Consider a European claim with maturity T whose payoff $\mathcal{X}$ is contingent to the price of the risky asset $\{S_t\}_{t\leq T}$. In this part, we study one of the fundamental problem of finance: pricing the contingent claim $\mathcal{X}$ such that, under the proposed pricing process $\{\Pi_t\}_{0\leq t\leq T}$, the market consisting of the claim, the underlying, and the money market account is arbitrage-free. This was the original problem set up in the landmark work of Black and Scholes 1973, and as in their formulation, we will analyze instead a simplified, but still reasonable, version of problem. We look for pricing procedures which are of the form

$$\Pi_t = F(t, S_t), \quad 0 \leq t \leq T, \tag{15}$$

for a function $F(t, s) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that is continuously differentiable in t and twice continuously differentiable in s. Such a price process is called *Markovian*. Obviously, since the final price Π_T of the claim must coincide with the payoff of the claim, for such a function F to exist, the claim must be simple. Hence, we assume that

$$\mathcal{X} = \Phi(S_T).$$

Black and Scholes key idea, made completely formal later on by Merton, was to “dynamically" trade in the risky asset so that to be able to “offset " the random (uncontrolled) movements of the claim. By offset, we meant to be able to produce a risk-free (instantaneous) rate of return at any time by taking two opposite risky positions. Let us formalized this idea.

Consider a *self-financing* trading strategy that takes a long position in the claim and a dynamic position in the stock. Concretely, let y_t be the number of shares in the stock held at time t, which as usual is assumed to depend only on the information available at time t. The value of the portfolio at time t is

$$V_t = \Pi_t + y_t S_t. \tag{16}$$

By the self-financing condition,

$$dV_t = d\Pi_t + y_t dS_t, \tag{17}$$

for all $0 \le t \le T$. Again, this has the natural interpretation that the change in the value of portfolio during a small time interval $[t, t + dt]$ is the result only of the change in the price of the claim $d\Pi_t$ and the loss/profit in the stock during $[t, t + dt]$ and not of any withdrawal or infusion of money into the portfolio.

Using the Markovian pricing assumption (15) and the dynamics (13), it follows from Itô's formula that

$$\begin{aligned} d\Pi_t &= \frac{\partial F}{\partial t}(t, S_t) \cdot dt + \frac{\partial F}{\partial s}(t, S_t) \cdot d\, S_t + \frac{1}{2}\frac{\partial^2 F}{\partial s^2}(t, S_t) \cdot (dS_t)^2 \\ &= \left\{ \frac{\partial F}{\partial t}(t, S_t) + \alpha(t, S_t) S_t \frac{\partial F}{\partial s}(t, S_t) + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 F}{\partial s^2}(t, S_t) \right\} dt + \frac{\partial F}{\partial s}(t, S_t) S_t \sigma(t, S_t) dW_t. \end{aligned}$$

Plugging the differential $d\Pi_t$ in (17), we get that

$$\begin{aligned} dV_t = &\left\{ \frac{\partial F}{\partial t}(t, S_t) + \alpha(t, S_t) S_t \frac{\partial F}{\partial s}(t, S_t) + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 F}{\partial s^2}(t, S_t) + y_t \alpha(t, S_t) S_t \right\} dt \\ &+ \left\{ \frac{\partial F}{\partial s}(t, S_t) + y_t \right\} S_t \sigma(t, S_t) dW_t. \end{aligned}$$

It is now evident that the riskyness of the portfolio can be hedged away entirely by setting

$$y_t = -\frac{\partial F}{\partial s}(t, S_t). \tag{18}$$

Plugging this y_t in the previous expression of dV_t, we get

$$dV_t = \left\{ \frac{\partial F}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 F}{\partial s^2}(t, S_t) \right\} dt. \tag{19}$$

This portfolio has an instantaneous risk-free rate of return at time t of

$$r_{_V}(t) = \frac{\frac{\partial F}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 F}{\partial s^2}(t, S_t)}{V_t}.$$

Absence of arbitrage will imply that the rate $r_{_V}(t)$ must coincide with the short rate of interest $r(t)$ at all time t. Intuitively, if for instance, $r(t) > r_{_V}(t)$, one can "short" the portfolio (short sell the claim and take the position $-y_t$ in the stock) at time t and invest the proceeding in the money market account so that, for a small enough period dt, one can lock a sure profit of $(r(t) - r_{_V}(t))dt$. We conclude that

$$\frac{\frac{\partial F}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 F}{\partial s^2}(t, S_t)}{V_t} = r(t), \quad 0 \le t \le T. \tag{20}$$

Plugging (17) into (20) and subsequently using (18) and (15), we come up with the following equation

$$\frac{\partial F}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2(t, S_t) S_t^2 \frac{\partial^2 F}{\partial s^2}(t, S_t) + r(t) S_t \frac{\partial F}{\partial s}(t, S_t) - r(t) F(t, S_t) = 0.$$

This equation holds true for any value s that can be reached by S_t at time t. Assuming that the range of possible values of S_t is $\mathbb{R}_+$, we obtain the following fundamental PDE, called *the Black-Scholes equation*,

$$\frac{\partial F}{\partial t}(t,s) + \frac{1}{2}\sigma^2(t,s)s^2\frac{\partial^2 F}{\partial s^2}(t,s) + r(t)s\frac{\partial F}{\partial s}(t,s) - r(t)F(t,s) = 0. \tag{21}$$

Obviously, the final price Π_T of the claim must coincide with the payoff $\mathcal{X} = \Phi(S_T)$. Thus, we obtain the following boundary condition:

$$F(T,s) = \Phi(s), \tag{22}$$

We summarize the previous reasoning in the following fundamental theorem of finance.

Theorem 2.1. *The only pricing procedure of the form (15) which is consistent with the absence of arbitrage is when F satisfies (21) for any $(t,s) \in [0,T) \times \mathbb{R}_+$ with the boundary condition (22).*

Exercise 2. What can be said about an American contingent claim with immediate time t payoff $\Phi(S_t)$? Argue intuitively that $r(t) < r_V(t)$ for some t leads to an arbitrage-opportunity, but this is not the case when $r(t) > r_V(t)$. Conclude that $F(t,s)$ must satisfy the problem

$$\frac{\partial F}{\partial t}(t,s) + \frac{1}{2}\sigma^2(t,s)s^2\frac{\partial^2 F}{\partial s^2}(t,s) + r(t)s\frac{\partial F}{\partial s}(t,s) - r(t)F(t,s) \leq 0, \tag{23}$$

with the boundary condition (22).

3. Risk-neutral option pricing - The martingale approach

Let us assume the set up of the previous part. Drawing from our experience with discrete-time models, in this part we look for an interpretation of the arbitrage-free pricing process (15) as the expected discounted payoff of the claim, given the current information, under a type of risk-neutral probability measure Q. Concretely, we investigate if

$$\Pi_t = e^{-\int_t^T r(u)du} E^Q\left[\Phi(S_T)\middle|\,\mathcal{F}_t^W\right], \tag{24}$$

where $\mathcal{F}_t^W$ is the information available at time t and Q is a probability measure such that, under Q, the rate of return of the stock is the same as the risk-free rate of return r; that is, under Q,

$$dS_t = S_t\left\{r(t)dt + \sigma(t,S_t)dW_t^Q\right\}, \quad 0 \leq t \leq T, \tag{25}$$

where W^Q is a Wiener process under Q. Formula (24) is called the *Discounted Risk-Neutral Valuation Formula (DRNVF)*.

Before we proceed to check the validity of (24), let us learn a few facts about how to use it.

(1) In the case of $t = 0$, formula (24) takes the following easy expression:

$$\Pi_0 = e^{-\int_t^T r(u)du} E^Q\left[\Phi(S_T)\right]. \tag{26}$$

(2) Also, notice that because of the Markovian nature of (47), Π_t is a function $F(t, S_t)$ of time t and of the spot stock price S_t:

$$\Pi_t = e^{-\int_t^T r(u)du} E^Q \left[\Phi(S_T) | \, S_t \right] = F(t, S_t), \quad \text{where}$$
$$F(t, s) := e^{-\int_t^T r(u)du} E^Q \left[\Phi(S_T) | \, S_t = s \right].$$

(3) To evaluate $F(t, s)$ above, one can apply for instance Monte Carlo methods by noticing that, under Q and conditioning on $S_t = s$, the dynamics of S is given by

$$dS_u = S_t \left\{ r(t)dt + \sigma(t, S_t) dW_t^Q \right\}, \quad t \le u \le T, \quad S_t = s.$$

There are a few things that we should resolve in order for (24-47) to hold. First, does such a probability measure Q exist? Notice that for (47) to be consistent with (13), we must have

$$dW_t^Q = dW_t + \sigma(t, S_t)^{-1} \left(\alpha(t, S_t) - r(t) \right) dt.$$

Let us denote

$$\lambda_t = \frac{\alpha(t, S_t) - r(t)}{\sigma(t, S_t)},$$

which is called the *market-price of risk* and is interpreted as the *risk premium per volatility unit* inherent in the risky asset. Then, the existence of the risk-neutral probability measure reduces to find a a probability Q such that the process

(27) $$W_t^Q := W_t + \int_0^t \lambda_s ds, \quad t \le T,$$

is a Wiener process under Q. The existence of such a Q is guaranteed by the *Girsanov's theorem*, one of the most fundamental results of stochastic calculus. This result states that under the so-called *Novikov's condition*

$$E \left(e^{\frac{1}{2} \int_0^T \lambda_s^2 ds} \right) < \infty,$$

there exists a probability measure Q such that $\{W_t^Q\}_{t \le T}$ in (27) is a Wiener process under Q. Q is also an *equivalent measure* to P, meaning that a set A is P-null if and only if A is Q-null.

When λ is set to be the market price of risk, the resulting Girsanov's measure Q will be such that $\{S_t\}_{t \le T}$ satisfies the dynamics (47) provided that λ satisfies the Novikov's condition. Clearly, Novikov's condition will hold if for instance $\alpha(t, s)$, $r(t)$, and $\sigma^{-1}(t, s)$ are uniformly bounded (for any $t \le T$ and s).

Let us turn to the second point: does the arbitrage-free pricing procedure (15), when F is smooth, satisfy (24)? To show that this is true, we recall two facts:

- First, by the Feyman-Kac stochastic representation (see Exercise 10 in Chapter 3), the solution of the PDE (21) has a stochastic representation of the form

(28) $$F(t, x) = e^{-\int_t^T r(u)du} E_{t,x} \left(\Phi(X_T^{t,x}) \right),$$

where $\{X_s^{t,x}\}_{t\leq s\leq T}$ is the solution of the following SDE:

$$\begin{aligned} dX_s^{t,x} &= \mu(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})\, dW_s^{t,x}, \quad s \geq t \\ X_t^{t,x} &= x. \end{aligned}$$

- On the hand, by the Markov's property of S (see Section 3.5.1 and formula (47) in Chapter 3), (24) can be written as

  $$\Pi_t = e^{-\int_t^T r(u)du} E^Q \left[\Phi(S_T) | \, S_t \right] = F(t, S_t),$$

  where F is exactly given by (28).

Summarizing, there are two way two compute an arbitrage-free price process for the claim of the form $\Pi_t = F(t, S_t)$ with F smooth:

(1) Either we solves the Black-Scholes PDE (21-22) (analytically or numerically), or
(2) we evaluate (analytically or numerically) the DRNVF (24).

We now give some other properties of the risk-neutral probability measure and an important particular case where (24) admits close form formulas: the original Black-Scholes setup.

3.1. **The original Black-Scholes formulation.**

Black and Scholes original set up consider constant volatility and short-rate of interest:

$$\begin{aligned} dS_t &= S_t \left\{ \alpha dt + \sigma dW_t \right\}, \quad S_0 = s_0, \\ dB_t &= rB_t dt, \quad B_0 = 1. \end{aligned}$$

In this case, we can substantially simplify the DRNVF (24). Recall that the dynamics of S under Q is given by

$$dS_t = S_t \left\{ rdt + \sigma dW_t \right\}, \quad S_0 = s_0,$$

which solution is $S_t = S_0 e^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t}$. Now, using the Markov's property and the solution of S,

$$\begin{aligned} \Pi_t &= e^{-r(T-t)} E^Q \left[\Phi\left(S_T\right) | \, S_t \right] \\ &= e^{-r(T-t)} E^Q \left[\Phi\left(S_t e^{\sigma(W_T - W_t) + (r - \frac{1}{2}\sigma^2)(T-t)} \right) \Big| \, S_t \right]. \end{aligned}$$

Since $W_T - W_t$ is independent of S_t (since S_t depends only on the values of W before time t), we can write

$$\Pi_t = e^{-r(T-t)} G_T(t, S_t), \quad \text{where} \quad G_T(t,s) = E^Q \left[\Phi\left(s e^{\sigma(W_T - W_t) + (r - \frac{1}{2}\sigma^2)(T-t)} \right) \right] \tag{29}$$

Since $W_T - W_t \sim \mathcal{N}(0, T-t)$, we obtain that

$$G_T(t,s) = E^Q \left[\Phi\left(s e^{\sigma\sqrt{T-t}\cdot Z + (r - \frac{1}{2}\sigma^2)(T-t)} \right) \right] \tag{30}$$

where Z stands for a standard normal random variable under Q.

Example 3.1. The previous formulas are very easy to approximate by a numerical technique called Monte Carlo estimation. The idea is to estimate the expected value by a simple average:

$$\Pi_t \approx e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^{N} \Phi\left(s e^{\sigma\sqrt{T-t}\cdot Z_i + (r-\frac{1}{2}\sigma^2)(T-t)} \right),$$

where $Z_1, \dots, Z_N$ are N independent standard normal variables.

Example 3.2. As an application, let us deduce the famous Black-Scholes formula for call options, where the payoff is $\Phi(S_T) = (S_T - K)_+$. Denoting

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t) \right\}, \quad d_2 = d_1 - \sigma\sqrt{T-t}, \tag{31}$$

we can write (30) as follows

$$\begin{aligned} G_T(t,s) &= E^Q\left[\left(s e^{\sigma\sqrt{T-t}\cdot Z + (r-\frac{1}{2}\sigma^2)(T-t)} - K \right) \mathbf{1}_{\{Z \geq -d_2\}} \right] \\ &= s e^{r(T-t)} E^Q\left[e^{\sigma\sqrt{T-t}\cdot Z - \frac{1}{2}\sigma^2(T-t)} \mathbf{1}_{\{Z \geq -d_-\}} \right] - KQ(Z \geq \, -d_2). \end{aligned}$$

It follows easily that

$$\begin{aligned} E^Q\left[e^{\sigma\sqrt{T-t}\cdot Z - \frac{1}{2}\sigma^2(T-t)} \mathbf{1}_{\{Z \geq -d_2\}} \right] &= \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{\sigma\sqrt{T-t}z - \frac{1}{2}\sigma^2(T-t)} e^{-\frac{1}{2}z^2} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-d_1} e^{-\frac{1}{2}u^2} du = Q(Z \geq \, -d_1), \end{aligned}$$

where we used the change of variables $u = z - \sigma\sqrt{T-t}$. Putting everything together and denoting $C(t,T)$ the resulting call option price, we obtain the famous *Black-Scholes Formula*,

$$C(t,s) := sN(d_1) - e^{-r(T-t)} K N(d_2), \tag{32}$$

where $N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2} dz$ is the cumulative distribution function of a standard normal variable. In summary, $C(t,s)$ above is the price of a call option with maturity T and strike K at time t when the spot stock price is s. The price process for the call option will be given by $\Pi_t = C(t, S_t)$ at time t.

Example 3.3. The arbitrage-free price of a put option can be derived in a similar manner as above. However, it is easier and more illustrative to deduce this formula from the *put-call parity* relationship. Let $P_t = P(t, S_t)$ be the time-t price of a put option with maturity T and strike K. From the RNVF (29) for the call and put options, one easily deduce that

$$C(t,s) = P(t,s) - s + e^{-r(T-t)} k. \tag{33}$$

Thus, the Black-Scholes formula for put options takes the form

$$P(t,s) := e^{-r(T-t)} K N(-d_2) - sN(-d_1). \tag{34}$$

3.2. **Martingale properties and arbitrage-freeness.**

As in the discrete-time case, the risk-neutral measure Q can be seen as a martingale measure. Concretely, the discounted price process

$$\tilde{S}_t := e^{-\int_0^t r(u)du} S_t, \quad 0 \leq t \leq T,$$

is a martingale under Q. This fact follows from finding the dynamics $d\tilde{S}_t$ via Itô's formula. Due to this property, Q is called a *martingale measure for the model (13-14).*

Example 3.4. The previous observation has a very natural interpretation in terms of the risk-neutral valuation formula (24). Indeed, intuitively the time-t price of a European option with payoff $\Phi(S_T) = S_T$ must be S_t (otherwise, there will be arbitrage, why?). Thus, the RNVF will imply that

$$S_t = e^{-\int_t^T r(u)du} E^Q \left[S_T \middle| \mathcal{F}_t^W \right], \tag{35}$$

which is exactly the definition of a martingale.

Notice also that, from (24) and the tower property, one can easily check that the discounted claim's price process

$$\tilde{\Pi}_t := e^{-\int_0^t r(u)du} \Pi_t,$$

is a martingale under Q.

The following is an important theorem of mathematical finance (especially useful when dealing with incomplete market where there are more than one martingale measure).

Proposition 3.1. *Let Q be an* equivalent martingale measure *for the market price process (2-3) and let Π be defined by (24). Then, the market consisting of the risk-free asset, the stock, and the claim with price process Π is arbitrage-free*

Let us give the general idea of the previous result. Suppose that h is an arbitrage opportunity; hence, the resulting portfolio value

$$V_t^h := x_t + y_t S_t + z_t \Pi_t,$$

is such that

$$V_0^h = 0, \quad V_T^h \geq 0, \quad \text{and} \quad P\left(V_T^h > 0\right) > 0.$$

Similar to Proposition 1.1, it follows that

$$\tilde{V}_t^h = V_0^h + \int_0^t y_u \, d\tilde{S}_u + \int_0^t z_u \, d\tilde{\Pi}_u.$$

In the same manner as why an Itô integral is a martingale, under certain integrability conditions on the trading strategies, the integrals $\int_0^t y_u \, d\tilde{S}_u$ and $\int_0^t z_u \, d\tilde{\Pi}_u$ are martingales under Q because $\tilde{S}$ and $\tilde{\Pi}$ are martingales under Q. Thus, $\tilde{V}^h$ is a martingale under Q and

$$E^Q\left(\tilde{V}_T^h\right) = \tilde{V}_0^T = 0.$$

Since $V_T^h \geq 0$, $Q(V_T^h > 0) = 0$. Because Q and P are equivalent, it follows that $P(V_T^h > 0) = 0$ and hence, h cannot be an arbitrage opportunity.

4. HEDGING AND COMPLETENESS

The second fundamental problem of finance is hedging (see definition 1.2). Again, we considered a simplified version of the problem. Suppose that $\mathcal{X}$ is a European T-claim with payoff

$$\mathcal{X} = \Phi(S_T).$$

We want to build a self-financing trading strategy h such that the resulting portfolio value V^h satisfies

$$V_T^h = \Phi(S_T).$$

The solution to the hedging problem will lead to a unique arbitrage-free price process for the claim. More specifically, if a claim $\mathcal{X}$ admits a replicating portfolio h, then the only arbitrage-free price process for $\mathcal{X}$ is

$$\Pi_t = V_t^h.$$

Otherwise, if for instance $\Pi_t > V_t^h$ at some time t, one can short-sell the claim for Π_t, start the portfolio with V_t^h, and put the rest in the money market account. At time T, the wealth of the portfolio coincides with the payoff and hence, if positive, we can cover it, and we get a sure profit without risk.

The hedging problem is again closely related to a PDE formulation. For instance, suppose that h is a hedging strategy for $\mathcal{X}$. In light of Proposition 1.1 and the fact that $\tilde{S}$ is a martingale under Q, the discounted portfolio value process $\{\tilde{V}_t^h\}_{t \leq T}$ is a martingale under Q. It follows that

$$\tilde{V}_t^h = E^Q\left\{ \tilde{V}_T^h \middle| \mathcal{F}_t^W \right\} = e^{-\int_0^T r(u)du} E^Q\left\{ \Phi\left(S_T\right) \middle| \mathcal{F}_t^W \right\},$$

for any $t \leq T$. From the Markov property of the model (13), $E^Q\left\{ \Phi\left(S_T\right) \middle| \mathcal{F}_t^W \right\}$ depends only on S_t and thus, there exists a deterministic function $F(t,s)$ such that

$$V_t^h = F(t, S_t).$$

Furthermore, if F is smooth, then F will solve (21-22) in light of Proposition 4.1 and the fact that $\Pi_t = V_t^h$ will be arbitrage-free. Surprisingly, the reciprocal statement is true.

Theorem 4.1. *Suppose that $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ solves the PDE (21) with boundary condition (22). Then, the European claim $\mathcal{X} = \Phi(S_T)$ is reachable and its replicating portfolio is given by*

(36) $$y_t = \text{(Number of shares of stock at time t)} = \frac{\partial F}{\partial s}(t, S_t),$$
$$x_t = \text{(Cash held in the money market account at time t)} = F(t, S_t) - y_t S_t.$$

The verification of Theorem 4.1 goes along the following lines:

(1) By definition, the value of the portfolio is $V_t^h = x_t + y_t S_t = F(t, S_t)$.
(2) In particular, $V_T^h = F(T, S_T) = \Phi(S_T)$ by (22). Thus, the portfolio is replicating $\mathcal{X}$.

(3) It remains to check that V is self-financing; that is,

$$dV_t^h = x_t r(t)dt + y_t dS_t. \tag{37}$$

(4) We computed each side of (37).For instance, the right-hand side becomes

$$\left\{ rF - S(r-\alpha)\frac{\partial F}{\partial s} + \right\} dt + S\frac{\partial F}{\partial s} dW_t,$$

where we replaced the definition of h and the dynamics of S.

(5) Using Itô's formula, the left-hand side of (37) becomes

$$\begin{aligned} dV_t^h = dF(t, S_t) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial s^2}(dS_t)^2 \\ &= \left\{ \frac{\partial F}{\partial t} + S\alpha\frac{\partial F}{\partial s} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 F}{\partial s^2} \right\} dt + \frac{\partial F}{\partial s}\sigma dW. \end{aligned}$$

Using the PDE (21), we obtain exactly the dynamics of (4).

It is a fundamental result (not trivial at all) that, under the model (13-14), any European claim $\mathcal{X}$ that is sufficiently integrable is replicable. Hence, the Black-Scholes model (13-14) is *complete*.

Theorem 4.1 gives sufficient conditions for a claim to be reachable. The strategy (36) is called *delta-hedging*. In practice, we apply the hedging strategy as follows:

(i) Find (numerically or analytically) a function $F(t, s)$ that solves (21-22);

(ii) Pick evenly spaced trading times $t_0 < t_1 < \cdots < t_n = T$ (e.g. typically daily for large portfolios of options);

(iii) At each time t_i, trade as much stock as necessary such that the position in the stock is

$$\Delta(t_i) := \frac{\partial F}{\partial s}(t_i, S_{t_i}) \tag{38}$$

number of shares.

(iv) Adjust the money market account accordingly so that the portfolio remains self-financing.

Notice that at step t_i, one shall need to buy or sell

$$\bar{\Gamma}(t_i) := \Delta(t_i) - \Delta(t_{i-1}),$$

shares of stock. Hence, the cash required for (or left of) this transaction will be $\bar{\Gamma}(t_i)S_{t_i}$, which results in the following time-t_i money-market-account position:

$$x_{t_i} = x_{t_{i-1}} e^{\int_{t_{i-1}}^{t_i} r(u)du} - (\Delta(t_i) - \Delta(t_{i-1}))S_{t_i}.$$

When the price of a given option or claim is a smooth function $F(t, s)$ of the time t and the spot price s of the stock, the quantity $\Delta(t, s) := \frac{\partial F}{\partial s}(t, s)$ is called the *delta* of the claim at time t when the stock price is s.

Example 4.1. Consider the original formulation of the Black-Scholes model of Section 3.1. From the Black-Scholes formula (31-32), the delta of a call option is

$$\Delta := N(d_1).$$

Then, the delta increases from 0 to 1 as s increases. When time to maturity $T - t$ is small, the transition is sharper. In fact, the graph of $s \to N(d_1)$ converges to the graph of step function $s \to \mathbf{1}_{s \geq K}$. What is the delta for a put option?

The delta of a portfolio of options can be defined by linearity. For instance, suppose that P_t denotes the value of a portfolio at time t consisting of n^i units of a claim $\mathcal{X}^i = \Phi^i(S_T)$, $i = 1, \dots, k$. Then, denoting the corresponding option prices $F^1, \dots, F^k$, the value of the portfolio at time t will be

$$P_t := F(t, S_t) = n_t^1 F^1(t, S_t) + \cdots + n_t^k F^k(t, S_t).$$

The delta of the portfolio at time t when the spot price of the underlying is s is defined by

$$\Delta(t) := n_t^1 \frac{\partial F^1}{\partial s} + \cdots + n_t^k \frac{\partial F^k}{\partial s}.$$

We remark that the portfolio consisting of the claim and the delta-hedging strategy has a delta which is identically always 0. A portfolio with this property is said to be *delta neutral*.

Example 4.2. *(See Hull, Section 14.4).* Consider a European call option on $100,000$ shares of a non-dividend paying stock. We assume the Black-Scholes model with the followings parameters (time is measure in years):

$$S_0 = \$49, \quad K = \$50/\text{share}, \quad r = 5\%, \quad \sigma = 20\%, \quad T = 20 \text{ weeks} = 0.3846, \quad \alpha = 13\%.$$

The Black-Scholes price of the call option is $\$240,000$. Suppose that we simulate the stock price process and apply daily delta hedging ($\Delta = 1$ *week* in a year-count convention of 365 days). Consider the Table 15.2 of Hull.

- The initial value of the delta is $\Delta(0) = .522$. Thus, as soon as the option is written, $\$2,557,800$ must be borrowed to buy $52,200$ shares at price $\$49$.
- The interest cost of the first week is $2,557,800 * (e^{.05/52} - 1) = \$2,460.6058$.
- Suppose that by the end of the first week, the stock price falls to $\$48.12$. The delta is now given by $\Delta = .458$. Then, $6,400$ shares are sold, realizing $\$307,968$ in cash.
- The cumulative borrowing at the end of week 1 is $\$2,252,292.6058$.
- At the end of 20th week, the stock price is $\$57.25$ and the cumulative borrowing is $\$5,263,300$ and the position in the stock is $100,000$ shares. The holder of the option exercise the option and hence, we receive $100,000 * 50 = \$5,000,000$ dollar. The total cost for hedging is $\$263,300$.
- If the writer charges the Black-Scholes price $\$240,000$ at time $t = 0$, then this money grows up to $240,000 * e^{(20/52)*.05} = \$244,660$. Then, the net cost for hedging and writing the option at maturity is $263,300 - 244,660 = \$18,640$.

- Theoretically and under a frictionless market, the present value of net cost will tend to the theoretical B-S price if rebalancing takes place more and more frequently.

In the Table 15.3 of Hull, the option closes out of the money and the total cost will $\$11,340$ is the writer charges the Black-Scholes price.

In the face of transaction costs, delta-hedging for a single option of the underlying is too expensive to maintain (at least at the theoretical required frequency to be effective). In the previous example, we might wish to reduce the rebalancing time from one week to one day to reduce the hedging cost. However, in reality, this might actually result in a higher costs due to the transaction costs. However, for a large portfolio of options, delta-hedging could actually be feasible since only one trade in the stock is needed to hedge all the options at once. Typically, the transaction costs of hedging are absorbed by the profits on the options of the portfolios.

Another application of delta hedging is in the creation of synthetic options. For a fund manager could be more attractive to create a required option (e.g. a put option) using delta hedging in the stock and the money market account due to the lack of liquidity (or absence) of options with atypical strikes.

5. Other hedging strategies

5.1. **Gamma hedging.**

Typically, an option trader deals with a large portfolio of options with a common underlying and he/she will look to reduce his risk exposure by running a delta hedging strategy. Thus, the whole portfolio of the hedger will consist of the options and the underlying (the money market account serves as a kind of saving account to finance the needed transaction in the portfolio). Suppose that $F(t,s)$ is the value of the option portfolio at time t when the price of the underlying is s. Then, at a given time t, the value of the portfolio of a trader holding a short position in F and looking to hedge using y_t unit of stock is

$$V_t = y_t S_t - F(t, S_t). \tag{39}$$

Let us call the previous portfolio the "writer's hedging portfolio".

If at a given time t, the position in the stock y_t is set according to the delta $\frac{\partial F}{\partial s}(t, S_t)$ of the portfolio of options, we say that the portfolio is *delta neutral*. This terminology follows from looking at the whole portfolio (39) as a portfolio of options (one option has the trivial payoff $\Phi(S_T) = S_T$ and hence, a time-t price function of $F_0(t,s) = s$). Then, the delta of the whole portfolio at that time is set to be 0.

Notice that making a portfolio delta neutral accounts to running a delta-hedging strategy for the portfolio of options F. Typically, a trader of options make themselves delta neutral at the end of each day. Making the writer's hedging portfolio delta neutral at a given time t_i entails to trade

$$\Delta(t_i) - \Delta(t_{i-1}) = \frac{\partial F}{\partial s}(t_i, S_{t_i}) - \frac{\partial F}{\partial s}(t_{i-1}, S_{t_{i-1}}),$$

units of the underlying, where $\Delta(t_{i-1})$ is the delta at the previous trading time. Assuming that $t_i - t_{i-1}$ is relatively small, we can estimate this change by

$$\Delta(t_i) - \Delta(t_{i-1}) \approx \frac{\partial^2 F}{\partial t \partial s}(t_{i-1}, S_{t_{i-1}})(t_i - t_{i-1}) + \frac{\partial^2 F}{\partial s^2}(t_{i-1}, S_{t_{i-1}})(S_{t_i} - S_{t_{i-1}}).$$

The first quantity is typically negligible and can be controlled, but the second component is more significant as the change in the asset price cannot be controlled. The factor of proportionality is called the Gamma of the portfolio of options F. Concretely, the *gamma of the portfolio of options* F at time t when the underlying is s is defined by

$$\Gamma(t, s) := \frac{\partial^2 F}{\partial s^2}(t, s).$$

The previous reasoning argues that it is important for a portfolio to have a small Gamma as in that case the transactions necessary to make the writer's hedging portfolio delta-neutral will be small. The gamma of the portfolio measures our ability to hedge the portfolio with minimal cost. The slower the delta change relative to the stock price level, the less frequent portfolio adjustments are needed (for delta-neutrality) and thus, the lower the transaction costs (which are typically proportional to the number of traded shares).

The following problem is then quite natural: Can we reduce the Gamma of a portfolio of options? The answer is yes by using a liquid option. Typically, the writer wishes to reduce the Gamma of a complex large portfolio of options, an exotic option, or a far out-of-the money (which are not liquid), using a simpler liquid option (e.g. an at-the-money call or put option). We can see this as a hedging strategy where we hedge complex options using simpler options. This procedure is called *Gamma immunization* and can be implemented as follows:

(1) Suppose that at a given time t, the portfolio of options F has a Gamma of $\Gamma := \Gamma(t, S_t)$ and a liquid traded option has a value of $F'(t, S_t)$ and a gamma of $\Gamma' := \Gamma'(t, S_t)$.

(2) If z units of the option are added to the portfolio, the gamma is

$$\Gamma + z\Gamma'.$$

Thus, to make the portfolio gamma neutral we need $z = -\frac{\Gamma}{\Gamma'}$, units of the option F'.

(3) The trader will subsequently correct the delta of the portfolio using stock. Concretely, the new Gamma neutral portfolio of options is

$$F + zF',$$

which has a delta of

$$\frac{\partial F}{\partial s} + z\frac{\partial F'}{\partial s}.$$

Then, the writer's delta neutral portfolio is give by

$$V := yS - (F + zF'),$$

with

$$z = -\frac{\Gamma}{\Gamma'} \quad \text{and} \quad y = \frac{\partial F}{\partial s} + z\frac{\partial F'}{\partial s}.$$

Notice that the portfolio remain gamma neutral as adding stock to a portfolio does not change the gamma.

Example 5.1. In the original Black-Sholes model formulation, the Gamma of a call option enjoys a close formula given by

$$\Gamma = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}},$$

where $\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$. The Gamma is bell shaped centered at the strike. Thus, the maximum Gamma is reached when the option is near at-the-money. Also, notice that the Gamma is always positive. For instance, when the price of the underlying has gone down, the change in the delta will be be negative. This translates into selling stock (when applying delta hedging). Thus, delta hedging entails (typically) selling stock just after the price goes down and buying stock stock just after the price has gone. A sound strategy that can in financial jargon is sometimes rephrased as “buy-high" and “sell-low".

5.2. **Other greeks.**

It is also desirable to reduce the exposure of the portfolio options to miss-specifications in the parameters of the model or to variations in underlying factors of the model. A natural way to measure the sensibility of the portfolio value F to a certain parameter is the partial derivative of the value function with respect to the parameter in question. These measures of sensibility are called *greeks*. The following are the most popular greeks (besides the delta and gamma already introduced):

$$\text{Vega } (\mathcal{V}) = \frac{\partial F}{\partial \sigma}, \quad \text{Rho } (\rho) = \frac{\partial F}{\partial r}, \quad \text{Theta } (\Theta) = \frac{\partial F}{\partial t}.$$

For a Black-Scholes model with constant volatility σ and constant short rate of return r, the greeks of a call option are given by the following formulas:

$$\mathcal{V} = s\varphi(d_1)\sqrt{T-t}, \quad \rho = K(T-t)e^{-r(t-t)}N(d_2), \quad \Theta = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}}.$$

We can interpret the greeks as a rate of change of the portfolio value per unit change of the corresponding parameter. For instance, if at a given time t, the $\mathcal{V} = 1$, we should expect that for each $\%1$ increase in the volatility, the portfolio value will increase $\$1$ dollar. More precise, for a small change $\Delta\sigma$ in volatility, the portfolio value will experience a change of

$$\Delta F \approx \mathcal{V} \cdot \Delta\sigma.$$

A portfolio for which one of the greeks is 0 at a given time t is said to be *neutral* with respect to that greek (e.g. Vega neutral). In principle, if we wish to immunize a portfolio of options against changes in k different parameters, we will need to trade into $k-1$ different options. In practice, traders delta neutral their portfolios at the end of each trading day, but they don’t typically immunize their portfolio against changes in other underlying parameters since it could be hard to find liquid options to do the job. However, the greeks are constantly computed and monitored.

6. ARBITRAGE-FREE PRICING AND HEDGING IN A MULTI-ASSET ITÔ MODEL

6.1. **The market model.**

In this part we consider a general multi-asset market model consisting of N risky assets and a risk-free asset driven by a d-dimensional Wiener process $W = (W^1, \ldots, W^d)^T$. Each Wiener process is interpreted as a source of riskiness. Concretely, under the statistical or objective probability measure P, the dynamics of the risky assets are governed by the dynamics (10), while the risk-free asset has dynamics (2). Here, $\sigma := (\sigma^{ij})_{i\leq N, j\leq d}$, $\alpha := (\alpha^i)_{i\leq N}$, and r are processes adapted to the information $\{\mathcal{F}_t^W\}_{t\geq 0}$ generated by W. As before, the interpretation of these processes is as follows:

- σ_t^{ij} is the volatility contribution of the j^{th} source of riskiness to the i^{th} stock at time t; the total volatility of the i^{th} stock at time t is $\bar{\sigma}_t^{i\cdot} := \sqrt{\sum_{j=1}^d \left(\sigma_t^{ij}\right)^2}$;
- α_t^i is the (instantaneous) rate of return of i^{th} stock at time t;
- $r(t)$ is the short-rate of return at time t.

As usual, we assume the frictionless conditions described at the beginning of Section 1.2. Below, $\sigma^{i,\cdot} = (\sigma^{i1}, \ldots, \sigma^{id})$ is the row vector of volatilities corresponding to the i^{th} stock.

There are two general models of the form (10) that have received some dealt of attention in the literature. In the first type, called a "local or markov" model, σ and α at time t depend on the spot price at that time:

$$dS_t^i = S_t^i \left\{\alpha(t, S_t)dt + \sigma^i(t, S_t)dW_t\right\}, \quad S_0^i = s_0^i, \quad i = 1, \ldots, N. \tag{40}$$

This was the model used in Section 2 to derive the Black-Scholes equation. The second type assumes that α and σ are driven by underlying factors $X := (X^1, \ldots, X^m)$:

$$dS_t^i = S_t^i \left\{\alpha(t, X_t)dt + \sigma^i(t, X_t)dW_t\right\}, \quad S_0^i = s_0^i, \quad i = 1, \ldots, N. \tag{41}$$

The factors are themselves random driven by W in the following manner:

$$dX_t^i = \alpha_X^i(t, X_t)dt + \sigma_X^i(t, X_t)dW_t, \quad i = 1, \ldots, m, \tag{42}$$

where $\alpha_X : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\sigma_X : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. A classical instance of this model is the so-called Heston model:

$$dS_t = S_t\left(\mu dt + S_t\sqrt{V_t}dB^1(t)\right),$$
$$dV_t = \delta(m - V(t))dt + \nu\sqrt{V(t)}dB^2(t),$$

where B^1 and B^2 are correlated Wiener processes.

6.2. **Equivalent martingale measures.**

The martingale method is built on the sufficiency part of the first fundamental theorem of finance. Namely, if there exists an equivalent martingale measure Q then the market is arbitrage-free. A probability measure Q on $\mathcal{F}_T^W$ is said to be an equivalent martingale measure for the market $(B, S^1, \ldots, S^N)$ if the following two conditions hold:

(1) Each discounted price process $\tilde{S}_t^i := B_t^{-1} S_t^i$, for $0 \leq t \leq T$, is a martingale under Q;
(2) $P(A) = 0$ if and only if $Q(A) = 0$.

Each equivalent martingale measure Q induces an arbitrage-free pricing procedure for contingent claims via the DRNVF (24). More precisely, let $\mathcal{X}$ be a random measurement which value is known at time T (that is, $\mathcal{X} \in \mathcal{F}_T^W$) and let

$$\Pi_t = E^Q \left[e^{-\int_t^T r(u)du} \mathcal{X} \middle| \mathcal{F}_t^W \right], \tag{43}$$

for any $t \leq T$. Here, $\mathcal{X}$ represents the payoff of a contingent claim. Then, the market consisting of the prices processes $(B, S_t^1, \ldots, S^N, \Pi)$ is arbitrage-free. The verification of this fact is similar to the arguments following Proposition 3.1.

Our goal in this part is to built equivalent martingale measures for the model (2) and (10). We shall need the following assumptions:

Assumption 6.1.

(i) *There exists a vector of processes* $\lambda_t := (\lambda^1(t), \ldots, \lambda^d(t))$ *such that*

$$\alpha_t^i - r(t) = \sum_{j=1}^{d} \sigma_t^{ij} \lambda_t^j, \tag{44}$$

for any $t \leq T$ *and* $i = 1, \ldots, N$*;*

(ii) *The so-called* Novikov condition *is satisfied:*

$$E^P \left\{ e^{\frac{1}{2} \int_0^T \|\lambda_u\|^2} \right\} < \infty. \tag{45}$$

The coefficient λ_j is called the *market price of risk* for the j^{th} risk factor per volatility unit. This terminology comes from the following observation. For each stock i, the risk premium $\alpha_t^i - r(t)$ can be decomposed into k components due to each of the k sources of risk. The contribution of the j^{th} source is proportional to the volatility σ^{ij} with a constant of proportionality λ^j. A large value of λ^j means a large premium per unit of volatility due to the j^{th} source of risk.

Let us recall the under the Novikov condition (45), Girsanov's theorem imply the existence of an equivalent probability measure Q^λ on $\mathcal{F}_T^W$ such that the process

$$W_t^\lambda := W_t + \int_0^t \lambda_s ds = \begin{bmatrix} W_t^1 + \int_0^t \lambda_s^1 ds \\ \vdots \\ W_t^d + \int_0^t \lambda_s^d ds \end{bmatrix}, \quad t \leq T, \tag{46}$$

is a $d-$dimensional Wiener process under Q^{λ}. It turns out that taking under Assumption 6.1 and taking λ as in (44), the probability measure Q^{λ} is an equivalent martingale measure:

Proposition 6.1. *The following statements hold true:*

(1) *In terms of W^{λ}, the dynamics of $S^1, \ldots, S^N$ are given by*

$$dS_t^i = S_t^i \left\{ r(t)dt + \sigma_t^{i\cdot} dW_t^{\lambda} \right\}, \quad i = 1, \ldots, N; \tag{47}$$

hence, the rate of return of each stock under Q^{λ} is $r(t)$

(2) *The dynamics of discounted stock price process $\tilde{S}^i$ is*

$$d\tilde{S}_t^i = \tilde{S}_t^i \sigma_t^{i\cdot} dW_t^{\lambda};$$

hence, Q^{λ} is an equivalent martingale measure for the market.

(3) *Any self-financing portfolio V^h is such that its discounted value process $\tilde{V} := B_t^{-1} V_t$ is a martingale measure under Q^{λ}.*

It is useful to write (44) in matrix form. Concretely, define

$$\alpha = \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^N \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma^{11} & \ldots & \sigma^{1,d} \\ \vdots & \vdots & \vdots \\ \sigma^{N,1} & \ldots & \sigma^{N,d} \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^d \end{bmatrix}. \tag{48}$$

Then, the market price of risk λ is such that

$$\sigma\lambda = \alpha - r(t)\mathbf{1}.$$

Thus, the existence of the market price of risk will be guaranteed (regardless of the rates of return α) if the image of $N \times d$ matrix $\sigma_t = (\sigma_t^{i,j})$ is $\mathbb{R}^N$ for any t:

$$\mathrm{Im}(\sigma) = \mathbb{R}^N \Longleftrightarrow \mathrm{Rank}(\sigma) = N.$$

A sufficient condition for this will be if $N = d$ and σ is invertible. If $d < N$ (number of risky sources is less than number of assets), there might exist rates of returns α that lead to arbitrage opportunities. A very simple instance of this arbitrage opportunities is e.g. if there are two stocks driven by independent Wiener processes and one of them has a larger rate of return.

Exercise 3. Consider the case of two assets driven by only one Wiener process. Suppose that the respective rate of returns α^i, volatilities σ^i, and short rate of return r are constant. Give necessary and sufficient conditions for the existence of the market price of risk λ. Intuitively, why will the violation of these conditions lead to arbitrage opportunities?

It turns out that the existence of the market price of risk is a necessary condition for market to be arbitrage-free.

6.3. **Hedging and completeness.**

It is possible to give precise conditions for completeness. In order to this, we need the following fundamental result of stochastic calculus:

Theorem 6.1 (**The martingale representation theorem**). *Let $\{M_t\}_{t\leq T}$ be a martingale relative to $\{\mathcal{F}_t^W\}_{t\geq 0}$. Then, there exists unique $\{\mathcal{F}_t^W\}_{t\leq T}$-adapted processes $g_1,\dots,g_k$ such that*

$$M_t = M_0 + \sum_{j=1}^{d}\int_0^t g_j(s)dW_s^j, \quad 0\leq t\leq T.$$

The following result gives a simple sufficient condition for the market to me complete.

Theorem 6.2. *Suppose that the Assumption 6.1 holds true and also that the image of the matrix σ^T is $\mathbb{R}^d$. Then, the market is complete; that is, for any claim $\mathcal{X}\in\mathcal{F}_T^W$, there exists a self-financing portfolio V such that $V_T=\mathcal{X}$.*

7. HEDGING IN MARKOV MODELS

As in the case of the Black-Scholes model (13), a Markov model allows explicit hedging strategies. Through this part, we consider the model

$$\begin{aligned} dS_t^i &= S_t^i\left\{\alpha(t,S_t)dt + \sigma^i(t,S_t)dW_t\right\}, \quad i=1,\dots,N \\ dB_t &= r(t)B_t dt, \end{aligned}$$

where $\alpha:\mathbb{R}^2\to\mathbb{R}$, $\sigma:\mathbb{R}^2\to\mathbb{R}_+$, and $r:\mathbb{R}_+\to\mathbb{R}_+$ are deterministic functions. We assume that the Assumption 6.1 is satisfied. The following is a multi-asset version of Theorem 4.1.

Theorem 7.1. *Let $c^{i,j}$ be the (i,j) component of the matrix $\sigma\sigma^T$, fix $s=(s_1,\dots,s_N)$, and $S_t=(S_t^1,\dots,S_t^N)$. Suppose that $F(t,s):\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}$ is a smooth solution of the PDE*

$$\frac{\partial F}{\partial t}(t,s) + \frac{1}{2}\sum_{i,j=1}^{N} c^{ij}(t,s)s_i s_j\frac{\partial^2 F}{\partial s_i\partial s_j}(t,s) + r(t)\sum_{i=1}^{N} s_i\frac{\partial F}{\partial s_i}(t,s) - r(t)F(t,s) = 0, \tag{49}$$

with boundary condition

$$F(T,s) = \Phi(s). \tag{50}$$

Then, the European claim $\mathcal{X}=\Phi(S_T^1,\dots,S_T^N)$ is reachable and its replicating portfolio is determined by the following positions:

$$y_t^i = \text{(Number of shares of the } i^{th} \text{ stock at time } t) = \frac{\partial F}{\partial s_i}(t,S_t), \tag{51}$$

$$x_t = \text{(Cash held in the money market account at time } t) = F(t,S_t) - \sum_{i=1}^{N} y_t^i S_t^i.$$

We can also express the DRNVF price of a European simple claim $\mathcal{X} := \Phi(S_T)$ in terms of the solution of a PDE. Concretely, suppose that λ satisfies (44-45) and Q^λ is the corresponding Girsanov's equivalent probability measure that make $\{W_t^\lambda\}_{t\leq T}$ in (46) a Wiener process. In light of Proposition 6.1, Q^λ is an equivalent martingale measure for the market $(B, S^1, \ldots, S^N)$. Consider an European simple claim with payoff $\mathcal{X} = \Phi(S_T)$ and its arbitrage-free price process induced by Q^λ:

$$\Pi_t := E^{Q^\lambda}\left\{ e^{-\int_t^T r(u)du}\Phi(S_T) \middle| \mathcal{F}_t^W \right\}.$$

Since S is a Markov process, we know that there exists a function $F(t,s)$ such that

$$\Pi_t = F(t, S_t) := E^{Q^\lambda}\left\{ e^{-\int_t^T r(u)du}\Phi(S_T) \middle| S_t \right\}.$$

As with the Feynman-Kac representation in Chapter 3, under the assumption that $F(t,s)$ is twice continuously differentiable in s and one-time differentiable in t, the function $F(t,s)$ satisfies (49-50).