Math515/Stat540 Spring 2009
Homework assignment 3 Figueroa-Lopez

1. Consider an arbitrage-free Binomial market model.

   (a) Find the price process $\{\Pi(t) : t = 0, \ldots, T\}$ of a forward contract on the stock.

   (b) Prove the so-called *put-call parity*, which can be stated as follows:

   > "A long position in a European put on the stock is equivalent to a long position in a European call on the same stock together with a short position in a forward contract on the stock."

2. Consider a discrete-time market model with one stock and a money-market account with constant interest-rate R per period. Let Q be a martingale measure for the price process $\{S_t : t = 0, \ldots, T\}$ of the stock and let $h = \{(x_t, y_t) : t = 1, \ldots, T\}$ be a trading strategy.

   (a) Show that if h is a self-financing strategy then

   $$V_t^h = V_{t-1}^h + \frac{x_t}{B_{t-1}} (B_t - B_{t-1}) + y_t (S_t - S_{t-1}),$$

   for all $t = 1, \ldots, T$, where $B_t = (1 + R)^t$.

   (b) Show that if h is a self-financing strategy then the discounted value process of the portfolio,

   $$V_t^{h,*} := \frac{1}{B_t} V_t^h, \quad t = 0, \ldots, T,$$

   is such that

   $$V_t^{h,*} = V_{t-1}^{h,*} + y_t \left(S_t^* - S_{t-1}^* \right), \tag{1}$$

   for all $t = 1, \ldots, T$, where $S_t^* := S_t / B_t$ is the discounted prices process of the stock.

   (c) Is the converse of (b) true? Justify or give a counterexample.

   (d) Also, prove that

   $$E^Q \left\{ V_u^{h,*} \middle| V_t^{h,*}, \ldots, V_0^{h,*} \right\} = V_t^{h,*},$$

   for any $t, u \in \{0, \ldots, T\}$ with $t \le u$.

   State explicitly which properties of conditional expectation given in the class notes are being used.

3. Consider the Cox-Ross-Rubistein (CRR) Binomial market model with n trading periods.

   (a) Show that under the risk-neutral measure Q_n, the drift and volatility of $\widetilde{S}^n$ per period are such that

   $$\frac{1}{\Delta_n} \cdot E^{Q_n} \left\{ \log \frac{\widetilde{S}_{t_i}^n}{\widetilde{S}_{t_{i-1}}^n} \right\} \overset{n \to \infty}{\longrightarrow} r - \frac{1}{2}\sigma^2, \quad \frac{1}{\Delta_n} \cdot \mathrm{Var}^{Q_n} \left\{ \log \frac{\widetilde{S}_{t_i}^n}{\widetilde{S}_{t_{i-1}}^n} \right\} \overset{n \to \infty}{\longrightarrow} \sigma^2.$$

(b) Remember that in the light of Donsker's invariance theorem $\widetilde{S}_T^n \xrightarrow{\mathfrak{D}} S_0 e^{\sigma W_T + (r - \frac{\sigma^2}{2})T}$. Justify that

$$\Pi_n(0) := e^{-rT} E^{Q_n} \left\{ \left(\widetilde{S}_T^n - K \right)_+ \right\} \longrightarrow \Pi^{\text{call}}(0) = e^{-rT} E^{Q} \left\{ \left(S_0 e^{\sigma W_T + (r - \frac{\sigma^2}{2})T} - K \right)_+ \right\},$$

as $n \to \infty$.

4. A nine-month American put option on a non-dividend-paying stock has a strike price of \$49. The stock price is \$ 50, the risk-free rate is 5% per annum, and the volatility is 30% per annum.

   (a) Use a three period CRR Binomial tree to calculate the option price.

   (b) Determine the hedging strategy from your tree.

   (c) Suppose that we used instead a 100 period Binomial CRR model. The drift μ of stock is such that the probability of an upward move is $p = .6$. (i) Determine μ and (ii) estimate the probability that the stock will increase at least 80% in value during the next 9 months.