Math515/Stat540 Spring 2009
Homework assignment 4 Figueroa-Lopez

1. Björk's book: Exercises 4.1, 4.2, 4.8, 5.1, 5.5, 5.6. 5.7, 5.8.

2. Let $\{W_t : t \geq 0\}$ be a Wiener process:

   (a) Find the covariance $\mathrm{Cov}(W_t, W_s)$, for $t, s \geq 0$.

   (b) Show that for any constant $c > 0$, the process $\{\sqrt{c}\, W_{t/c}\}_{t\geq 0}$ is a Wiener process. *Note: Such a property of the Wiener process is called self-similarity.*

3. Verify that the following two conditions are equivalent:

$$M_t := \left(\int_0^t f_s dW_s\right)^2 - \int_0^t f_s^2 ds \text{ is a martingale relative to } \{\mathcal{F}_t\}_{t\geq 0},$$

$$E\left[\left(\int_t^u f_s dW_s\right)^2 \big| \mathcal{F}_t^W\right] = E\left[\int_t^u f_s^2 ds \,\big| \mathcal{F}_t^W\right], \quad \text{for any } 0 \leq t \leq u.$$

4. Let $X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$ and $Y_t = Y_0 + \int_0^t \tilde{\mu}_s ds + \int_0^t \tilde{\sigma}_s dW_s$ be two Itô processes.

   (a) Find the following differentials:

$$d(X_t + Y_t)^2, \quad dX_t^2, \quad dY_t^2.$$

   (b) Using parts (a) and that $xy = \{(x+y)^2 - x^2 - y^2\}/2$, show that

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t \sigma_s \tilde{\sigma}_s ds. \tag{1}$$

5. Let $S_t = S_0 e^{\sigma W_t + \mu t}$, $t \geq 0$, be the geometric Brownian motion. Suppose that there is another process $\tilde{S}$ such that

$$d\tilde{S}_t = \tilde{S}_t \left\{ \left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma dW_t \right\}, \tag{2}$$

   with $\tilde{S}_0 = S_0$. Apply the integration by parts formula (1) and Itô's formula to show that

$$d(S_t^{-1} \tilde{S}_t) = 0.$$

   Conclude that $S_t = \tilde{S}_t$, for all t.