Ma 515/Stat 540. Practice Final 2.

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1. Consider a general discrete-time model with one risky asset, n states $\Omega = \{\omega_1, \dots, \omega_n\}$, and a saving account. The asset price process is denoted by $\{S_t : t = 0 \dots T\}$, and the interest rate prevailing during the period $[t, t+1)$ is denoted by R_t.

   (a) *(Fill the blank)* The first fundamental theorem of finance states that a necessary and sufficient condition for the market to be arbitrage-free is that there exists a probability measure Q such that

   $$Q(\omega_i) > \underline{\qquad} \text{ for all } i, \text{ and } E^Q\left\{ \frac{S_{t+1}}{1+R_t} \middle| \mathcal{F}_t^S \right\} = \underline{\qquad\quad}, \text{ for all } t.$$

   (b) *(Fill the blank)* The probability measure Q in the previous part is called an equivalent-martingale measure because the process $\underline{\qquad\qquad\qquad}$ is a *martingale* under Q, where B_t is

   $\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$.

   (c) *(Fill the blank)* Another terminology for the probability measure Q above is *risk-neutral probability measure* because, relative to measure Q, the expected $\underline{\qquad\qquad\qquad\qquad\qquad}$ of any

   asset during any time period $[t, t+1)$ is the same as $\underline{\qquad\qquad\qquad}$.

   (d) Consider a one-period model with one risky asset, one bond with rate of return $R = 1/9$ and a stock with initial price $S_0 = 5$. Suppose that the stock can take $M = 3$ states as follows:

   $$S_1(\omega_1) = \frac{20}{3}, \quad S_1(\omega_2) = \frac{40}{9}, \quad S_1(\omega_3) = \frac{30}{9}.$$

   i. Is the market arbitrage-free?
   ii. Is the market complete? Explain.
   iii. Let $\mathcal{X}$ be a contingent claim. Find a linear equation on $\mathcal{X}(\omega_1), \mathcal{X}(\omega_2), \mathcal{X}(\omega_3)$ for the claim to be reachable.

   (e) A necessary condition for the market to be arbitrage free is that, for any t, $R_t + 1 \leq \frac{S_{t+1}}{S_t}$, sometimes. Devise an arbitrage opportunity if, for some time t, $R_t + 1 > \frac{S_{t+1}}{S_t}$, <u>always</u>.

2. Consider a two period discrete-time model with one risk-free asset, and a risky asset. The interest rate of the risk-free asset per period is $R = 0$. The price evolution can be represented by the following tree:

		$S_2 = 9$
	$S_1 = 8$	
		$S_2 = 5$
$S_0 = 5$		
		$S_2 = 5$
	$S_1 = 4$	
		$S_2 = 3$

   (a) State and use the risk-neutral valuation formula to compute the *European lookback call option*, which pays at time $T = 2$ the payoff $\mathcal{X} := (\max\{S_0, S_1, S_2\} - 7)_+$.

(b) An agent sell the option of part (a) for the arbitrage-free price V_0. Suppose that during the first period the stock moves to $S_1 = 8$. What position Δ_1, should the agent take in the stock to be sure that she will replicate the claim.

(c) Using Backward induction, compute the price of the American version of the option in part (a); that is, the American option that pays $Z_t = (\max\{S_0, \ldots, S_t\} - 7)_+$ at time t when it is exercised.

(d) Is it optimal to exercise early? If so, when?

3. Answer the following questions:

(a) What is a speculator expecting if he goes short on a portfolio with negative *Vega*?

(b) Why is it good to have a portfolio with small Gamma?

(c) How many additional claims are necessary to make a portfolio Gamma neutral, and vega neutral?

(d) Devise an arbitrage opportunity if a European call option with delivery price K and maturity T is sold by less than $S_0 - Ke^{-rT} > 0$, where r is the current short-interest rate of a bond with maturity T. Explain carefully.

(e) In pricing an option using a binomial approximation to the Black-Scholes model, the volatility of the stock can be determined using two methods. Briefly describe them (two lines each method).

4. Regarding continuous-time trading, answer the following questions:

(a) Heuristically, the stochastic integral

$$\int_0^t h_1(u) dS_u,$$

is defined as the limit of the Riemann sums

$$\sum_{t_i} h_1(t_i) \left\{ \underline{\hspace{3cm}} \right\}$$

when the time points t_i are such that ____________________

If $h_1(t)$ represent the number of shares held at time t in a "dynamical trading strategy", the financial interpretation of the integral is ____________________

(b) **Let $\{(x_t, y_t)\}_{t \geq 0}$ be a portfolio trading strategy so that the value of the portfolio at time t is**

$$V_t = x_t + y_t S_t.$$

What "Stochastic Differential Equation" should V satisfy for the strategy to be self-financing?

____________________.

The above condition formalize the idea that there is neither ____________________

into the portfolio nor ____________________.

(c) **Suppose that the "short-rate of interest" at time t in the money market account is $r(t)$. Let B_t be the value of the money market account at time t if 1 dollar is invested at time 0. Give an expression for B_t in terms of $r(t)$.**

(d) **Assuming $r(t)$ is the short rate of interest and $dS_t = S_t\{\alpha(t)dt + \sigma(t)dW(t)\}$, show that if $\{(x_t, y_t)\}_{0\le t\le T}$, is a self-financing strategy, then**

$$V_t^* = V_0^* + \int_0^t y_u dS_u^*,$$

where $V_t^* := V_t/B_t$ and $S_t^* := S_t/B_t$ are the corresponding discounted processes.

5. Consider a market consisting of one money market account with constant interest rate r and one risky asset. The dynamics of the risky asset follows the factor model

$$dS(t) = S(t)\{\alpha dt + \sigma(X(t))dW_1(t)\},$$

where $\sigma : \mathbb{R} \to (0, \infty)$ and the factor X, that drives the volatility, is determined by the dynamics

$$dX(t) = \mu_X(t, X(t), S(t))dt + \nu_X(t, X(t), S(t))dW_2(t).$$

Here, μ_X and ν_X are deterministic functions, and W_1 and W_2 are independent Wiener processes.

Consider a fixed claim with payoff $\mathcal{Y} := \Gamma(S(T))$. Suppose that $G(t, X(t), S(t))$ is the pricing function of the claim $\mathcal{Y}$, where G is assumed to be a smooth function. By the Meta-Theorem, it is expected that the market consisting of the money market account, the tradable asset S, and the claim with price process $P(t) := G(t, X(t), S(t))$ is complete. This market is denoted by $[B, S, P]$. Suppose that $[\sigma_1^P, \nu_2^P]$ are the volatilities of process P such that

$$dP(t) = P(t)\left\{\alpha^P(t)dt + \sigma_1^P(t)dW_1(t) + \sigma_2^P(t)dW_2(t)\right\}.$$

(a) What requirements σ_1^P, and σ_2^P must satisfy for the market to be **arbitrage-free regardless of** α^P?

(b) Give sufficient conditions on σ_1^P and σ_2^P for the market $[B, S, P]$ to be complete no matter the values of α^P.

(c) **Determine the hedging portfolio in the market $[B, S, P]$ for a claim with payoff $\mathcal{X} = \Phi(X(T))$, assuming that the time-t price of the claim is given by $F(t, S(t), P(t))$ for a function $F(t, s, p)$. Explain the assumption on F.**