**Math515/Stat540** Spring 2009
**Homework Assignment 2** Figueroa-Lopez

1. Consider an arbitrage-free one-period market model.

   (a) Show that for any risk-neutral probability measure $Q$ and any trading strategy $h$,
   $$E^Q\left\{\frac{V_1^h}{1+R}\right\} = V_0^h.$$

   (b) Use (a) to show that the market is arbitrage-free.

   (c) Show that if $\mathcal{X}$ is replicable, then
   $$V_0^h = \frac{1}{1+R}E^Q\{\mathcal{X}\},$$
   for all replicating portfolio $h$ and all equivalent risk-neutral measures $Q$.

2. A stock price is currently \$100. Over each of the next two 1-month periods the stock is expected to go up or down by 10%. The risk-free interest rate is 3% per month.

   (a) What is the value of a two month European call option on this stock with strike price \$100?

   (b) Find the replicating trading strategy for this claim at time $t = 0$ and $t = 1$.

3. Consider a three-period Binomial market model with parameters $R = 2\%$ per period, $u = 1.08$, $d = .96$, and $S_0 = 100$. A *lookback call option* is a type of *path-dependent* contingent claim that pays
   $$\mathcal{X} := \left(\max\{S_0, \ldots, S_T\} - K\right)_+,$$
   at maturity.

   (a) Using Backward induction, determine the initial price $\Pi(0; \mathcal{X})$ of the lookback call option with strike $K = 100$.

   (b) Compute also the time $t = 2$ hedging portfolios for the claim (recall that the hedging strategies will depend not only upon the current stock price but also the past price evolution).

   (c) Compute the initial price of the claim using the risk-neutral valuation formula
   $$\Pi(t; \mathcal{X}) = E^Q\left\{\frac{\mathcal{X}}{(1+R)^T}\right\}.$$

4. Consider a stock and a risk-free asset following the Binomial market model with $u = 2$, $d = 1/2$, and $R = 1/4$. The stock is currently at \$4.

   (a) For $t = 0$ and $t = 1$, compute the time $t$ value, denoted by $V_t$, of an American put option maturing at $T = 2$ with strike $K = 5$.

   (b) Find the optimum exercise time $\tau^*$ for the buyer. You need to write $\tau^*(\omega_1, \omega_2)$ for each $(w_1, w_2)$ of the sample space $\Omega = \{(w_1, w_2) : w_i = 0, 1\}$.

5. Consider a two period discrete-time model $\{S_t : t = 0, 1, 2\}$ and an American contingent claim with immediate payoff process $\{Z_t : t = 0, 1, 2\}$. In other words, if the holder of the claim exercise the option at time $t$, then he receives a payoff of $Z_t$, which depends on the current asset price. The stock price evolution and immediate payoffs are given as follows:

$$S_2 = 9, Z_2 = 4$$

$$S_1 = 8, Z_1 = 4$$

$$S_0 = 5, Z_0 = 1 \qquad\qquad S_2 = 6, Z_2 = 1$$

$$S_1 = 4, Z_1 = 0$$

$$S_2 = 3, Z_2 = 0$$

   Suppose that the rate of return of the risk-free asset per period is $R = 0$.

   (a) Using Backward induction, compute the value process $\{V_t : t = 0, 1, 2\}$ of the American option.

   (b) Is it optimal to exercise early? If so, when?

6. Consider an American option with immediate payoff process $\{Z_t : t = 0, \ldots, T\}$ and the corresponding European option with time $T$ payoff $\mathcal{X} = Z_T$. Let $V_t^e$ denote the time $t$ value of this European option. Explain why if $V_t^e(\omega) \geq Z_t(\omega)$ for all $t$ and state $\omega$, then, for all $t$, $V_t^e$ is equal to $V_t^a$, the time $t$ value of the American option, and it is optimal to wait until time $T$ to exercise. You can assume a discrete arbitrage-free Binomial market model, though the result is model free.

7. Consider a discrete-time arbitrage-free complete market model with maturity $T$. Let $P_0$ and $C_0$ be the time-zero price of an American put and call options with maturity $T$, respectively. Let $U(0)$ be the price at time zero of an American *straddle* that expires at time $T$ and has immediate payoff $(S_t - K)_+ + (K - S_t)_+$ at time $t$. Explain why $U(0) \leq P(0) + C(0)$.