Ma 515/Stat 540. Practice Final 1.

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1. Consider two stocks S_1 and S_2 driven by the following dynamics:

$$d\,S_1(t) = S_1(t)\left\{\alpha_1 dt + \sigma_1 dW_1(t)\right\},$$
$$d\,S_2(t) = S_2(t)\left\{\alpha_2 dt + \sigma_2 \rho dW_1(t) + \sigma_2\sqrt{1-\rho^2}dW_2(t)\right\},$$

where W_1 and W_2 are independent Wiener processes, and all parameters are presumed constant.

   (a) Find S_2 in terms only of W_1, W_2, α_2, σ_2, and ρ. What is the expected log return of S_2 during a time interval $(s,t]$?

   (b) Indicate how to estimate the volatility σ_1 based on historical prices $S_1(0)$, $S_1(\Delta t)$, ..., $S_1(k\Delta t)$ of the of stock 1.

   (c) Indicate how to estimate the parameter ρ based in the following observations of the two stock prices $(S_1(0), S_2(0))$, $(S_1(\Delta t), S_2(\Delta t))$, ..., $(S_1(k\Delta t), S_2(k\Delta t))$.

2. Suppose that $F(x,s)$ satisfies the partial differential equation

$$\frac{\partial F}{\partial t} + x\frac{\partial F}{\partial x} + 2x^2\frac{\partial^2 F}{\partial x^2} = -F\log x,$$

with boundary condition $F(T,x) = 1$.

   (a) Write down a Feyman-Kac (stochastic) representation formula for F, showing that it involves a geometric Bownian motion X. DEDUCE your formula.

   (b) Show that the function of part (a) is of the form

$$F(t,x) = g(x, T-t)E\left\{e^{\mathbf{2}\int_t^T (W(s)-W(t))ds}\right\},$$

   where $g(t,x)$ is a deterministic function which you will need to find.

3. Consider the American corporation ACME INC. The price process S for a share of ACME is of course denoted in US dollars and has dynamics:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB_1(t),$$

where α and σ are constants. The currency ratio Euros/US \$ at time t, denoted by $Y(t)$, is modeled by the dynamics

$$dY(t) = \beta Y(t)dt + \delta Y(t)dB_2(t),$$

where B_1 and B_2 are correlated Wiener processes with $dB_1 \cdot dB_2 = \rho dt$ and β and δ are constants. A European company sells a contract that gives the right to sell at time T, one share of the ACME stock at a price of K Euros. Let $r_{US}(t)$ and $r_E(t)$ be the time t risk-free short rates in the USA and Euro markets, respectively. Find the arbitrage-free price at time t of the previous contract.

Hint: You can leave your formula expressed in terms of $N(z)$ the cumulative distribution function of the standard Normal random variable.

4. A company sells options that are adjusted to inflation. Concretely, if $S(t)$ is the price of the underline at time t and $Y(t)$ describes the time t inflation, then the option pays

$$\mathcal{X} = \Phi(Z(T)),$$

at time T, where $Z(T) = S(T)/Y(T)$. Suppose that X and Y are modeled via the following dynamics:

$$d\,S(t) = \alpha S(t)dt + \sigma S(t)dW(t),$$
$$d\,Y(t) = \gamma Y(t)dt + \delta Y(t)dV(t),$$

where W and V are independent Wiener processes.

(a) Determine a stochastic differential equation for Z.

(b) Find the price at time t of the option that pays

$$\mathcal{X} = \ln\left\{Z(T)^2\right\}.$$

Assume that $\{Z(t)\}_{t\geq 0}$ represents the price of a tradable asset, that the short-rate r is constant, and that the marker is arbitrage-free.

5. **At the initial time $t = 0$, an option F, maturing in 1 year, is selling for the price of $F_0 = 20$, and its the delta, gamma, rho, and vega are respectively $\Delta_F = -1$, $\Gamma_F = 2$, $\rho_F = 1$, and $\mathcal{V}_F = 3$. The underlying has a volatility of 20% and the short rate r of interest is 5% per annum.** The following part are independent.

(a) At the initial time, you have a short position in the option F and you wish to hedge the option F using the stock and at-the-money call options with strike $K = 100$, so that the final portfolio is both delta neutral and Gamma neutral. You finance your strategy using a money market account. If you rebalance your position every Δt, find a formula for your position in the money market account at time $t = \Delta t$ (before rebalancing again at $t = \Delta t$).

Your formula should be as explicit as possible, expressed maybe in terms of the parameters, $N(\cdot)$, and the values of the underlying and options.

(b) Suppose that you hedge the option F using a "delta hedging strategy" (in the stock) financed by the money market account. If you rebalance your position every Δt, find a formula for your position in the money market account at time $t = 2\Delta t$ (after you rebalance your position and the market prices change). **Assume that at time Δt, the Δ_F changes to $.5$, and also assume that $S_{\Delta t} = 30$, and $S_{2\Delta t} = 25$ (r and the rest of the Greeks of F remain unchanged)**.

(c) During a small time period (say one day), what is the change in the option that you expect if the underlying change \$1 dollar?

(d) If you made an error of $\pm.05$ in the computation of your volatility and an error of $\pm.01$ in the computation of the short interest rate, approximately what error do you expect to make in the computation of the value of the portfolio?

6. An stock has price dynamics (under the objective probability measure P) given by

$$dS_t = S_t\alpha(S_t)dt + S_t\sigma(S_t)dW_t,$$

where W is a Wiener process and α and σ are deterministic functions. The stock pays a continuous dividend of $S_t\delta(S_t)dt$ per share during a small time interval $[t, t+dt)$. We write a T-claim with payoff $\Phi(S_T)$ on the stock.

(a) Consider a hypothetical self-financing portfolio V consisting of a short position in the claim, and a position of $\Delta(t)$ (shares) in the stock. Find the dynamics dV of the value of the portfolio (don't forget the fact that the stock pays dividend).

(b) Assuming that the time t price of the claim is of the form $\Pi(t) := F(t, S(t))$ for a smooth function F. Find the position in the stock so that the portfolio in part (a) is (locally) risk-free.

(c) Determine a partial differential equation for $F(t, s)$.

(d) Determine a stochastic representation for $\Pi(t)$.

(e) Give a formula for the price of a call option on the stock with delivery price K and maturity T (Hint: You don't need to derive your formula. You can leave it in terms of the the Black-Scholes formula $C_{BS}(t, T, s; \sigma, r)$ for a call option on a non-paying stock).