MATHEMATICS OF FINANCE

CHAPTER 2
DISCRETE-TIME MODELS

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CONTENTS

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1. THE ONE-PERIOD BINOMIAL MARKET MODEL

In this part, we introduce a toy model for a financial market with the goal of motivating the key concepts and methods of arbitrage-free pricing. We shall see that this basic model can be improved substantially by “concatenation".

The one-period Binomial market model is determined by the following conditions:

(1) There are two primary assets: a risky asset (hereafter, called *stock*) and a bond, also called risk-free asset.
(2) Two business times: the initial time $t = 0$ where all trading takes place, and the final time $t = 1$, where the asset prices change.
(3) **(Dynamics of the stock:)** The initial price S_0 of one share of the stock is a known constant. The final price S_1 of one share could be either uS_0 with probability p_u, or dS_0 with probability p_d, where $d < u$ *are known constants*:
$$S_1 = \begin{cases} uS_0, & \text{with probability } p_u \\ dS_0, & \text{with probability } p_d. \end{cases}$$
(4) **(Dynamics of the bond:)** The zero-coupon bond has known constant rate of return R during $[0, 1]$. By convention[1], the holder of one unit of the bond, worth $B_0 = 1$ at time $t = 0$, receives
$$B_1 = 1 + R,$$
at time $t = 1$.
(5) Trading takes place only at time $t = 0$ under the following conditions (that we refer from now on as *frictionless market conditions*):
    (a) *Short-positions* are allowed.
    (b) Assets are sell at the same bid/offer price.
    (c) No transaction costs.
    (d) One can buy/sell any amount (even fractional) of the assets.

It is convenient to think of the market as a random experiment with two possible outcomes $\Omega = \{u, d\}$. The sample space is equipped with a probability measure P, called the *statistical probability measure*, such that $P(\{u\}) := p_u$ and $P(\{d\}) := p_d$. In that case, S_1 is a random variable on Ω defined by
$$S_1(u) = uS_0, \quad S_1(d) = dS_0.$$
Hence, we can think of u as *“bull market"* state, while d as a *“bear market"* state.

Naturally, the most fundamental purpose of a market is trading. Let us formally define a *trading strategy*.

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[1] Notice that this definition of bond differs from the zero-coupon bond notation of Chapter 1 with face value 1. However, this bond and the one of Chapter 1 both have rate of return R if $P(0, T) = 1/(1 + R)$.

Definition 1.1. A *trading strategy* or *portfolio* is determined by a pair of numbers $h := (x, y)$ representing, respectively, the number of bonds and shares held at time $t = 0$. The value of the portfolio is defined by

$$V_t := V_t^h := xB_t + yS_t, \quad t = 0, 1.$$

We proceed to analyze the different aspects of the one-period Binomial market model.

1.1. **Absence of arbitrage.**

The following definition clearly expresses our intuitive idea of arbitrage as a trading opportunity to make money without any initial net investments and without any risk of loosing money.

Definition 1.2. We say that a trading strategy $h = (x, y)$ is an *arbitrage opportunity* if

(1) $V_0^h = 0$;
(2) $V_1^h \geq 0$, but V_1^h is not identically zero.

Intuitively, the risky asset shouldn't always outperform bond. Indeed, if $1 + R \leq d < u$, then one should borrow money to invest in the stock and be able to make money. Concretely, consider the trading strategy $y = 1$ and $x = -S_0$ so that $V_0 = 0$ and

$$V_1 = \begin{cases} S_0(d - (1 + R)), & \text{in a bear market} \\ S_0(u - (1 + R)), & \text{in a bull market,} \end{cases}$$

which is never negative and sometimes is even positive: *an arbitrage opportunity*. By a similar argument, if $1 + R \geq u > d$, one should be able to make money without any initial net investment by going short in the stock and long in the bond. In summary, a necessary condition for the one-period binomial market model to be arbitrage-free is

$$d < 1 + R < u. \tag{1}$$

One can also see that it also sufficient.

Exercise 1. Show that (1) is a sufficient condition for the absence of arbitrage.

1.2. **Pricing of contingent claims via hedging.**

In a one-period market model, a (European) contingent claim is only determine by its payoff $\mathcal{X}$ at time $t = 1$, which is a function of the stock price:

$$\mathcal{X} := \Phi(S_1).$$

It is convenient to see $\mathcal{X}$ as just a random variable on the underlying sample space $\Omega = \{u, d\}$ where S_1 is defined. For instance, $\mathcal{X}(u)$ is the payoff of the claim if the stock goes up by a factor u.

As it was previously mentioned, there are two fundamental problems related to contingent claims: hedging and pricing. These problems are actually connected as a replicable claim $\mathcal{X}$ should be priced according to the initial net investment of the replicating portfolio. Concretely, if there exists a portfolio $h = (x, y)$ such that

$$V_1^h = \mathcal{X},$$

then any price $\Pi(0)$ for the claim $\mathcal{X}$ different from V_0^h will lead to an arbitrage opportunity.

By equating the final value of a portfolio with the payoffs of the derivative, the replicating portfolio of a contingent claim $\mathcal{X}$ is given by

$$y = \frac{\mathcal{X}(u) - \mathcal{X}(d)}{S_1(u) - S_1(d)}, \quad \text{and} \quad x = \frac{1}{1+R}\frac{S_1(u)\mathcal{X}(d) - S_1(d)\mathcal{X}(u)}{S_1(u) - S_1(d)}. \tag{2}$$

Then, the arbitrage-free price of the contingent claim is given by

$$V_0 = x + yS_1, \tag{3}$$

with x and y given by (2). Indeed, if for instance an agent were willing to buy and sell the contingent claim $\mathcal{X}$ at a price $F_0 > V_0$, then an investor could sell the claim to the agent at time $t = 0$ for F_0, hedge away the position in the option with the replicating strategy, and put the reminder $F_0 - V_0$ in bonds. This strategy requires no initial net investment, but produces a final positive gain for sure. The case $F_0 < V_0$ is similar.

A market where any contingent claim is replicable is said to be *complete*. We conclude the following result:

Proposition 1.1. *The one-period Binomial market model is complete.*

1.3. **Pricing via risk-neutral discounted payoff expectation.**

The arbitrage-free price obtained in the previous part has a very natural interpretation. Upon substitution of (2) into (2.1), we can write

$$V_0 = Q(u) \cdot \frac{\mathcal{X}(u)}{1+R} + Q(d) \cdot \frac{\mathcal{X}(d)}{1+R}, \tag{4}$$

where

$$Q(u) = \frac{1+R-d}{u-d}, \quad Q(d) = \frac{u-(1+R)}{u-d}. \tag{5}$$

Under the arbitrage-freeness condition (1), one can interpret Q as a probability measure in the sample space $\Omega = \{u, d\}$. The formula (5) can be viewed as the *expected discounted payoff of the claim* under the probability measure Q, denoted by

$$V_0 = E^Q\left\{\frac{\mathcal{X}}{1+R}\right\}.$$

The probability measure Q in turn has another natural financial interpretation. Notice that

$$R = E^Q\left\{\frac{S_1 - S_0}{S_0}\right\}. \tag{6}$$

Hence, under Q, *the expected rate of return in the risky asset is the same as the rate of return in the risk-free asset*. The measure Q is said to be the *risk-neutral probability measure* and (5) is called the *risk-neutral valuation formula*.

The formula (6) is often write in the following equivalent manner:

$$S_0 = E^Q \left\{ \frac{S_1}{B_1} \right\}. \tag{7}$$

Exercise 2. Show that Q is the unique probability measure on Ω satisfying either (6) or (8).

2. THE FUNDAMENTAL THEOREMS OF FINANCE - THE GENERAL ONE-PERIOD MODEL

In this part we generalized the previous model in two ways: we consider multiple assets and multiple states of the market. We shall show that the apparently coincidental connection between arbitrage-freeness and the existence of a risk-neutral probability measure is quite general. This result is known as *the first fundamental theorem of finance* (FFTF). Furthermore, the observation in Exercise 2 will turn out to be related to the completeness of the market, a result known as *the second fundamental theorem of finance* (SFTF).

Again, we consider only two business times $t = 0, 1$. There are N assets, with respective price processes $S_t^1, \ldots, S_t^N$. As usual, the initial asset prices $S_0^1, \ldots, S_0^N$ are known to all agents, while the final prices $S_1^1, \ldots, S_1^N$ are random variables defined in a sample space Ω endowed with a probability measure P. We assume that Ω can take M possible states $\{\omega_1, \ldots, \omega_M\}$, each with positive probability to occur, and the potential possible values of the assets are known. In other words, the $N \times M$ matrix below is known

$$D := \begin{bmatrix} S_1^1(\omega_1) & \ldots & S_1^1(\omega_M) \\ \vdots & \ldots & \vdots \\ S_N^1(\omega_1) & \ldots & S_N^1(\omega_M) \end{bmatrix}. \tag{8}$$

In the one-period Binomial market model, $N = 2$ and $M = 2$, and

$$D := \begin{bmatrix} 1+R & 1+R \\ uS_0 & dS_0 \end{bmatrix}.$$

We assume that the *frictionless market conditions* of Section 1 holds.

The concept of trading strategy generalizes in a straightforward way. For instance, a trading strategy or portfolio is defined in terms of an N-tuple $h := (h^1, \ldots, h^N)$ representing the holdings of assets i in the portfolio at time $t = 0$. Thus, the value process is defined by

$$V_t := V_t^h := \sum_{i=1}^{N} h^i S_t^i, \quad t = 0, 1.$$

The definition of *arbitrage opportunity* is exactly the same as in Definition 1.2.

The *first fundamental theorem of finance* (FFTF) is a milestone in the development of mathematical finance. It characterizes the arbitrage-freeness of market model in terms of the existence of a *risk-neutral probability measure*.

Definition 2.1. A probability measure Q on $\Omega = \{\omega_1, \ldots, \omega_N)$ is said to be an equivalent *risk-neutral measure* for the stock price processes $S^1, \ldots, S^N$ if (i) $Q(\omega_j) > 0$ for all $j = 1, \ldots, M$, and there exists a constant R such that

$$(ii) \quad R := E^Q\left\{\frac{S_1^i - S_0^i}{S_0^i}\right\}, \quad i = 1, \ldots, N. \tag{9}$$

One can interpret R above as the risk-neutral expected rate of return of the market. Equivalently, one can write (9) as follows:

$$(ii) \quad S_0^i := E^Q\left\{\frac{S_1^i}{1+R}\right\}, \quad i = 1, \ldots, N. \tag{10}$$

Theorem 2.1 (**[The first fundamental theorem of finance**). *The stochastic market* $(S^1, \ldots, S^n)$ *is arbitrage-free if and only if there exists an equivalent risk-neutral measure* Q.

The sufficiency part of the FFTF is not very hard and it is left as an exercise below. For the necessity, please refer to Appendix A

Exercise 3. Suppose that market admits an equivalent risk-neutral measure Q.

(a) Show that for any risk-neutral probability measure Q and any trading strategy h,

$$\mathbb{E}^Q\left\{\frac{V_1^h}{1+R}\right\} = V_0^h.$$

(b) Use this fact to show that the market is arbitrage-free.

The FFTF has several implications for the pricing of contingent claims. Consider a contingent claim that pays the amount $\mathcal{X}(\omega_j)$ at time $t = 1$ if the market takes the state ω_j.

Proposition 2.1. *The only prices* $\Pi(0)$ *for the contingent claim* $\mathcal{X}$ *which are consistent with the absence of arbitrage are of the form*

$$\Pi(0) := \frac{1}{1+R} E^Q\{\mathcal{X}\}, \tag{11}$$

for a risk-neutral measure Q.

Indeed, consider the expanded market consisting of the primary assets S^j, $j = 1, \ldots, N$, and the contingent claim with time $t = 0$ price $\Pi(0)$. We can think of the contingent claim as an additional asset, say S^{N+1}, such that

$$S_0^{N+1} = \Pi(0), \quad S_1^{N+1} = \mathcal{X}.$$

Then, this expanded market is arbitrage-free if and only if there exists a probability measure Q satisfying (9) for $j = 1, \ldots, N+1$. In particular, Q is a risk-neutral measure for $S^1, \ldots, S^N$, and also,

$$R = E^Q \left\{ \frac{S_1^{N+1} - S_0^{N+1}}{S_0^{N+1}} \right\},$$

which simplifies to (11).

Notice that Proposition 2.1 does not mean that there exists a unique arbitrage-free price for a claim as there might be more than one risk-neutral measures Q. However, if the contingent claim is replicable by a portfolio h, then the only arbitrage-free price should be the initial value of the portfolio. The following result shows that this is indeed the case even if the risk-neutral measure is not unique. Recall that $h = (h^1, \ldots, h^N)$ is a *replicating* or *hedging portfolio* for $\mathcal{X}$ if

$$V_1^h(\omega_j) = \mathcal{X}(\omega_j),$$

for all possibles states ω_j of the market.

Exercise 4. Suppose that the market is arbitrage-free. If $\mathcal{X}$ is replicable, then

$$V_0^h = \frac{1}{1+R} E^Q \{\mathcal{X}\},$$

for all replicating portfolio h and all equivalent risk-neutral measures Q.

The *second fundamental theorem of finance* (SFTF) concerns with complete markets. As before, a *complete market* is one where all contingent claims are replicable.

Theorem 2.2 (The second fundamental theorem of finance). *Suppose that the financial market $(S^1, \ldots, S^N)$ is arbitrage-free. Then, the market is complete if and only if there exists a* unique *equivalent risk-neutral measure Q.*

It is useful to go over a few points of the proof which, by the way, relies completely on well-known results of linear algebra. Notice that if Q is risk-neutral measure then

$$D\mathbf{q} = \mathbf{b},$$

where D is the matrix of final prices in (8),

$$\mathbf{q} = [Q(\omega_1), \ldots, Q(\omega_M)]^T, \quad \text{and} \quad \mathbf{b} = [(1+R)S_0^1, \ldots, (1+R)S_0^N]^T.$$

Also, notice that $\mathcal{X}$ is replicable if there exists a vector $\mathbf{h} = [h^0, \ldots, h^N]^T$ such that

$$D^T \mathbf{h} = \mathbf{x},$$

where $\mathbf{x} = [\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_M)]^T$. Then, the market is complete if and only if the rank of D^T is M (or what is the same, the image of D^T is $\mathbb{R}^M$). On the other hand, we recall that a solvable system $D\mathbf{q} = \mathbf{p}$ has a unique solution if and only if nullity of D is 0. Finally, since rank$(D^T) =$ rank(D), and

$$\text{nullity}(D) + \text{rank}(D) = \# \text{ of columns of } A,$$

we conclude that there exists a unique risk-neutral measure if and only if the market is complete.

Notice that if $N < M$, then it is impossible that the market is complete. We can interpret this remarks as follows:

> If number of sources of randomness M is more than number of assets N, then the market is incomplete.

3. THE MULTI-PERIOD BINOMIAL MARKET MODEL

In the two previous sections, we have built enough intuition about the pricing of contingent claims via hedging and risk-neutral valuation formula. At the same time, we have shown the close connection between arbitrage-freeness and completeness and the existence of a unique-risk neutral measure. We are ready to see a more realistic stock market model, which in turn will work later on as a discrete-time approximation of the most important model of mathematical finance: the celebrated Black-Scholes model.

The multi-period Binomial market model is essentially built by concatenation or chaining of the one-period model of Section 1 as follows:

1. There are $T+1$ trading times $t = 0, \dots, T$.
2. **(Dynamics of the stock:)** There is a risky asset, say a stock. The initial stock price S_0 per share is known at time $t = 0$. At any time $t = 1, \dots, T$, the stock price can move up by a factor u with probability p_u or move down by a factor d with probability p_d, independently from the past history of the stock price. Concretely, we assume that
$$S_t = Z_t S_{t-1},$$
where $Z_1, \dots, Z_n$ are independent random variables such that
$$Z_t = \begin{cases} u, & \text{with probability } p_u \\ d, & \text{with probability } p_d. \end{cases}$$
3. **(Bond market or the money-market account:)** At each time $t = 0, \dots, T-1$, there is available a zero-coupon bond maturing at the next trading time with rate of return R. Equivalently, we assume that there exists a *money-market account* for borrowing or lending of money at a constant interest rate R (per unit time and per dollar).
4. The *frictionless market conditions* of Section 1 holds.

For future reference, we think of the stock prices $\{S_t : t = 0, \dots, T\}$ as random variables defined on the sample space $\Omega := \{(w_1, \dots, w_T) : w_i = 0, 1\}$ as follows:

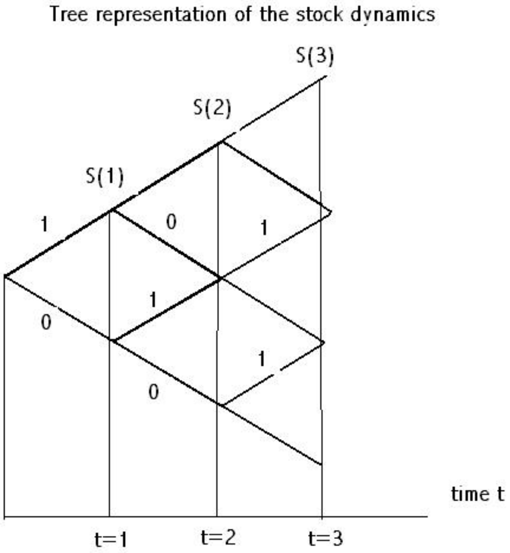
$$S_t(w_1, \dots, w_T) := u^{w_t} d^{1-w_t} S_{t-1}(w_1, \dots, w_T) = S_0 u^{\sum_{i=1}^t w_i} d^{t - \sum_{i=1}^t w_i},$$

(under the convention that $\sum_{i=1}^0 = 0$). Notice that Z_t is defined by $Z_t(w_1, \dots, w_T) = u^{w_t} d^{1-w_t}$. The sample space Ω is equipped with the probability measure

$$P((w_1, \dots, w_n)) := p_u^{\sum_{i=1}^T w_i} p_d^{\sum_{i=1}^T (1-w_i)}.$$

It is an easy exercise to check that P is the right probability measure that makes $Z_1, \dots, Z_T$ independent with $P(Z_t = u) = p_u$.

The best representation of the stock price dynamics is in terms of a tree such as the following:



We can think of each outcome $(w_1, \dots, w_T) \in \Omega$ as a path of the tree from its root to one of the terminal nodes or what is the same, a path or trajectory of the stock price. For a three period model ($T = 3$), there are 8 possible paths of the stock price, even though at the final time, there are only four possible final stock prices. The outcome $(1, 0, 1)$ is associated with the following path: the stock price first moves upward, then downward, and finally upward again. Hence, $S_2(1, 0, 1) = S_0 u^2 d$.

3.1. **Trading strategies, self-financing portfolios, and hedging.**

Again, trading strategies will play a crucial role in the sequel.

Definition 3.1. A *trading strategy* or *portfolio* is determined by a sequence of random bivariates:

$$\{h_t := (x_t, y_t) : t = 1, \dots, T\},$$

such that x_t and y_t is a function of $S_0, \dots, S_{t-1}$. The variables x_t and y_t represent the amount of money invested in the money-market account at time $t - 1$ and the number of shares of stock held at time $t - 1$, respectively. The time-t *value of the portfolio* is defined by

$$V_t := V_t^h := \begin{cases} x_1 + y_1 S_0, & t = 0 \\ x_t(1 + R) + y_t S_t, & t = 1, \dots, T. \end{cases} \tag{12}$$

The value of the portfolio at time $t = 0$ is quite natural. For $t = 1, \dots, T$, V_t represents the value of the portfolio before transactions or rebalancing of the portfolio takes place, but after the asset prices change and the interest-rate is paid. Also, notice that by definition, V_t depends only on $(w_1, \dots, w_t)$, the trajectory of the stock up to time t.

In order to define arbitrage-opportunities and hedging strategies, we will focus on those trading strategies that does not require any intermediate infusion or withdraw of money into the portfolio.

Definition 3.2. We say that a trading strategy h is *self-financing* if the following condition holds true:

$$V_t^h = x_{t+1} + y_{t+1} S_t, \tag{13}$$

for $t = 1, \dots, T-1$.

Notice that the right-hand side on (13) is the amount of money needed at time t to hold the position x_{t+1}, y_{t+1}, while the left-hand side is the money available from the previous position. Hence, condition (13) means that no extra money is needed to keep the portfolio running and also, that all the money is invested in the market. t is not hard to see that the previous definition of self-financing is equivalent to

$$V_{t+1} - V_t = x_{t+1} R + y_{t+1}(S_{t+1} - S_t), \tag{14}$$

for $t = 0, \dots, T-1$.

Once we have specified the relevant trading strategies, the definitions of arbitrage and hedging strategies are straightforward:

Definition 3.3. An arbitrage opportunity is a *self-financing* trading strategy h such that

(1) $V_0^h = 0$;
(2) $V_T^h \geq 0$;
(3) $P(V_T^h > 0) > 0$.

Definition 3.4.

(1) A *contingent claim* is a contract between two parties where, for a suitable initial price $\Pi(0, \mathcal{X})$, the holder receives a random payoff $\mathcal{X}(w_1, \dots, w_T)$ at time T if the state of the market is $(w_1, \dots, w_T) \in \Omega$.
(2) The claim is said to be *simple* if $\mathcal{X} = \Phi(S_T)$ for a function Φ.
(3) We say that the claim is *replicable* or *reachable* if there exists a self-financing trading strategy h such that
$$V_T^h = \mathcal{X},$$
regardless of the state of the market. The portfolio h is called the *hedging portfolio*.

Based upon our experience with the one-period model, it is expected that a necessary condition for the market to be arbitrage-free is that

$$d < 1 + R < u, \tag{15}$$

otherwise one could devise an arbitrage-opportunity at each period. It is expected that the above condition is sufficient too, but we shall analyze these issues in a more systematic manner later on. For the moment we dealt with the hedging and pricing problem using a very intuitive technique called *backward induction*.

3.2. **The binomial algorithm: European and American Options.**

The binomial algorithm is an iterative method to determine the hedging strategy and the price of a claim. Given that the final goal of a hedging strategy h is that

$$V_T^h(w_1, \ldots, w_T) = \mathcal{X}(w_1, \ldots, w_T),$$

the algorithm works backwards to determine both the necessary value of the hedging portfolio and the arbitrage-free price of the claim at the previous trading time. The following result is the key theoretical justification of the algorithm:

Proposition 3.1. *Let*

$$q_u = \frac{1 + R - d}{u - d}, \quad q_d = \frac{u - (1 + R)}{u - d}.$$

Then, a trading strategy $h = \{h_t = (x_t, y_t) : t = 1, \ldots, T\}$ *is a self-financing if and only if its value process* $\{V_t^h : t = 0, \ldots, T\}$ *defined by (12) is such that*

$$V_{t-1}^h(w_1, \ldots, w_{t-1}) = q_u \cdot \frac{V_t^h(w_1, \ldots, w_{t-1}, 1)}{1 + R} + q_d \cdot \frac{V_t^h(w_1, \ldots, w_{t-1}, 0)}{1 + R}, \tag{16}$$

for any time $t = 1, \ldots, T$ *and any state* $(w_1, \ldots, w_{t-1})$.

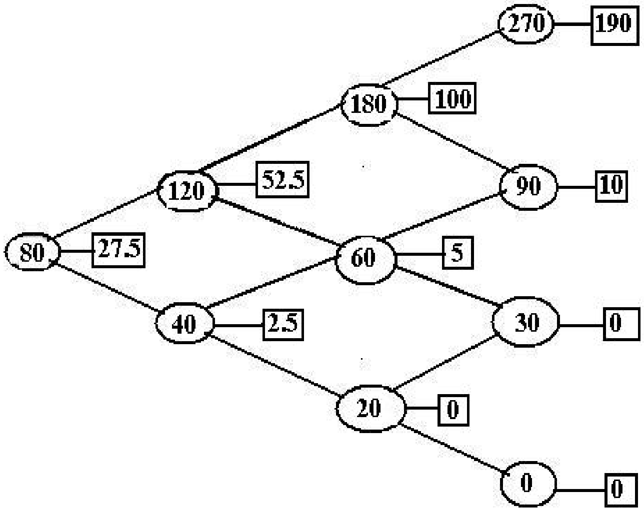
We postpone the verification of the above result to Section 3.3.1. We now illustrate the power and heuristics of the Binomial algorithm with examples.

3.2.1. *Simple European claims.*

Consider a three-period Binomial market model with initial stock price $S_0 = 80$, $u = 1.5$, and $d = .5$, and interest rate $R = 0$. We want to determine the hedging strategy and the price of a European call option with expiration time $T = 3$ and strike $K = 80$. Notice that

$$q_u = \frac{1 + R - d}{u - d} = \frac{1}{2}, \quad q_d = \frac{1}{2}.$$

The following tree describes the stock prices and the prices of the option at each time.



Let us describe how this tree is obtained and the very natural intuition behind the Binomial algorithm:

(1) At maturity $T = 3$, the option price is obviously the same as the payoff of the option. Thus, if the final asset price is say 90, the option is worth $\mathcal{X} = (S_T - K)_+ = 10$.

(2) Next, we determine the price of the option at time $t = T - 1 = 2$.

   (a) Suppose that the stock price is 180 at that moment. How much is worth the option? The option can give only two possible payoffs at maturity: either 190 or 10. From the one period case, one can easily find a replicating portfolio for such a payoff profile with a net investment of

$$w = \frac{1}{2} \cdot 190 + \frac{1}{2} \cdot 10 = 100.$$

   The buyer of the option won't be willing to pay any price higher than w as he can always go into the market and obtain the same payoffs as the option. The seller should charge at least w, as this price will allow him to hedge away the risk of losing money by using the follwing portfolio:

$$y = \frac{190 - 10}{270 - 90} = 1, \quad x = 100 - (1)180 = -80.$$

   We conclude that the time $t = 2$ price of the option and the hedging portfolio if the current price is $S_2 = 180$ are

$$\Pi(2) = V_2 = 100, \quad x_3 = -80, \quad y_3 = 1.$$

   (b) We proceed in a similar manner for each of the other possible stock prices at time $t = 2$:

S_2	180	60	20
$\Pi(2)$	100	5	0
y_3	1	1/6	0
x_2	-80	-5	0

(3) We determine the price of the option at time $t = 1$.
  (a) Suppose that the stock price is 120 at that moment. Same question: How much is worth the option in the case? The next trading day, the option will be worth either 100 or 5 (which can be thought as payoffs since one can sell the option for those prices). The net initial investment of the corresponding replicating portfolio is

$$V_2 = \Pi(2) = \frac{1}{2} \cdot 100 + \frac{1}{2} \cdot 5 = 52.5.$$

  The hedging portfolio for a the writer will be

$$y(2) = \frac{100 - 5}{180 - 60} \approx .79, \quad x(2) = 52.5 - \frac{19}{24} \cdot 120 \approx 42.5.$$

  We conclude that the time $t = 2$ price of the option and hedging portfolio if the current price is $S_2 = 180$ are

$$\Pi(2) = V_2 = 100, \quad x_3 = -80, \quad y_3 = 1.$$

  (b) We proceed in a similar manner for each of the other possible stock prices at time $t = 1$.

(4) Finally, at the initial time $t = 0$, the arbitrage-free price of the option is

$$\Pi(0) = \Pi(2) = \frac{1}{2} \cdot 52.5 + \frac{1}{2} \cdot 2.5 = 27.5.$$

Thus, the hedging portfolio is

$$y(1) = \frac{52.5 - 2.5}{120 - 40} = \frac{5}{8}, \quad x(1) = 27.5 - \frac{5}{8} \cdot 80 = 22.5.$$

We summarize the Binomial algorithm for a simple European contingent claim $\mathcal{X} = \Phi(S_T)$ through the following two steps:

(1) At time $t = T$, the claim's price is

$$\Pi(T; s) = \Phi(s),$$

when $S_T = s$ for each $s \in \{S_0 u^T, S_0 u^{T-1} d, \ldots, S_0 d^T\}$;

(2) For each $t = T - 1, \ldots, 1$, in that order,
  (a) the time t price of the claim when $S_t = s$ is given by

$$\Pi(t; s) = q_u \cdot \frac{\Pi(t+1; us)}{1+R} + q_d \cdot \frac{\Pi(t+1; ds)}{1+R},$$

  for each $s \in \{S_0 u^t, S_0 u^{t-1} d, \ldots, S_0 d^t\}$;

(b) the time t hedging strategy for the claim when $S_t = s$ is given by

$$y_t(s) = \frac{\Pi(t+1; us) - \Pi(t+1; ds)}{us - ds},$$
$$x_t(s) = \Pi(t; s) - y_t s.$$

for each $s \in \{S_0 u^t, S_0 u^{t-1} d, \ldots, S_0 d^t\}$;

We remark that by definition, the above strategy $h = \{h_t = (x_t, y_t) : t = 1, \ldots, T\}$ is such that

$$V_t^h = \Pi(t), \quad \text{and}$$
$$V_{t-1}^h(w_1, \ldots, w_{t-1}) = q_u \cdot \frac{V_t^h(w_1, \ldots, w_{t-1}, 1)}{1+R} + q_d \cdot \frac{V_t^h(w_1, \ldots, w_{t-1}, 0)}{1+R},$$

for any $t = 1, \ldots, T$. From Proposition 3.1, the strategy h is self-financing and hence, a replicating for the claim $\mathcal{X}$.

3.2.2. *Path-dependent European claims.*

Consider the three-period Binomial market model of the previous section with $S_0 = 80$, $u = 1.5$, and $d = .5$, and interest rate $R = 0$. We want to price of a type of call option with a special provision: the contract is canceled if the stock price ever goes below a level $H = 55$. As before we set the strike to be $K = 80$. This is a special type of path-dependent derivative, called a *knockout barrier call option* or *down-and-out call option*.

The analysis of the previous part will essentially hold in this context with the additional feature that the prices and hedging portfolio are path dependent. For instance, at maturity $T = 3$ there are eight possible paths with the following payoffs:

Path	(1,1,1)	(1,1,0)	(1,0,1)	(1,0,0)	(0,1,1)	(0,1,0)	(0,0,1)	(0,0,0)
Payoff	190	10	10	0	0	0	0	0

The price of the option at maturity will of course coincide with these payoff.

The next step of the backward induction is to determine the price at $t = T - 1 = 2$. However, in this kind of contingent claims, it does not suffices to fix the current price because it is also relevant how the stock price got there. Suppose that at time $t = 2$, the price of the stock is 60 coming from a time $t = 1$ price of 120. Then, there are two possible payoffs 10 or 0 at time $t = 3$, which can be replicated with a net initial net investment of

$$w = \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 = 5.$$

Hence, the time $t = 2$ price of the option and the hedging portfolio, *for that particular path*, is

$$\Pi(2; 1, 0) = V_2(1, 0) = 5, \quad x_3(1, 0) = \frac{1}{6}, \quad y_3(1, 0) = 5 - \frac{1}{6} \cdot 60 = -5.$$

However, if at the time $t = 2$ the price of the stock is 60 coming from a time $t = 1$ price of 40, then the possible payoffs at $t = 3$ are zero. Hence,

$$\Pi(2; 0, 1) = V_2(0, 1) = 0, \quad x_3(0, 1) = 0, \quad y_3(0, 1) = 0.$$

The following table shows the time $t = 2$ option prices:

Paths at $t = 2$	(1,1)	(1,0)	(0,1)	(0,0)
Option Prices V_2	100	5	0	0

The time $t = 1$ prices are now given by $V_1(1) = 52.5$ and $V_1(0) = 0$. Hence, the initial price of the down-and-out call option is $V_0 = 52.5/2 = 26.25$, rather than 27.5 as the vanilla call option of the previous section.

The following program summarizes the general steps of the Binomial algorithm for a European contingent claim $\mathcal{X}(w_1, \dots, w_T)$:

(1) At time $t = T$, the claim's price coincides with the payoff:

$$\Pi(T; w_1, \dots, w_T) = \mathcal{X}(w_1, \dots, w_T),$$

for all paths $(w_1, \dots, w_T) \in \Omega$;

(2) For each $t = T - 1, \dots, 1$, in that order,

  (a) the time t price of the claim given the past path evolution $(w_1, \dots, w_t)$ is

$$\Pi(t; w_1, \dots, w_t) = q_u \cdot \frac{\Pi(t+1; w_1, \dots, w_t, 1)}{1+R} + q_d \cdot \frac{\Pi(t+1; w_1, \dots, w_t, 0)}{1+R}.$$

  (b) the time t hedging strategy for the claim given the past path evolution $(w_1, \dots, w_t)$ is

$$\begin{aligned} y_t(w_1, \dots, w_t) &= \frac{\Pi(t+1; w_1, \dots, w_t, 1) - \Pi(t+1; w_1, \dots, w_t, 0)}{S_t(w_1, \dots, w_t, 1) - S_t(w_1, \dots, w_t, 0)}, \\ x_t(w_1, \dots, w_t) &= \Pi(t; w_1, \dots, w_t) - y_t(w_1, \dots, w_t) S_t(w_1, \dots, w_t). \end{aligned}$$

In light of Proposition 3.1, the above strategy $h = \{h_t = (x_t, y_t) : t = 1, \dots, T\}$ is a (self-financing) hedging strategy since by construction

$$V_T^h = \mathcal{X}, \quad \text{and} \tag{17}$$

$$V_{t-1}^h(w_1, \dots, w_{t-1}) = q_u \cdot \frac{V_t^h(w_1, \dots, w_{t-1}, 1)}{1+R} + q_d \cdot \frac{V_t^h(w_1, \dots, w_{t-1}, 0)}{1+R}. \tag{18}$$

3.2.3. *American claims.*

Consider again a three-period Binomial market model. The parameters are given as follows:

$$S_0 = 100, \quad u = 1.1, \quad d = 1/u, \quad R = 0.$$

Notice that the one-period risk-neutral probability is approximately

$$q_u = .4762, \quad q_d = .5238.$$

We want to determine the hedging strategy and price of an American put option with strike $K = 100$. In other words, the holder of the option has the right to exercise the option at any time prior to expiration or at expiration. How does this early exercise feature affects the price of the option?

In addition to the two traditional problems of pricing and hedging, there is a new important problem for an American option: determine when to optimally exercise the option. Backward induction is again an effective method to solve these problems.

At maturity $T = 3$, the price of the option coincides with the payoff of the option (which depend only upon the stock price at that moment):

S_T	133.1	110	90.91	75.13
$\Pi(2) = \mathcal{X}$	0	0	9.09	24.87

As before the next step is to determine the option prices at time $t = 2$ depending on wether S_2 is 121, 100, or 82.64, the possible time $t = 2$ stock prices. Suppose that the stock price is 82.64. The holder of the option has two possible actions:

- If the holder decides to hold the option, he/she can aspire to a payoff of either 9.09 or 24.87. This one-period claim is worth:
$$w = .4762 \cdot 9.09 + .5238 \cdot 24.87 = 17.35.$$
- On the other hand, if the holder decides to exercise at the moment, he/she would receive an immediate payoff of $100 - 82.64 = 17.36$ dollars.

Even though it is a small difference (which can get bigger if the holder buy a large number of options), it is optimal for the holder to exercise the option at that time. Hence, the option is worth its immediate payoff 17.36 dollars. Of course, the writer won't be willing to charge less than this amount if he want to hedge his position.

Let us assume that the stock price is at the level 100 at time $t = 2$:

- If the holder of the option decides to hold, he/she can aspire to a payoff of either 0 or 9.0909, which is worth
$$w = .4762 \cdot 0 + .5238 \cdot 9.09 = 4.76.$$
- If the holder decides to exercise, he/she would receive nothing.

Then, the holder shouldn't exercise at that moment and he shouldn't be willing to pay more than 4.76 for the option. The writer will ask for no less than 4.76, hence this is the arbitrage-free price. The time $t = 2$ option price for the other stock price is 0. We obtain the following time $t = 2$ option prices:

S_2	121	100	82.64
$\Pi(2)$	0	4.76	17.36

Let us determine the option prices at time $t = 1$ for each of the two possible stock prices: 90.91 and 110. Suppose that the stock price is at the level of 90.91:

- If the holder decides to wait, he/she can get either 4.76 or 17.36. Such a one-period claim is worth
$$\Pi(1) = .4762 \cdot 4.76 + .5238 \cdot 17.36 = 11.36$$

- If instead the holder exercise the option, he/she will get $100 - 90.91 = 9.09$.

Thus, it is optimal to hold the option, and the price of the option at time $t = 1$ when $S_1 = 90.91$ is $\Pi(1) = 11.36$. If $S_1 = 110$, it is of course optimal to hold and receive either 0 or 4.76. We obtain the following time $t = 1$ option prices:

S_1	110	90.91
$\Pi(1)$	2.26	11.36

Finally, the arbitrage-free price of the American option at $t = 0$ is

$$\Pi(0) = .4762 \cdot 2.26 + .5238 \cdot 11.36 = 7.02.$$

The previous arguments yield not only the price and hedging strategy of the American option, but also the optimal exercise policy for the holder. We saw that the buyer of the option should use the early exercise feature at time $t = 2$ if the stock price at that moment is 82.64. The following table shows the optimal exercise time τ for each of possible stock price evolutions. Below, the value $\tau = \infty$ means that the holder will never exercise the option. For instance, in the path $(1, 1, 0)$, the final stock price is 110 and it would be foolish to exercise the option and sell the stock for 100.

Path	(1,1,1)	(1,1,0)	(1,0,1)	(1,0,0)	(0,1,1)	(0,1,0)	(0,0,1)	(0,0,0)
Payoff	$\tau = \infty$	$\tau = \infty$	$\tau = \infty$	$\tau = 3$	$\tau = \infty$	$\tau = 3$	$\tau = 2$	$\tau = 2$

The case of an American contingent claim can be treated is a similar manner. Recall that in an American contingent claim, the holder can choose to receive the payoff at a time prior to maturity. If the buyer decides to exercise at time $\tau \in \{0, \ldots, T\}$, he will receive a payoff $\mathcal{X}_\tau$. The process $\{\mathcal{X}_t : t = 1, \ldots, T\}$ is called the *immediate payoff process*. The immediate payoff $\mathcal{X}_t$ at time t is a function of the underlying stock price up to that time:

$$\mathcal{X}_t = \Phi(S_0, \ldots, S_t),$$

for a function Φ. In the case that the immediate payoff at time t depends only on the price of the underlying at that moment, we say that the contingent claim is *simple*. Since a contingent claim entails a one time payoff at some time of the duration of the contract, the exercise time τ can take only values on $\{0, 1, \ldots, T\}$ and $\tau = \infty$ is not allowed. In other words, the holder of the contingent claim receives the payoff $\mathcal{X}_T$ at maturity if he/she has not done so already.

We summarize the Binomial algorithm for a simple American contingent claim with immediate payoff process $\{\mathcal{X}_t := \Phi(S_t) : t = 0, \ldots, T\}$ through the following two steps:

(1) At time $t = T$, the claim's price is

$$\Pi(T; s) = \Phi(s),$$

when $S_T = s$ for each $s \in \{S_0 u^T, S_0 u^{T-1} d, \ldots, S_0 d^T\}$;

(2) For each $t = T-1, \dots, 1$, in that order, the time t price of the claim when $S_t = s$ is given by

$$\Pi(t; s) = \max\left\{ q_u \cdot \frac{\Pi(t+1; us)}{1+R} + q_d \cdot \frac{\Pi(t+1; ds)}{1+R}, \Phi(s) \right\},$$

for each $s \in \{S_0 u^t, S_0 u^{t-1} d, \dots, S_0 d^t\}$.

3.3. **The risk-neutral measure and the risk-neutral valuation formula.**

The Binomial algorithm provides an iterative method to built the hedging portfolio and hence, to determine the arbitrage-free price of an European contingent claim. Drawing on our experience with the one-period Binomial market model, we now interpret the arbitrage-free price as the expected payoff discounted to present value, under a suitable risk-neutral measure Q:

$$\Pi(0) = E^Q \left\{ \frac{\mathcal{X}}{(1+R)^T} \right\}.$$

We shall use several properties of conditional expectations which we collect in Appendix B for reader's facility.

To deduce the risk-neutral measure Q, consider the hedging portfolio (17)-(18). Working backwards, one can see that

$$\begin{aligned} V^h_{T-1}(w_1, \dots, w_{T-1}) &= q_u \cdot \frac{\mathcal{X}(w_1, \dots, w_{T-1}, 1)}{1+R} + q_d \cdot \frac{\mathcal{X}(w_1, \dots, w_{T-1}, 0)}{1+R} \\ V^h_{T-2}(w_1, \dots, w_{T-1}) &= q_u^2 \cdot \frac{\mathcal{X}(w_1, \dots, w_{T-1}, 1, 1)}{(1+R)^2} + q_d q_u \cdot \frac{\mathcal{X}(w_1, \dots, w_{T-2}, 0, 1)}{(1+R)^2} \\ &\quad + q_d q_u \cdot \frac{\mathcal{X}(w_1, \dots, w_{T-2}, 1, 0)}{(1+R)^2} + q_d^2 \cdot \frac{\mathcal{X}(w_1, \dots, w_{T-2}, 0, 0)}{(1+R)^2}. \end{aligned}$$

By induction, we arrive to the formula

$$V^h_t(w_1, \dots, w_t) = \sum_{(w_{t+1}, \dots, w_T)} q_u^{\sum_{u=t+1}^T w_u} q_d^{\sum_{u=t+1}^T (1-w_u)} \cdot \frac{\mathcal{X}(w_1, \dots, w_t, w_{t+1}, \dots, w_T)}{(1+R)^{T-t}}. \tag{19}$$

At time $t = 0$, this formula takes the very intuitive form

$$V_0 = \Pi(0) = E^Q \left\{ \frac{\mathcal{X}}{(1+R)^T} \right\}, \tag{20}$$

with Q defined on the probability space $\Omega = \{(w_1, \dots, w_T) : w_i = 0, 1\}$ by

$$Q((w_1, \dots, w_T)) = q_u^{\sum_{u=1}^T w_u} q_d^{\sum_{u=1}^T (1-w_u)}. \tag{21}$$

Notice that under Q, the probability that the stock price goes up is q_u, while the probability that the stock goes down is q_d. The only requirement for Q to be a valid probability measure is that the condition (15) holds true.

The probability measure Q is called the *risk-neutral measure* of the market. The formula (20) is called the *risk-neutral valuation formula*, which has the following very natural interpretation:

> The price of a contingent claim is the risk-neutral expected payoff of the claim discounted to present value.

Using the definition of conditional expectation, we can also interpret (19) as the expected discounted payoff at time t given the past path $Z_1 = w_1, \dots, Z_t = w_t$ of the stock:

$$V_t^h(w_1, \dots, w_t) = E^Q \left\{ \frac{\mathcal{X}}{(1+R)^{T-t}} \middle| Z_1 = w_1, \dots, Z_t = w_t \right\}. \tag{22}$$

The above formula gives both, the time-t value of the replicating portfolio and the arbitrage-free price $\Pi(t)$ of the claim given the past path of the stock.

Example 3.1. We illustrate the use of the risk-neutral valuation formula for the down-and-out call option of Section 3.2.2. We take the same parameters, but with $R = .05$. In that case, the risk-neutral probability of an up jump is $q_u = .55$. We have the following paths, discounted payoffs, and probabilities:

Path	(1,1,1)	(1,1,0)	(1,0,1)	(1,0,0)	(0,1,1)	(0,1,0)	(0,0,1)	(0,0,0)
Payoff	190	10	10	0	0	0	0	0
Disc. Payoff	164.21	8.64	8.64	0	0	0	0	0
Risk-neutral prob.	.16	.13	.13	-	-	-	-	-

The price of the option is then given by

$$\Pi(0) = (164.21)(.16) + 2(8.64)(.13).$$

3.3.1. *The risk-neutral measure or martingale measure.*

In what sense is the probability measure Q in (21) a risk-neutral measure? We answer this question in this part. First notice that under Q, the random variables

$$Z_t(\omega_1, \dots, w_T) = u^{w_t} d^{1-w_t}, \quad t = 1, \dots, T,$$

are independent random variables with $Q(Z_t = u) = q_u$ and $Q(Z_t = d) = q_d$. Recall that

$$S_{t+1} = Z_{t+1} S_t, \quad t = 0, \dots, T-1.$$

We deduce the following property:

$$E^Q \left\{ \frac{S_{t+1} - S_t}{S_t} \middle| Z_1, \dots, Z_t \right\} = R, \tag{23}$$

for $t = 0, \dots, T-1$. Notice that the information contained in $Z_1, \dots, Z_t$ is the same as that contained in $S_1, \dots, S_t$. Thus,

$$E^Q \left\{ \frac{S_{t+1} - S_t}{S_t} \middle| S_1, \dots, S_t \right\} = R. \tag{24}$$

We can interpret (24) as risk-neutral formula similar to (6). Under Q, the expected rate of return of the stock at any time period given the past stock evolution is constant, and equal to the risk-free rate of return. Notice that by the tower property:

$$E^Q\left\{\frac{S_{t+1}-S_t}{S_t}\right\} = R.$$

We can simplify even further (24) since given S_t, the random variable S_{t+1} is independent of $S_0, \dots, S_{t-1}$ and thus,

$$E^Q\left\{\left.\frac{S_{t+1}-S_t}{S_t}\right| S_t\right\} = R. \tag{25}$$

Equations (23) and (25) are equivalent to the following formulas:

$$E^Q\left\{\left.\frac{S_{t+1}}{1+R}\right| Z_1, \dots, Z_t\right\} = S_t, \tag{26}$$

$$E^Q\left\{\left.\frac{S_{t+1}}{1+R}\right| S_t\right\} = S_t, \tag{27}$$

which can be interpreted by saying that *the risk-neutral measure Q is the probability measure such that the expected value of tomorrow's stock price discounted to present value is the same as the current stock price*.

Yet, another equivalent formulation of (23) and (25) are in terms of the process

$$S_t^* = \frac{S_t}{(1+R)^t},$$

called *the discounted prices process*. S_t^* denotes the time-t stock price discounted to current dollars. Then, (23) and (25) are equivalent to

$$E^Q\left\{\left. S_{t+1}^*\right| Z_1, \dots, Z_t\right\} = S_t^*, \tag{28}$$

$$E^Q\left\{\left. S_{t+1}^*\right| S_t^*\right\} = S_t^*. \tag{29}$$

A sequence of random variables $\{S_0^*, \dots, S_T^*\}$ satisfying either of the two equation above is called a *martingale*. Concretely, if (28) is satisfied for each $t = 0, \dots, T-1$, then we say that $\{S_t^* : t = 0, \dots, T\}$ is a Q-martingale with respect to the information generated by the process $\{Z_t : t = 0, \dots, T\}$. Due to the previous property, Q is commonly called the *martingale measure* of the stock price process. To summarize,

Under the risk-neutral measure Q, the discounted stock price process is a martingale.

It turns out that the discounted value process, defined by

$$V_t^{*,h} := \frac{V_t^h}{(1+R)^t},$$

of any *self-financing trading strategy* h is also a martingale under Q. Concretely,

$$E^Q\left\{\left. V_{t+1}^{*,h}\right| Z_1, \dots, Z_t\right\} = V_t^{*,h}, \tag{30}$$

for any $t = 0, \ldots, T-1$ and any self-financing trading strategy h. Let us verify this property. Clearly, (30) is equivalent to

$$E^Q \left\{ \frac{V_{t+1}^h}{1+R} \middle| Z_1, \ldots, Z_t \right\} = V_t^h. \tag{31}$$

By definition V_t^h,

$$\begin{aligned} E^Q \left\{ \frac{V_{t+1}^h}{1+R} \middle| Z_1, \ldots, Z_t \right\} &= E^Q \left\{ x_t + y_t \cdot \frac{S_t}{1+R} \middle| Z_1, \ldots, Z_t \right\} \\ &= x_t + y_t E^Q \left\{ \frac{S_t}{1+R} \middle| Z_1, \ldots, Z_t \right\} \\ &= x_t + y_t S_t, \end{aligned}$$

where we used that x_t and y_t depends only on $S_0, \ldots, S_{t-1}$ and the martingale property (26). Finally, we see that by the definition 13, $E^Q \left\{ V_{t+1}^h/(1+R) \middle| Z_1, \ldots, Z_t \right\} = V_t^h$ for each $t = 0, \ldots, T-1$ if and only if h is self-financing.

Notice that (31) is equivalent to

$$q_u \frac{V_{t+1}^h(w_1, \ldots, w_t, 1)}{1+R} + q_d \frac{V_{t+1}^h(w_1, \ldots, w_t, 0)}{1+R} = V_t^h(w_1, \ldots, w_t),$$

for any $(w_1, \ldots, w_t)$. Hence, we just verified Proposition 3.1, which is the theoretical fundament of the Binomial algorithm.

Another application of (31) is to verify that (15) is sufficient for the Binomial market model to be arbitrage-free. Indeed, under (15), the risk-neutral measure Q is a well-defined probability measure such that

$$Q(w_1, \ldots, w_T) > 0. \tag{32}$$

Suppose that there exists an arbitrage-opportunity h as in Definition 3.3. Since h is self-financing and $V_0^h = 0$, (31) implies that

$$E^Q \left\{ V_T^h \right\} = (1+R)^T V_0^h = 0,$$

in light of the tower property of conditional expectations and backward induction. However, since $V_T^h \geq 0$ and $Q(w_1, \ldots, w_T) > 0$ always, the only possibility is that $V_T^h(w_1, \ldots, w_T) = 0$ always. This contradicts that $P(V_T^h > 0) > 0$. We conclude that there is no such an arbitrage-opportunity.

One consequence of the Binomial algorithm is that the *Binomial market model is complete*. That is, all (European) contingent claims $\mathcal{X}$ are replicable. It is no surprise that the Second Fundamental Theorem of Finance is valid for muti-period market models and hence, Q is indeed the only probability measure satisfying (32) and (24) (or any of the other equivalent formulas such as (23), (26) or (28)).

4. ON THE FITTING THE BINOMIAL MARKET MODEL

The Binomial market model is a simple model that allows us to price contingent claims based on the fundamental concepts of arbitrage-freeness and replicating portfolios. However, real stock prices seems to be changing “all the time", rather than at equally spaced discrete times as in the Binomial model. Still, there is hope that by taking a very large number of time steps and adjusting the parameters of the Binomial model, one can approximately replicate the observed statistical behavior of a real stock price process. But how to fit the parameters?.

From the introduction, we recall that a typical stock price exhibit high randomness driven by a high number of transactions, most of them resulting in small changes of the stock price. The historical path of the stock price in time periods of medium size (say 6 months) seems to evolve, for the most part, continuously but in a non-smooth, very erratic manner. Of course, some features of the historical stock prices will be lost when modeling them with the Binomial model or any other model for that matter. But, what statistical features should we try to incorporate and how to adjust the parameters of the Binomial model to reflect these features?

Two of the main features of stock prices are *drift* and *volatility*. The *drift* μ during a particular time period $[u, v]$ is a measure of the average rate of return of the stock per unit time. In other words, denoting S_t the time-t stock price, the drift μ is such that

$$\mu = \frac{1}{\Delta} E\left\{\frac{S_{t+\Delta} - S_t}{S_t}\right\}, \tag{33}$$

for $u < t < t + \Delta < v$. So, if we record the stock price at evenly spaced times $u = t_0 < t_1 < \cdots < t_n = v$, then

$$\bar{\mu}_n := \frac{1}{\Delta_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \overset{n\to\infty}{\longrightarrow} \mu, \tag{34}$$

where Δ_n is the time span between observations. We are assuming that the stock prices exhibit such a stabilizing property.

Volatility σ is the most important feature of stock prices. σ measures the variability or randomness inherent in the stock prices during a time period $[u, v]$. Higher volatility means more rapidly changing stock prices. Being variance a standard measure of variability, the volatility σ is typically defined by

$$\sigma^2 = \frac{1}{\Delta} \mathrm{Var}\left\{\frac{S_{t+\Delta} - S_t}{S_t}\right\}, \tag{35}$$

for $u < t < t + \Delta < v$. Hence, we expect to observe the following stabilizing property:

$$\hat{\sigma}_n^2 := \frac{1}{\Delta_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} - \Delta_n \bar{\mu}_n\right)^2 \overset{n\to\infty}{\longrightarrow} \sigma^2. \tag{36}$$

Given evenly spaced historical stock prices (say, daily prices), the statistic $\hat{\sigma}_n$ above is called the *historical volatility* of the stock. The historical volatility depends on the number of observations n

and the time span Δ_n between observations. Volatility is usually expressed as percentage; e.g. a stock with a 20% volatility, means that $\sigma = .2$. Stocks typically have volatility between 15% and 60%.

Suppose that we want to adjust the parameters of a Binomial model to recover historically observed volatilities σ and drift μ. For this purpose, consider a Binomial market model with evenly spaced "trading" times $u = t_0 < t_1 < \cdots < t_n = v$ where $t_i = (v-u)/n$. The parameters of the model $d_n < 1 < u_n$ and p_u^n are assumed to depend on n. Suppose that $\{\widetilde{S}_t^n\}_{t\geq 0}$ is the stock price process with initial price S_0 that jumps only at times t_i according to a Binomial model and remains unchanged between trading times $t_0, \ldots, t_n$. Thus,

$$\widetilde{S}_0^n = S_0, \quad \widetilde{S}_{t_i}^n = \begin{cases} u_n \widetilde{S}_{t_{i-1}}^n & \text{with prob. } p_u^n, \\ d_n \widetilde{S}_{t_{i-1}}^n & \text{with prob. } p_d^n, \end{cases} \qquad \text{and} \qquad \widetilde{S}_t^n = \widetilde{S}_{t_{i-1}}, \qquad \text{if} \qquad t \in [t_{i-1}, t_i).$$

Let $\Delta_n = t_i - t_{i-1}$ be the time span between consecutive observations. We want to choose u_n, d_n, p_u^n such that

$$\frac{1}{\Delta_n} \cdot E\left\{ \frac{\widetilde{S}_{t_i}^n - \widetilde{S}_{t_{i-1}}^n}{\widetilde{S}_{t_{i-1}}^n} \right\} = \frac{1}{\Delta_n} \left(p_u^n u_n + p_d^n d_n - 1 \right) \overset{n\to\infty}{\longrightarrow} \mu, \tag{37}$$

and

$$\frac{1}{\Delta_n} \cdot \mathrm{Var}\left\{ \frac{\widetilde{S}_{t_i}^n - \widetilde{S}_{t_{i-1}}^n}{\widetilde{S}_{t_{i-1}}^n} \right\} = \frac{1}{\Delta_n} \left(p_u^n \cdot p_d^n (u_n - d_n)^2 \right) \overset{n\to\infty}{\longrightarrow} \sigma^2. \tag{38}$$

There are many ways to accomplish (37)-(38). One simple was is to set $p_u^n = 1/2$, and then, to solve the equations:

$$\frac{u_n + d_n}{2} = 1 + \mu \Delta_n, \quad u_n - d_n = 2\sigma\sqrt{\Delta_n}.$$

This will yield the solutions:

$$\begin{aligned} u_n &= 1 + \mu\Delta + \sigma\sqrt{\Delta_n}, \\ d_n &= 1 + \mu\Delta - \sigma\sqrt{\Delta_n}. \end{aligned}$$

This method of fitting the Binomial model is called *Hull-White method*. Notice that this method has the drawback that u_n and d_n depends on the drift μ.

A different method is to work with log returns instead of simple returns. The *log return* of the stock during the time interval $[t, t+\Delta]$ is defined by

$$\log \frac{S_{t+\Delta}}{S_t}.$$

One of the advantages of working with log returns is that log returns are *additive*. The log return during the time interval $[u, v]$ is the sum of the log returns during $[u, t]$ and $[t, v]$. We can define measures of volatility and drift in terms of log returns:

$$\mu = \frac{1}{\Delta} E\left\{ \log \frac{S_{t+\Delta}}{S_t} \right\}, \quad \sigma^2 = \frac{1}{\Delta} \mathrm{Var}\left\{ \log \frac{S_{t+\Delta}}{S_t} \right\}.$$

It turns out that the assumption that the log returns of the stock exhibit constant mean and variance is empirically more accurate than the previous assumption that the simple returns have constant mean and variance. As before, one can empirically estimate the volatility σ using the so-called *historical volatility*, based on $S_0, S_\Delta, \dots, S_{n\Delta}$, which is defined by

$$\hat{\sigma} := \sqrt{\frac{1}{\Delta}\sum_{i=1}^{n}\left(R_i - \bar{R}\right)^2}, \tag{39}$$

where R_i is the i^{th} log return $\log(S_{i\Delta}/S_{(i-1)\Delta})$ and $\bar{R}$ is the sample average of the R_i's.

Again, we would like to choose the parameters of the Binomial market model to replicate certain desired drift and volatility in the following sense

$$\frac{1}{\Delta_n}\cdot E\left\{\log\frac{\widetilde{S}^n_{t_i}}{\widetilde{S}^n_{t_{i-1}}}\right\} \overset{n\to\infty}{\longrightarrow} \mu, \qquad \frac{1}{\Delta_n}\cdot \mathrm{Var}\left\{\log\frac{\widetilde{S}^n_{t_i}}{\widetilde{S}^n_{t_{i-1}}}\right\} \overset{n\to\infty}{\longrightarrow} \sigma^2. \tag{40}$$

One can check that the following choice of parameters comply with (40):

$$u_n = \frac{1}{d_n} = e^{\sigma\sqrt{\Delta_n}}, \quad p^n_u = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta_n}. \tag{41}$$

This method is due to *Cox, Ross, and Rubinstein* (CRR). One advantage is that u_n and d_n depend only on the volatility σ and not on the drift. Since the option prices of the Binomial model depends only on u_n, d_n, the option prices will be independent of the drift of the stock price process.

Of course, the main application of a model should be prediction. Suppose that we want to price now at $t = 0$ an option maturing at time T. We can presume that the Binomial market model will be a good fit for the real stock price evolution during the next time period $[0, T]$. There is no way to know for sure this now, but from past experience, we believe that the volatility and drift of the stock for the time period we are working on will be more or less steady. To estimate the volatility, we gather historical daily stock prices for last few months and apply the estimator (39).

We also need to predict the interest rate r for borrowing and lending prevailing during the life time of the option. This can be done using, for instance, a treasury bill with maturity close to T. A $t-$bill is a short-term zero coupon bond with principal $\$1$ dollar issued by the US Treasury department. Market prices of t-bills are widely available. Suppose that $p(0, T) < 1$ is the price of a t-bill maturing at time T. The way to determine the interest rate r per unit time depends on our assumptions about the bond price during $[0, T]$. The simplest assumption is that

$$P(t, T) = e^{-r(T-t)},$$

in which case r is called the *continuously-compounding interest rate* of the bond. Thus, the rate of return for lending or borrowing during a time interval $[t, t + \Delta]$ is

$$R = e^{r\Delta} - 1.$$

The implementation for the pricing of an option maturing at time T using the CRR Binomial market model will look like this:

(1) Estimate the stock's volatility σ either using historical stock price data (*historical volatility*) or matching the market price of a liquid option (*implied volatility*).

(2) Determine the interest rate r prevailing during the lifetime of the option. This can be done using a treasury bill with maturity close to the expiration of the option. If $p(0,T)$ is the time-zero price of such a t-bill, then r is given by

$$r = \frac{1}{T} \log \frac{1}{p(0,T)}.$$

(3) Evaluate the option price using the Binomial algorithm or the risk-neutral pricing formula with the parameters

$$\Delta_n = T/n, \quad u_n := \frac{1}{d_n} = e^{\sigma\sqrt{\Delta_n}}, \quad R_n := e^{r\Delta_n} - 1, \tag{42}$$

The number of steps n is chosen large enough that the option price stabilizes.

Let us finish with some important remarks. Why should the option price obtained from the Binomial algorithm stabilize when n is large? For instance, suppose we want to price a put option with strike K and maturity T and Q_n is the risk-neutral probability measure associated with the parameters (42). Our approach is to compute

$$\Pi_n(0) := \frac{1}{(1+R_n)^n} E^{Q_n}\left\{\left(K - \widetilde{S}^n_{T}\right)_+\right\} = e^{-rT} E^{Q_n}\left\{\left(K - \widetilde{S}^n_{T}\right)_+\right\},$$

for an n large enough that $\Pi_n(0)$ does not change significantly. However, how do we know that the option prices converge to a number $\Pi(0)$?

It turns out that the option prices based on the Binomial market model indeed converges. The limit option price will take the form:

$$\Pi(0) = e^{-rT} E^{Q}\left\{\left(K - S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T}\right)_+\right\},$$

where, under Q, W_T is Normally distributed with mean 0 and variance T. The previous formula is the celebrated *Black-Scholes price formula*, which is justified next Chapter. The following exercise explains at least partially where $r - \sigma^2/2$ and σ come from.

Exercise 5. Show that under Q_n, the drift and volatility of $\widetilde{S}^n$ are such that

$$\frac{1}{\Delta_n} \cdot E^{Q_n}\left\{\log \frac{\widetilde{S}^n_{t_i}}{\widetilde{S}^n_{t_{i-1}}}\right\} \overset{n\to\infty}{\longrightarrow} r - \frac{1}{2}\sigma^2, \quad \frac{1}{\Delta_n} \cdot \mathrm{Var}^{Q_n}\left\{\log \frac{\widetilde{S}^n_{t_i}}{\widetilde{S}^n_{t_{i-1}}}\right\} \overset{n\to\infty}{\longrightarrow} \sigma^2.$$

As a consequence, argue that

$$E^{Q_n}\left\{\log \frac{\widetilde{S}^n_{T}}{\widetilde{S}^n_{0}}\right\} \overset{n\to\infty}{\longrightarrow} (r - \frac{1}{2}\sigma^2)T, \quad \mathrm{Var}^{Q_n}\left\{\log \frac{\widetilde{S}^n_{T}}{\widetilde{S}^n_{0}}\right\} \overset{n\to\infty}{\longrightarrow} \sigma^2 T.$$

APPENDIX A. ABOUT THE FIRST FUNDAMENTAL THEOREM OF FINANCE

The first fundamental theorem of finance relies on a fundamental theorem of analysis: The *separating hyperplane theorem.* There are different versions, but we will use the following version, which proof can be found in Laberton and Lapeyre.

Theorem A.1. *Let K be a convex, closed subset of $\mathbb{R}^M$ and let W be a subspace from $\mathbb{R}^M$ such that $K \cap W = \emptyset$. Then, there exists a vector $\mathbf{c} \in \mathbb{R}^M$ such that the following two conditions hold:*

(i) $\mathbf{c} \cdot \mathbf{x} > 0,$ *for all $x \in K$;*
(ii) $\mathbf{c} \cdot \mathbf{x} = 0,$ *for all $x \in W$.*

The idea is two consider the following sets in $\mathbb{R}^M$:

$$K = \left\{[x_1, \dots, x_M] \in \mathbb{R}^M : x_i \geq 0\}\backslash\{0\right\},$$
$$W = \left\{[V_1^h(\omega_1), \dots, V_1^h(\omega_M)] : V_0^h = 0\right\}.$$

The set W represents the possible final wealths of trading strategies that start with zero net investment. Clearly, no arbitrage opportunity is equivalent to the condition

$$W \cap K = \emptyset.$$

It is not hard to see that W is a subspace of W since for any h_1, h_2 s.t. $V_0^{h_i} = 0$, we have that $V_0^{ch_1+h_2} = 0$ and moreover, $V_1^{ch_1+h_2} = cV_1^{h_1} + V_1^{h_2}$.

By the above separating hyperplane theorem, no-arbitrage will imply that there exists $\mathbf{c} \in \mathbb{R}^M$ such that (i) $c_1, \dots, c_M > 0$ and $(ii) \quad c_1 V_1^h(\omega_1) + \cdots + c_M V_1^h(\omega_M) = 0$, for any h satisfying that $V_0^h = 0$. We then define $Q(\{\omega_i\}) = c_i / \sum_k c_k$. Thus, (ii) can be written as

$$(ii) \quad \mathbb{E}^Q\left\{V_1^h\right\} = 0,$$

for any h satisfying that $V_0^h = 0$. It is a nice exercise to show that Q satisfies the risk-neutral condition (9). The idea is to take two stocks at a time $i \neq j$, and consider a trading strategy h that use stock i to finance one unit of stock j (so that the net initial investment is 0).

APPENDIX B. FORMULAS OF CONDITIONAL EXPECTATION

Basic definitions

- Intuition: $E(Y|X)$ is the best predictor of the random variable Y based on the value of the random variable X.
- When $E|Y|^2 < \infty$, $E(Y|X)$ is the function of X, say $f(X)$, that is "closest" to Y on average; that is,
$$\min_g E\left(Y - g(X)\right)^2 = E\left(Y - f(X)\right)^2.$$
- For discrete or (abs.) continuous r.v.'s, there are explicit formulas in term of the joint density function $p(x, y)$ and the marginal $p_X(x)$:

- Discrete: $E(Y|X) = f(X)$, where

  $$f(x) = E(Y|X = x) := \sum_y yP(Y = y|X = x) = \sum_y y\frac{p(x,y)}{p_X(x)}.$$

- Continuous: $E(Y|X) = f(X)$, where

  $$f(x) = E(Y|X = x) := \int yp(y|X = x)dy = \int y\frac{p(x,y)}{p_X(x)}dy.$$

Main properties

- *$E(Yg(X)|X) = g(X)E(Y|X)$.*
- *Linearity: $E(aY_1 + bY_2|X) = aE(Y_1|X) + bE(Y_2|X)$.*
- *Positivity: $E(Y|X) \geq 0$, if $Y \geq 0$.*
- *Tower Property:*

  $$E(E(Y|X)) = E(Y)$$
  $$E(E(Y|X,Z)|X) = E(E(Y|X)|X,Z) = E(Y|X).$$

- *"Expected-based definition property": $E(Y|X)$ is the only function of X satisfying that*

  $$E(g(X)E(Y|X)) = E(g(X)Y),$$

  for any function g.
- *Independence Properties:*
  - *If Y is independent of X, then all the following formulas hold true*

    $$E(Y|X = x) = E(Y), \quad E(Y|X) = E(Y),$$
    $$E(g(X,Y)|X = x) = E(g(x,Y)),$$
    $$E(g(X,Y)|X) = f(X), \quad \text{with } f(x) = E(g(x,Y)).$$

  - *If W is independent of Y given X, then*

    $$E(Y|X,W) = E(Y|X).$$

- *Equivalence property: Let h be a one-to-one mapping. Then,*

  $$E(Y|X) = E(Y|h(X)).$$

  In particular, for any function g,

  $$E(Y|X) = E(Y|X, g(X)).$$