MATHEMATICS OF FINANCE

CHAPTER I
INTRODUCTION

JOSÉ E. FIGUEROA-LÓPEZ

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1. FINANCIAL MARKETS AND THE MOST IMPORTANT ASSETS

The goal of this course will be to develop a mathematical theory for the pricing of certain financial instruments traded in the market, called *derivatives*. These instruments are contracts that promise certain payoff to the holder or buyer of the contract. The amount of the payoff depend on the market price of other underlying assets (such as stocks or bonds). The timing of the payoffs are set either at the end of the expiration of the contract or at a time of choice of the holder prior to the expiration. Our main concern will be to price the derivatives in a consistent way that precludes opportunities to make money for free without any risk, via transactions on the derivatives and the primary assets. The primary assets are typically the underlying asset of the derivative and *bonds*, which can liquidly be traded in financial markets. These "free-lunch" opportunities, called *arbitrage opportunities*, are expected to be very rare in reality as they will lead to unstable markets.

A (zero-coupon) *bond* is a contract where, for a suitable price now, the buyer or holder gets a fixed payment, called the *principal value* L or *face value*, at a prespecified fixed time T, called the maturity. Clearly, one can realized any principal L by buying L units of a zero-coupon bond with a face value of 1 dollar. The time t price of this primary bond is denoted $P(t,T)$, which obviously satisfies that

$$P(t,T) \leq 1, \quad \text{for all } t \leq T.$$

Bonds are instruments that allows the market participants to borrow or lend money. For instance, if an agent wishes to borrow L dollars at time $t = 0$, he/she will have to sell $L/P(0,T)$ zero-coupon bonds now.

Stocks or equity is a typical underlying for derivative securities. The holder of a share of a stock owns a fraction of a public company; however, the shareholder is not liable for the debt of the company. Equity are traded in stock markets or in the over-the-counter market where a large number of participants, called the *market makers* or *dealers*, quote a *bid* price (a price at which they are prepared to buy) and an *offer* or *ask* price (a price at which they are prepared to sell). The

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largest stock markets in the United States are the Nasdaq stock market (Nasdaq), the New York Stock Exchange (NYSE), and the American Stock Exchange (Amex). The *over-the-counter* market is larger (in total trading volume) than exchanges and consists of a network of dealers transacting through a telephone- or computer-linked network.

Stock and bond prices are unpredictable quantities whose value evolves in an erratic manner in time due to the large number of market participants. For instance, Nasdaq includes over 300 market makers and the bond market for Treasury bill includes over 1,500 dealers. At a given time, the realized stock price is set when the highest quoted bid price matches the lowest offer price. By the end of the trading day, the closing price is the cumulative effect of a large number of *shocks* or transactions. The discrete movements of a stock are quite frequent, typically small in size, resulting in a price process that seems to evolve almost "continuously" in time. For instance, Microsoft experienced 480,000 transactions in 316 business days (from 01/01/93 to 03/31/94), an average of $1,500$ transactions per day. Due to the above considerations, our mathematical models for the prices of the underlying assets will be *stochastic or random*, and furthermore, we shall see that a good model for such a random process is closely related to a *Brownian motion* whose main virtue is that its value at a given time is normally distributed. Typically, $S_t := S(t)$ will denote the time t price of the stock or more generally, the underlying of a derivative.

2. DERIVATIVES

A *derivative* is a security whose values are tied to the price of other primary assets, called *underlyings*, at one or more points of the duration of the contract T. There are roughly two general (non-exclusive) concepts of derivatives: *contingent claims* and *options*. A contingent claim is a contract that pays its holder (the buyer) a one-time payment $\mathcal{X}$ either at the expiration date T (in that case, we say the contract is *European*) or at a time τ prior to the expiration T which is decided by the buyer (in that case, we say the contract is *American* and τ is called the *exercise time*). The payoff $\mathcal{X}$ at the time of exercise will depend on the market price of the underlying at that moment. For instance, the payoff of a European contingent claim can be set to be equal to $\mathcal{X} = \Phi(S_T)$, where Φ is a given function and $S(t)$ denotes the time-t value of an asset (say a stock). Notice that in this case, the value $\Pi(t)$ of the derivative at time t is tied to the price of the underlying asset at time T since $\Pi(T) = \Phi(S(T))$.

Example 1. A *forward contract* is a very popular derivative. In this contract, the holder agrees to buy an asset at a prescribed date T for an amount F, called the forward price, agreed upon today at time $t = 0$. The forward price F of the asset is set so that the time$t = 0$ value of the contract is zero. The party who will receive the asset (the buyer) is said to take a *long-position* in the contract, while the counterparty (the seller) is said to take a *short-position*. What will be the payoff (of a long position) of this contingent claim in terms of the asset price process $\{S(t)\}_{t\geq 0}$?

The most important derivatives are the put and call options. In a *call option* (resp. a *put option*), the holder or buyer of the option has the *right* (not the obligation) to buy (resp. to sell) certain quantity of an asset during a specified period of time $[0, T]$, at a price K set today at time $t = 0$. Obviously, the seller (also, called the *writer*) receives a compensation V_0 for selling the option. The time T is called the expiration or maturity date, while K is called the *strike price* or *exercise price*. If the holder is allowed to exercise the option only at maturity T, we say that the contract is European, while if the holder has the freedom to exercise at any time τ prior to maturity, we say the option is *American*. In an option contract, the buyer is said to take the *long position* or to go long in the option contract and the seller is said to take a *short position* or to go short. The Chicago Board Options Exchange (CBOE) is the largest U.S. options exchange, offering options on over 2,200 companies and 22 stock indexes.

The option meets the description of derivative given above. Indeed, if $\Pi(t)$ and $S(t)$ denote the price of the option and the underlying asset, respectively, at time t, then the option price is tied to the asset price at the exercise time τ since

$$\Pi(\tau) = (S(\tau) - K)_+ := \begin{cases} S(\tau) - K, & \text{if } S_\tau \geq K, \\ 0, & \text{if } S_\tau < K, \end{cases}$$

for a call option, while

$$\Pi(\tau) = (K - S(\tau))_+ := \begin{cases} K - S(\tau), & \text{if } S_\tau \leq K, \\ 0, & \text{if } S_\tau > K, \end{cases}$$

for a put option. Notice that one can associate a contingent claim to an option. For instance, the value of a European call option with maturity T and strike K must coincide with the value of the contingent claim with expiration T and payoff

$$\mathcal{X} := (S(T) - K)_+.$$

There are two main usages of derivatives and options: risk management and speculative investment. Financial companies use options to increase or reduce certain type of risk. This practice is called *hedging*. For instance, the holder of an asset, worried that its value will decreased substantially in value, might wish to buy a put option that allows him to sell his asset at a fixed specified value. In this case, the investor is hedging its position on the underlying. However, an investor might wish to buy a put option only for merely profitable reasons on the presumption that the asset price will decline in value. In this case, the market participant is acting as a *speculator*.

3. Arbitrage-free pricing

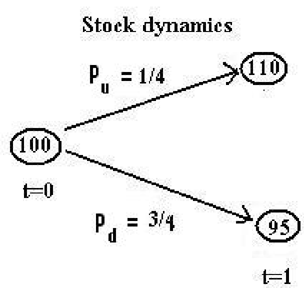
As it was previously mentioned, an arbitrage-free pricing theory for the derivative market aims at precluding potential opportunities to make money without any net initial money. Such arbitrage opportunities are rare and even if they exist, they last for a very short-period. Let us consider two motivational examples to illustrate how arbitrage-opportunities may arise and at the same time, motivate the mechanism that we will exploit through this course to determine reasonable prices for the derivatives.

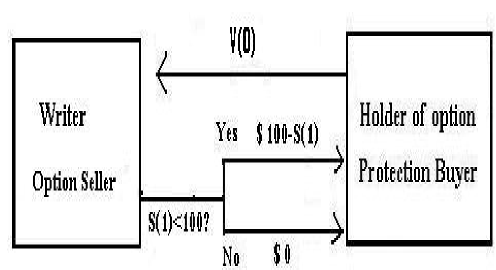
Example 2 (**A forward contract**). Suppose that an asset is selling now for the price of $S_0 := 50$ dollars. A one-year zero-coupon bond (with a principal of 1 dollar) is selling for $P(0,1) := .95$ dollars. Suppose that a forward contract on the asset is available in the market with a forward price of 53 dollars. We claim that this set of prices is inconsistent as they will allow an arbitrage opportunity. Which one? At time $t = 0$,

(1) Sell bonds in the amount of \$50 dollars. So, needs to sell $50/.95$ zero-coupon bonds;
(2) Buy the asset;
(3) Sell (take a long position in) the forward contract.

What will be the final net profit/loss of the above strategy?

Example 3 (**A European call option**). Suppose that one wishes to price a European call option with maturity $T = 1$ year and strike K equal to the current price of the underlying S_0 (in that case, the option is said to be *at-the-money*). Suppose that $S_0 = 100$ and that there is a $p_u = 1/4$ chance that the asset will end up at $S_1 = 110$ and a $p_d = 3/4$ chance that the asset will end up at $S_1 = 95$. The following two graphs describe the dynamics of the asset and the cashflow of the call option.

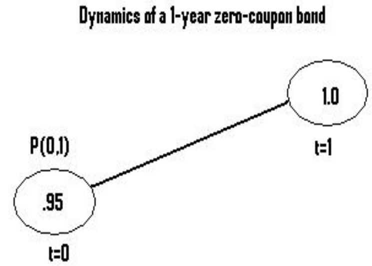




A seemly reasonable approach to determine a fair price for the above option is to see the option as a random game. In that case, the expected payoff of the option will be

$$\frac{1}{4} \cdot 10 + \frac{3}{4} \cdot 0 = 2.5;$$

in other words, if the writer of the option sells a large number of options, his average payoff per option will be 2.5. However, the market typically is equipped with bonds which create a *time-value-of-money* effect. For instance, a one-year zero-coupon bond selling today at $P(0) = .95$ (see the following graph) means that L dollars today are worth $L/P(0)$ in one-year or what is the same, L dollar in one year has a present or discounted value of $P(0)L$ dollars today. Since the payoff of the option takes place at expiration, the writer will need to charge only $(2.5)(.95)$ dollars per option today so that to be ready to cover (on average) the payoff of the option.



It turns out that the writer of the option can do better than cover its average profit/loss per option. There is a price V_0 that will allow him to completely hedge away the risk of the option by running a trading strategy in the underlying and in the bond whose final profit/loss is always the same as the payoff of the option. We say that this is a *hedging* or *replicating strategy*. Indeed, consider the following transactions:

(i) Buy/sell x zero-coupon bonds;
(ii) Buy/sell y shares of stock.

We should clarify that negative x means selling bonds (hence, borrowing money), while negative y means selling stock. The net initial investment of the strategy is

$$V_0 := x \cdot P(0) + y \cdot S_0,$$

and the resulting wealth at time $t = 1$ will be of the form

$$V_1 = \begin{cases} x + y \cdot 110, & \text{if stock goes up,} \\ x + y \cdot 95, & \text{if stock goes down.} \end{cases}$$

By simple arithmetic, it can be seeing that buying $y = 2/3$ shares of stock and selling $x = -(2/3)(95) \approx -63.3$ bonds result in the final wealth

$$V_1 = \begin{cases} -\frac{2}{3} \cdot 95 + \frac{2}{3} \cdot 110 = 10, & \text{if stock goes up} \\ -\frac{2}{3} \cdot 95 + \frac{2}{3} \cdot 95 = 0, & \text{if stock goes down,} \end{cases}$$

which is exactly the same as the payoffs of the call option. The net initial investment is

$$V_0 = -\frac{2}{3}(95)(.95) + \frac{2}{3}(100) \approx 6.5.$$

In summary, charging $V_0 = 6.5$ dollars for the option, the writer of the option will be able to hedge away the risk of the option by borrowing $(2/3)(95)(.95) \approx 60.1$ additional dollars (via bonds) and buying $2/3$ shares of stock. Notice also that it would be foolish for the buyer to pay more than V_0 as he/she can always realized the option's payoff by running the previous hedging strategy.

What is the connection of the above economical reasonings with the concept of arbitrage? Everything. If an agent of the market is ready to buy and sell the above option at a price F_0 different from the V_0 above, then there will be an arbitrage opportunity in the market. For instance, if $F_0 > V_0$, then one can sell the option to the agent at time $t = 0$ for F_0, hedge away the position in the option with V_0, and put the reminder $F_0 - V_0$ in bonds. This strategy requires no initial net investment, but produces a final positive gain. We can conclude that the only possible arbitrage-free price Π_0 for the option is V_0.

Another fundamental question is the following: Is the previous arbitrage-free price consistent with the “expected discounted payoff" of the option? The answer is affirmative if one adjust the probability measure of the different random outcomes of the stock (in this case, only two possible outcomes). Indeed, if q_u and q_d stands for the probability that the stock will go up or down, respectively, then the arbitrage-free price $V_0 = 6.5$ is such that

$$V_0 := (.95)(10) \cdot q_u + (.95)(0) \cdot q_d,$$

if

$$q_u = \frac{2}{3}, \quad \text{and} \quad q_d = \frac{1}{3}.$$

The probabilities q_u and q_d have another important interpretation. Consider the *rate of return* (or *interest rate*) in the bond, defined as the money earned per dollar invested:

$$R := \frac{\text{Final value} - \text{Initial value}}{\text{Initial value}} = \frac{1 - .95}{.95} \approx 5.2\%.$$

Notice that, under the probabilities weights $q_u = \frac{2}{3}$ and $q_d = \frac{1}{3}$, the expected rate of return in the stock is

$$\frac{110 - 100}{100} \cdot q_u + \frac{95 - 100}{100} \cdot q_d = R.$$

In that case, we say that the probability measure is *risk-neutral* in the sense that, under this probability measure, all assets have the same expected rate of return regardless of the riskiness. The fact that the arbitrage-free price of the derivative is equal to its expected discounted payoff under a risk-neutral probability measure is not by chance. We shall see that if the derivatives are priced according to this procedure, then the market consisting of the derivatives and the primary assets is arbitrage-free.