1. Consider the Cox-Ross-Rubinstein (CRR) Binomial market model with \( n \) trading periods.

(a) Show that under the risk-neutral measure \( Q_n \), the “drift” and volatility of \( \tilde{S}^n \) per period are such that

\[
\frac{1}{\Delta_n} \cdot E^{Q_n} \left\{ \log \frac{\tilde{S}^n_i}{\tilde{S}^n_{i-1}} \right\} \xrightarrow{n \to \infty} r - \frac{1}{2} \sigma^2, \quad \frac{1}{\Delta_n} \cdot \text{Var}^{Q_n} \left\{ \log \frac{\tilde{S}^n_i}{\tilde{S}^n_{i-1}} \right\} \xrightarrow{n \to \infty} \sigma^2.
\]

(b) Argue that

\[
\Pi_n(0) := e^{-rT} E^{Q_n} \left\{ (\tilde{S}_T^n - K)^+ \right\} \rightarrow \Pi^{\text{call}}(0) = e^{-rT} E^Q \left\{ \left( S_0 e^{\sigma W_T + (r - \frac{\sigma^2}{2})T} - K \right)^+ \right\},
\]

as \( n \to \infty \).

2. A nine-month American put option on a non-dividend-paying stock has a strike price of $49. The stock price is $50, the risk-free rate is 5% per annum, and the volatility is 30% per annum.

(a) Use a three period CRR Binomial tree to calculate the option price.

(b) Determine the hedging strategy at time 0.

(c) Suppose that we used instead a 100 period Binomial CRR model and that the expected rate of return per unit time on the stock, denoted by \( \mu \), is set to be 0.1. Estimate the probability that the stock will increase at least 80% in value during the next 9 months.

3. Implementation of the Binomial market model (this part can be done in groups of 3 students or less).

In the following we consider June vanilla options for two stocks: Google Inc or Microsoft.

(a) Use appropriate current Treasury bill data to determine a continuously compounded interest rate suitable to price June options.

Note: You need to indicated your sources of data and document your procedure.

(b) Find the historical volatilities \( \hat{\sigma} \) of the stock estimated from all closing prices of the stock for the last 3 months, 6 months, and 12 months.

Notes: You need to indicated your sources and show your computations.

(c) Using an appropriate Binomial model approximation that is consistent with the observed historical volatilities and the observed interest rate of the t-bill, find the price of liquid American and European call or put options of your choice on one of the stocks.

Note: You need to code the Binomial algorithm. Take large enough number of steps \( n \) so that the option price stabilizes or converges.
(d) The implied volatility \( \hat{\sigma}(K) \) corresponding to the market price \( \Pi^*(K) \) of an option with strike \( K \) is the value of \( \sigma \) such that \( \Pi_{n}^{Bin}(\sigma) = \Pi^*(K) \), where \( \Pi_{n}^{Bin}(\sigma) \) denotes the theoretical price of the option under an \( n \)-period Binomial model with volatility \( \sigma \). Using a CRR parameterization and a large enough \( n \), determine the so-called implied volatility smile, which is the graph of \( \hat{\sigma}(K) \) versus \( K \).

Note: Need to create a program that automatically is able to recover the implied volatility \( \hat{\sigma}(K) \) given \( n \) and \( \Pi^*(K) \). Then, use this program to obtain the implied volatilities corresponding to June options of one of the stocks.