Parametric Estimation of Geometric Lévy Models under High-Frequency Data

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Modeling High Frequency Data in Finance II
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Outline

1. Motivation
2. Objectives
3. Statistical Methodology
4. Case study 1: Variance Gamma Process
   - Finite-sample performance
   - Empirical results
5. Case study 2: Normal Inverse Gaussian
   - Finite-sample performance
   - Empirical results
6. Overview of preliminary theoretical results
7. Conclusions and open problems
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6 Overview of preliminary theoretical results
7 Conclusions and open problems
Stylized empirical features of asset prices

1. Asset prices follow a jiggling motion at different “resolutions” and scales
2. “Sudden large” price changes (Jumps)
3. Empirical distributions of log returns exhibit heavy-tails and high-kurtosis
4. Volatility clustering (intermittency)
5. Leverage phenomenon
6. Weak correlation of returns
7. Strong correlation of absolute returns
Asset price modeling

1 Continuous time:

- \( S_t \) = Stock price at any time \( t \in [0, \infty) \).
- \( X_t = \log \frac{S_t}{S_0} \) = Log return at any time \( t \in [0, \infty) \).

2 Discrete time:

- Time series of stock prices:

\[ S_{\delta}, \ldots, S_{n\delta}, \] for a given time mesh \( \delta \);

- Time series of simple returns:

\[ r_1^\delta = \frac{S_{\delta} - S_0}{S_0}, \ldots, r_n^\delta = \frac{S_{n\delta} - S_{(n-1)\delta}}{S_{(n-1)\delta}} \]

- Time series of log returns:

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Popular models

1. **Black-Scholes model** (Geometric Brownian Motion):

   \[
   S_t = S_0 \exp \left\{ bt + \sigma W_t \right\} \iff R_1^\delta, \ldots, R_n^\delta \overset{\text{i.i.d.}}{\sim} \mathcal{N}(b\delta, \sigma^2 \delta)
   \]

2. **Stochastic volatility models**:

   \[
   S_t = S_0 \exp \left\{ \int_0^t b_u du + \int_0^t \sigma_u dW_u \right\} \iff R_{i+1}^\delta \mid \mathcal{F}_i \overset{\mathcal{D}}{\approx} \mathcal{N}(b_i \delta, \sigma_{i\delta}^2 \delta)
   \]

3. **Geometric Lévy models**:

   \[
   S_t = S_0 \exp \left\{ bt + \sigma W_t + Z_t \right\} \iff R_1^\delta, \ldots, R_n^\delta \overset{\text{i.i.d.}}{\sim} f_\delta(\cdot)
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- **Drift**
- **Brownian**
- **Pure-jump Lévy**
- **Heavy tail**
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   \[ \text{Pure-jump Lévy} \]

   \[ \text{Heavy tail} \]
Key features of geometric Lévy models

1. Incorporation jumps ("sudden" price changes but no "reversals");
2. Every time series of log returns are independent with common distribution (facilitating the statistical estimation of the model by traditional methods; e.g. Maximum Likelihood Estimation and Method of Moments);
3. Large class of possible distributions for log returns;
4. Some classes have good fitting properties:
   - Variance Gamma
   - CGMY model
   - Normal Inverse Gaussian
   - Generalized hyperbolic
   (Note: Most empirical evidence based only on daily returns)
5. Lack of volatility clustering and leverage;
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Problems of interest

1. Analyze in depth the capability of GLM to describe intraday log returns:
   - Note: Microstructure noise will deem GLM useless for very “high” frequencies; Key question: How high is high?
   - Determine a range of frequencies for which a given GLM might be suitable.

2. Analyze the performance of traditional estimation methods using intraday returns:
   - Do we gain anything (in terms of estimation error) by considering intraday returns?
   - Do we expect better estimation performance when \( \delta \to 0 \) (disregarding the “Microstructure noise” and possible dependence between returns)?
   - Can we attain consistency and efficiency of estimation when \( \delta \to 0 \)?
   - Are the estimators “stable” for different \( \delta \)?
   - Expected behavior of the estimators under microstructure noise?
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Maximum Likelihood Estimation (MLE)

1. Maximum Likelihood Principle (Fisher):
   *The most sensible values of the parameter of the model are those that maximize the likelihood (chance) of observing the sample data at hand.*

2. General implementation method:
   - $r^\delta_1, \ldots, r^\delta_n$ are the sample observations of $n$ equally-spaced log returns (with time-span $\delta$).
   - Compute the Likelihood Function defined by
     \[
     L_\delta(\theta; r_1, \ldots, r_n) = \prod_{i=1}^{n} f_\delta(r_i; \theta).
     \]
   - The *Maximum Likelihood Estimate* (when it exists) is defined by
     \[
     \hat{\theta} = \hat{\theta}^\delta(r_1, \ldots, r_n) = \arg\max_\theta L_\delta(\theta; r_1, \ldots, r_n).
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Lévy-based models are often described in terms of their marginal characteristic function or Fourier transform $\hat{f}_\delta$; but, the density $f_\delta$ is unknown or "intractable".

A possible solutions:

- Inversion formula.

$$f_\delta (r ; \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izr} \hat{f}_\delta (z ; \theta) \, dz.$$  

- Coupled with an approximation of the integral via Fast Fourier Transform.

In many cases, the maximization problem is numerically hard to solve due to the non-concavity and flatness of the likelihood function;
Implementation issues

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Method of Moment Estimators

1. General idea:
   \textit{Choose the parameter values that match the theoretical moments with the sample empirical moments.}

2. General implementation method:
   \begin{itemize}
   \item $r_1^\delta, \ldots, r_n^\delta$ are the sample observations of $n$ equally-spaced returns (with time-span $\delta$).
   \item Compute the theoretical and sample (central) moments (as many as parameters are in the model)
   \begin{align*}
   m_k^\delta(\theta) &= \mathbb{E}[(R_i^\delta)^k], \quad \hat{m}_k = \frac{1}{n} \sum_{i=1}^{n} (r_i^\delta)^k.
   \end{align*}
   \item The \textit{Method of Moment Estimate} (when it exists) is defined as the solution $\hat{\theta}$ of the system of equations:
   \begin{align*}
   m_k^\delta(\hat{\theta}) &= \hat{m}_k, \quad k = 1, \ldots, d.
   \end{align*}
   \end{itemize}
Method of Moment Estimators

**General idea:**

*Choose the parameter values that match the theoretical moments with the sample empirical moments.*

**General implementation method:**

- $r_1^\delta, \ldots, r_n^\delta$ are the sample observations of $n$ equally-spaced returns (with time-span $\delta$).
- Compute the theoretical and sample (central) moments (as many as parameters are in the model)

\[
m_k^\delta(\theta) = \mathbb{E}[(R_i^\delta)^k], \quad \hat{m}_k = \frac{1}{n} \sum_{i=1}^{n} (r_i^\delta)^k.
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- The *Method of Moment Estimate* (when it exists) is defined as the solution $\hat{\theta}$ of the system of equations:

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Method of Moment Estimators

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Choose the parameter values that match the theoretical moments with the sample empirical moments.

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Outline

1 Motivation
2 Objectives
3 Statistical Methodology
4 Case study 1: Variance Gamma Process
   Finite-sample performance
   Empirical results
5 Case study 2: Normal Inverse Gaussian
   Finite-sample performance
   Empirical results
6 Overview of preliminary theoretical results
7 Conclusions and open problems
Variance Gamma Model

- **Definition:** (Carr, Madan, and Chang 98)

  \[
  \text{Log return on time-span } t = X_t = \sigma W_{\tau_t} + \theta \tau_t + b t,
  \]

  where \( W \) standard Brownian motion and \( \tau_t \) (Business Clock) Gamma Lévy process; that is, a Lévy process such that \( \tau_t \overset{D}{\sim} \text{Gamma} \left( \frac{t}{\kappa}, \kappa \right) \) (in particular, \( \mathbb{E}\tau_t = t \) and \( \text{Var}(\tau_t) = \kappa t \)).

- **Log returns:**

  \[
  R_1^\delta, \ldots, R_n^\delta \bigg| (\tau_t)_t \sim \mathcal{N}(\theta \tau^\delta + b \delta, \sigma^2 \tau^\delta)
  \]

- **Simulation:**

  - Generate \( s_1, \ldots, s_n \overset{\text{i.i.d.}}{\sim} \text{Gamma} \left( \frac{\delta}{\kappa}, \kappa \right) \);
  - Generate \( z_1, \ldots, z_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1) \);
  - \( R_i^\delta := b \delta + \theta s_i + \sigma \sqrt{s_i} z_i, \; i = 1, \ldots, n \);
Case study 1: Variance Gamma Process

Variance Gamma Model

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  \[ \text{Log return on time-span } t = X_t = \sigma W_{\tau_t} + \theta \tau_t + b \, t, \]

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**Case study 1: Variance Gamma Process**

**Variance Gamma Model**

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Variance Gamma Model

- Moments:

\[ \mu_1^\delta := \mathbb{E}(X_\delta), \quad \mu_2^\delta := \text{Var}(X_\delta) \]
\[ \mu_1^\delta = (\theta + b)\delta, \quad \mu_2^\delta = (\sigma^2 + \theta^2\kappa)\delta, \]
\[ \mu_3^\delta := \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^3 = (3\sigma^2\theta\kappa + 2\theta^3\kappa^2)\delta, \]
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- Interpretation for \( \theta \) is “small” (which is typically the case):

\[ \text{Var}(X_\delta) \approx \sigma^2\delta, \quad \text{Kurt}(X_\delta) := \frac{\mu_4^\delta}{\mu_2^\delta} - 3 \approx \frac{3\kappa}{\delta}, \quad \text{Skew}(X_\delta) := \frac{\mu_3^\delta}{\mu_2^{3/2}} \approx \frac{3\kappa}{\sigma \delta^{1/2} \theta}. \]

- \( \sigma \) controls the “volatility” (per time unit) of the log returns;
- \( \kappa \) controls the kurtosis (fatness of tails) of log returns;
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Variance Gamma Model

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  \[ \mu_1^\delta := \mathbb{E}(X_\delta), \quad \mu_2^\delta := \text{Var}(X_\delta) \quad \mu_1^\delta = (\theta + b)\delta, \quad \mu_2^\delta = (\sigma^2 + \theta^2 \kappa)\delta, \]

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  \mu^\delta_1 = (\theta + b)\delta, \quad \mu^\delta_2 = (\sigma^2 + \theta^2\kappa)\delta,
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Variance Gamma Model

- Density has a “closed form” in terms of “Bessel functions of second kind”:

\[
R_i^\delta \sim f_\delta(x) = \frac{2e^{\theta(x-b\delta)/\sigma^2}}{\sigma \sqrt{2\pi \kappa^{\delta/\kappa}} \Gamma\left(\frac{\delta}{\kappa}\right)} \left( \frac{|x - b\delta|}{\sqrt{\frac{2\sigma^2}{\kappa} + \theta^2}} \right)^{\frac{\delta}{\kappa} - \frac{1}{2}} \\
\times K_{\frac{\delta}{\kappa} - \frac{1}{2}} \left( \frac{|x - b\delta| \sqrt{\frac{2\sigma^2}{\kappa} + \theta^2}}{\sigma^2} \right)
\]

- Most high-level mathematical software provides built-in functions to compute Bessel functions;
Method of Moment Estimators

1. Once the sample moments $\hat{\mu}_i^\delta$ are computed, the MME solves

$$
\hat{\mu}_1^\delta = (\theta + b)\delta, \quad \hat{\mu}_2^\delta = (\sigma^2 + \theta^2 \kappa)\delta, \quad \hat{\mu}_3^\delta = (3\sigma^2 \theta \kappa + 2\theta^3 \kappa^2)\delta,
$$

$$
\hat{\mu}_4^\delta = (3\sigma^4 \kappa + 12\sigma^2 \theta^2 \kappa^2 + 6\theta^4 \kappa^3)\delta + 3(\hat{\mu}_2^\delta)^2.
$$

2. The solution in the symmetric case $\theta = 0$ is

$$\hat{\sigma}^2 = \frac{\hat{\mu}_2^\delta}{\delta} = \frac{1}{T} \sum_{i=1}^{n} (X_{i\delta} - X_{(i-1)\delta})^2, \quad \hat{\kappa} := \delta \frac{\hat{\mu}_4}{\hat{\mu}_2^2} - 3\delta = \frac{\delta}{3} \text{Kurt}, \quad \hat{b} := \frac{\hat{\mu}_1}{\delta} = \frac{X_T}{T}$$

3. Exact method:

- Express all the equation in terms of $\varepsilon := \theta^2 \kappa / \sigma^2$:

$$\hat{\mu}_2 = \delta \sigma^2 (1 + \varepsilon), \quad \hat{\mu}_3 = \delta \sigma^2 \theta \kappa (3 + 2\varepsilon), \quad \frac{\hat{\mu}_4}{3\hat{\mu}_2^2} - 1 = \frac{\kappa}{\delta} \frac{1 + 4\varepsilon + 2\varepsilon^2}{(1 + \varepsilon)^2}.$$

- Note that $\frac{3\hat{\mu}_3^2}{\hat{\mu}_2 (\hat{\mu}_4 - 3\hat{\mu}_2^2)} = \frac{\varepsilon (3 + 2\varepsilon)^2}{(1 + 4\varepsilon + 2\varepsilon^2)(1 + \varepsilon)} := f(\varepsilon)$, concave and increasing.
Method of Moment Estimators

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Case study 1: Variance Gamma Process

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- Note that

$$\frac{3\hat{\mu}_3^2}{\hat{\mu}_2(\hat{\mu}_4 - 3\hat{\mu}_2^2)} = \frac{\varepsilon(3+2\varepsilon)^2}{(1+4\varepsilon+2\varepsilon^2)(1+\varepsilon)} := f(\varepsilon),$$

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\hat{\mu}_2 &= \delta\sigma^2(1 + \varepsilon), \\
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1. Once the sample moments $\hat{\mu}^\delta_i$ are computed, the MME solves

$$
\hat{\mu}^\delta_1 = (\theta + b)\delta, \quad \hat{\mu}^\delta_2 = (\sigma^2 + \theta^2\kappa)\delta, \quad \hat{\mu}^\delta_3 = (3\sigma^2\theta\kappa + 2\theta^3\kappa^2)\delta, \\
\hat{\mu}^\delta_4 = (3\sigma^4\kappa + 12\sigma^2\theta^2\kappa^2 + 6\theta^4\kappa^3)\delta + 3(\hat{\mu}^\delta_2)^2.
$$

2. The solution in the symmetric case $\theta = 0$ is

$$
\hat{\sigma}^2 = \frac{1}{T} \sum_{i=1}^{n} (X_{i\delta} - X_{(i-1)\delta})^2, \quad \hat{\kappa} = \frac{\delta}{3} \text{Kurt}, \quad \hat{b} = \frac{X_T}{T}.
$$

3. Exact method:

- Express all the equation in terms of $\mathcal{E} := \theta^2\kappa/\sigma^2$:

$$
\hat{\mu}_2 = \delta\sigma^2(1 + \mathcal{E}), \quad \hat{\mu}_3 = \delta\sigma^2\theta\kappa(3 + 2\mathcal{E}), \quad \frac{3\hat{\mu}_3^2}{\hat{\mu}_2^3 - 1} = \frac{\kappa}{\delta} \frac{1 + 4\mathcal{E} + 2\mathcal{E}^2}{(1 + \mathcal{E})^2}.
$$

- Note that

$$
\frac{3\hat{\mu}_3^2}{\hat{\mu}_2(\hat{\mu}_4 - 3\hat{\mu}_2^2)} = \frac{\mathcal{E}(3+2\mathcal{E})^2}{(1+4\mathcal{E}+2\mathcal{E}^2)(1+\mathcal{E})} := f(\mathcal{E}), \text{ concave and increasing.}
$$
Finite-sample performance

1. Parameter values (time units are days):

   \[ \sigma = 0.0127, \quad \kappa = 0.2873, \quad b = -0.0017, \quad \theta = 0.0013. \]

2. Sample mean and standard error of the estimators based on:
   - 200 simulations of 5-second log returns during one year (252 days);
   - Trading period per day is 6.5 hours;
   - Overnight returns are not considered;

3. Maximum Likelihood is found numerically using Powell’s optimization scheme with starting point given by the exact moment estimators;

4. Compute the MME and MLE based on log returns with each of the following time meshes:

   \[ \delta = 10, 20, 30 \text{ min}, \quad \text{and} \quad 1/6, 1/4, 1/3, 1/2, 1 \text{ day}. \]
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Parameter values (time units are days):

\[ \sigma = 0.0127, \quad \kappa = 0.2873, \quad b = -0.0017, \quad \theta = 0.0013. \]

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- 200 simulations of 5-second log returns during one year (252 days);
- Trading period per day is 6.5 hours;
- Overnight returns are not considered;

Maximum Likelihood is found numerically using Powell’s optimization scheme with starting point given by the exact moment estimators;

Compute the MME and MLE based on log returns with each of the following time meshes:

\[ \delta = 10, 20, 30 \text{ min}, \text{ and } \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1 \text{ day}. \]
MME and MLE for $\sigma$ in the VG model

\[ \delta = \text{time span between observations} \]

<table>
<thead>
<tr>
<th>Mean of MLE</th>
<th>Mean + Std of MLE</th>
<th>Mean - Std of MLE</th>
<th>Mean of MME</th>
<th>Mean + Std of MME</th>
<th>Mean - Std of MME</th>
<th>True value = 0.0127</th>
</tr>
</thead>
<tbody>
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<td>0.011</td>
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</tr>
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</table>
MME and MLE for $\kappa$ in the VG model

![Graph showing MME and MLE for $\nu$ in the Variance Gamma Model](image-url)

- **Mean of MLE**
- **Mean + Std of MLE**
- **Mean − Std of MLE**
- **Mean of MME**
- **Mean + Std of MME**
- **Mean − Std of MME**

True value = 0.2873
MME and MLE for $\theta$ in the VG model

MME and MLE for $\theta$
Variance Gamma Model

$\delta$ = time span between observations

Mean of MME
Mean + Std of MME
Mean − Std of MME
Mean of MLE
Mean + Std of MLE
Mean − Std of MLE
True value = 0.0013
MME and MLE for $b$ in the VG model

\[ \delta = \text{time span between observations} \]

True value = $-0.0017$
An empirical case study: INTEL

1. The data was obtained from the NYSE TAQ database of 2005 trades via Wharton WRDS database.

2. Data was preprocessed to get rid of “irregular” and “reversal” quotes:


![“Clean” Intel 5-Second Stock Prices (Jan. 2, 2005 – Dec. 30, 2005)](image2)
MLE and MME for INTC Stock Data 2005

Table: Top panel MLE, Middle panel is Exact MME, and Bottom panel is Approx. MME (fixing $\theta = 0$).
## MLE and MME for INTC Stock Data 2005

<table>
<thead>
<tr>
<th>δ</th>
<th>10 sec</th>
<th>20 sec</th>
<th>30 sec</th>
<th>1 min</th>
<th>5 min</th>
<th>10 min</th>
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<tr>
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<td>-0.0004</td>
<td>-0.0004</td>
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</tr>
<tr>
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<td>0.0052</td>
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<td>0.0282</td>
</tr>
<tr>
<td>(\hat{\sigma})</td>
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<td>0.0152</td>
<td>0.0145</td>
<td>0.0138</td>
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<td>0.0121</td>
</tr>
<tr>
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<td>0.0014</td>
<td>0.0025</td>
<td>-0.0040</td>
<td>-0.0013</td>
<td>0.0011</td>
</tr>
<tr>
<td>(\hat{b})</td>
<td>-0.0003</td>
<td>-0.0018</td>
<td>-0.0029</td>
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<td>(\hat{\sigma})</td>
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<td>0.0036</td>
<td>0.0009</td>
<td>-0.0015</td>
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</tbody>
</table>

**Table:** Top panel MLE, Middle panel is Exact MME, and Bottom panel is Approx. MME (fixing \(\theta = 0\)).
Empirical distribution vs. fitted density

INTC Stock Data 2005

Histogram vs. Fitted Variance Gamma
INTC log returns with $\delta = 1/6$

Histogram
Fitted VG density
Fitted Normal Distribution
Outline

1 Motivation
2 Objectives
3 Statistical Methodology
4 Case study 1: Variance Gamma Process
   Finite-sample performance
   Empirical results
5 Case study 2: Normal Inverse Gaussian
   Finite-sample performance
   Empirical results
6 Overview of preliminary theoretical results
7 Conclusions and open problems
Normal Inverse Gaussian

- **Definition:** (Barndorff-Nielsen)

\[ X_t = \sigma W_{\tau_t} + \theta \tau_t + b t, \]

\( \text{W standard Brownian motion and } \tau_t \text{ is an Inverse Gaussian subordinator such that } \mathbb{E}\tau_t = t \text{ and } \text{Var}(\tau_t) = \kappa t. \)

- **Moments:**

\[ \mu_1^\delta := \mathbb{E}(X_\delta) = (\theta + b)\delta, \]

\[ \mu_2^\delta := \text{Var}(X_\delta) = (\sigma^2 + \theta^2 \kappa)\delta, \]

\[ \mu_3^\delta := \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^3 = (3\sigma^2 \theta \kappa + 3\theta^3 \kappa^2)\delta, \]

\[ \mu_4^\delta := \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^4 \]

\[ = (3\sigma^4 \kappa + 18\sigma^2 \theta^2 \kappa^2 + 15\theta^4 \kappa^3)\delta + 3(\mu_2^\delta)^2. \]
Normal Inverse Gaussian

- **Definition:** (Barndorff-Nielsen)

\[ X_t = \sigma W_{\tau_t} + \theta \tau_t + b t, \]

\( W \) standard Brownian motion and \( \tau_t \) is an Inverse Gaussian subordinator such that \( \mathbb{E} \tau_t = t \) and \( \text{Var}(\tau_t) = \kappa t \).

- **Moments:**

\[
\begin{align*}
\mu_1^\delta := & \mathbb{E}(X_\delta) = (\theta + b)\delta, \\
\mu_2^\delta := & \text{Var}(X_\delta) = (\sigma^2 + \theta^2 \kappa)\delta, \\
\mu_3^\delta := & \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^3 = (3\sigma^2 \theta \kappa + 3\theta^3 \kappa^2)\delta, \\
\mu_4^\delta := & \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^4 \\
&= (3\sigma^4 \kappa + 18\sigma^2 \theta^2 \kappa^2 + 15\theta^4 \kappa^3)\delta + 3(\mu_2^\delta)^2.
\end{align*}
\]
Normal Inverse Gaussian. Cont.

- **Method of Moments Estimators:**
  - Express all the equation in terms of \( \mathcal{E} := \theta^2 \kappa / \sigma^2 \):
    
    \[
    \hat{\mu}_2 = \delta \sigma^2 (1 + \mathcal{E}), \quad \hat{\mu}_3 = \delta \sigma^2 \theta \kappa (3 + 3 \mathcal{E}), \quad \frac{\hat{\mu}_4}{3 \hat{\mu}_2^2} - 1 = \frac{\kappa}{\delta} \frac{1 + 5 \mathcal{E}}{1 + \mathcal{E}}.
    \]
  - Note that \( \frac{3 \hat{\mu}_2^2}{\hat{\mu}_2 (\hat{\mu}_4 - 3 \hat{\mu}_2^2)} = \frac{9 \mathcal{E}}{5 \mathcal{E} + 1} := f(\mathcal{E}) \), which can be solved for \( \mathcal{E} \).
  - Density has a “closed form” in terms of “Bessel functions of second kind”:
    
    \[
    f_\delta(x) = \frac{Ce^{\theta (x-b\delta)/\sigma^2}}{\sqrt{(x-b\delta)^2 + \delta^2 \sigma^2 / \kappa}} K_1 \left( B \sqrt{(x-b\delta)^2 + \delta^2 \sigma^2 / \kappa} \right),
    \]
Method of Moments Estimators:

Express all the equation in terms of $\mathcal{E} := \theta^2 \kappa / \sigma^2$:

$$\hat{\mu}_2 = \delta \sigma^2 (1 + \mathcal{E}), \quad \hat{\mu}_3 = \delta \sigma^2 \theta \kappa (3 + 3\mathcal{E}), \quad \frac{\hat{\mu}_4}{3\hat{\mu}_2^2} - 1 = \frac{\kappa}{\delta} \frac{1 + 5\mathcal{E}}{1 + \mathcal{E}}.$$

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Density has a “closed form” in terms of “Bessel functions of second kind”:

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Normal Inverse Gaussian. Cont.

- Method of Moments Estimators:
  - Express all the equation in terms of $\mathcal{E} := \theta^2 \kappa / \sigma^2$:
    \[
    \hat{\mu}_2 = \delta \sigma^2 (1 + \mathcal{E}), \quad \hat{\mu}_3 = \delta \sigma^2 \theta \kappa (3 + 3 \mathcal{E}), \quad \frac{\hat{\mu}_4}{3\hat{\mu}_2^2} - 1 = \frac{\kappa}{\delta} \frac{1 + 5 \mathcal{E}}{1 + \mathcal{E}}.
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- **Method of Moments Estimators:**
  - Express all the equation in terms of $\mathcal{E} := \theta^2 \kappa / \sigma^2$:
    
    $\hat{\mu}_2 = \delta \sigma^2 (1 + \mathcal{E})$,  
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  - Note that $\frac{3 \hat{\mu}_3^2}{\hat{\mu}_2 (\hat{\mu}_4 - 3 \hat{\mu}_2^2)} = \frac{9 \mathcal{E}}{5 \mathcal{E} + 1} := f(\mathcal{E})$, which can be solved for $\mathcal{E}$.

- Density has a "closed form" in terms of "Bessel functions of second kind":
  
  $f_\delta(x) = \frac{Ce^{\theta(x-b\delta)/\sigma^2}}{\sqrt{(x-b\delta)^2 + \delta^2 \sigma^2/\kappa}} K_1 \left( B \sqrt{(x-b\delta)^2 + \delta^2 \sigma^2/\kappa} \right)$,
MME and MLE for $\sigma$ in the NIG model

True Value = 0.0080; $\delta = 1, 1/2, 1/3, 1/6, 1/12, \ldots, 1/(6 \times 60 \times 12)$
MME and MLE for $\kappa$ in the NIG model

True Value=0.422; $\delta = 1, 1/2, 1/3, 1/6, 1/12, \ldots, 1/(6 \times 60 \times 12)$
MME and MLE for $\theta$ in the NIG model

True Value=-0.00015; $\delta = 1, 1/2, 1/3, 1/6, 1/12, ..., 1/(6 \times 60 \times 12)$
MLE and MME for INTC Stock Data 2005

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<th>30 min</th>
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Table: Top panel MLE, Middle panel is Exact MME, and Bottom panel is Approx. MME.
### MLE and MME for INTC Stock Data 2005

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<th>δ</th>
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<th>30 sec</th>
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<th>30 sec</th>
<th>1 min</th>
<th>5 min</th>
<th>10 min</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{\kappa})</td>
<td>0.0003</td>
<td>0.0007</td>
<td>0.0012</td>
<td>0.0031</td>
<td>0.0160</td>
<td>0.0255</td>
</tr>
<tr>
<td>(\hat{\sigma})</td>
<td>0.0194</td>
<td>0.0161</td>
<td>0.0148</td>
<td>0.0134</td>
<td>0.0120</td>
<td>0.0114</td>
</tr>
<tr>
<td>(\hat{\theta})</td>
<td>0.0194</td>
<td>0.0187</td>
<td>0.0159</td>
<td>0.0132</td>
<td>0.0069</td>
<td>0.0042</td>
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<tr>
<td>(\hat{b})</td>
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<td>-0.0188</td>
<td>-0.0161</td>
<td>-0.0134</td>
<td>-0.0070</td>
<td>-0.0044</td>
</tr>
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</table>

**Table:** Top panel MLE, Middle panel is Exact MME, and Bottom panel is Approx. MME.
Empirical results

Empirical distribution vs. fitted density

INTC Stock Data 2005

Histogram vs. Fitted NIG model
INTC log returns with $\delta = 1/6$
Outline

1 Motivation
2 Objectives
3 Statistical Methodology
4 Case study 1: Variance Gamma Process
   Finite-sample performance
   Empirical results
5 Case study 2: Normal Inverse Gaussian
   Finite-sample performance
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6 Overview of preliminary theoretical results
7 Conclusions and open problems
Infill asymptotic behavior for the MME

1. Assuming for simplicity that \( \theta = 0 \),

\[
\hat{\sigma}_n^2 = \frac{1}{T} \sum_{i=1}^{n} (X_{i\delta_n} - X_{(i-1)\delta_n})^2, \quad \hat{\kappa} - n = \delta \frac{\hat{\mu}_4}{(\hat{\mu}_2^\delta_n)^2} - 3\delta_n, \quad \hat{b} = \frac{X_T}{T},
\]

where the sampling horizon is \( T \) and the sampling time mesh is \( \delta_n = \frac{T}{n} \).

2. Bias and variance for \( \hat{\sigma}_n \):

\[
\mathbb{E}\hat{\sigma}_n^2 = \frac{1}{\delta_n} \text{Var}(X_{\delta_n}) = \sigma^2; \quad \text{Var}(\hat{\sigma}_n^2) = \frac{1}{T} c_4(X_1) + \frac{2}{n-1} \text{Var}(X_1)^2.
\]

3. Bias and variance for \( \hat{\kappa}_n \)

\[
\mathbb{E}\hat{\kappa}_n \xrightarrow{n \to \infty} \kappa + O(T^{-1}) \quad \text{and} \quad \text{Var}(\hat{\kappa}_n) \xrightarrow{n \to \infty} O(T^{-1}).
\]
Infill asymptotic behavior for the MME

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\hat{\sigma}_n^2 = \frac{1}{T} \sum_{i=1}^{n} (X_i \delta_n - X_{(i-1)\delta_n})^2, \quad \hat{\kappa} - n = \delta \frac{\hat{\mu}_4 \delta_n}{(\hat{\mu}_2 \delta_n)^2} - 3\delta_n, \quad \hat{b} = \frac{X_T}{T},
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where the sampling horizon is \( T \) and the sampling time mesh is \( \delta_n = \frac{T}{n} \).

2. Bias and variance for \( \hat{\sigma}_n \):

\[
\mathbb{E} \hat{\sigma}_n^2 = \frac{1}{\delta_n} \var(X_{\delta_n}) = \sigma^2; \quad \text{Var}(\hat{\sigma}_n^2) = \frac{1}{T} c_4(X_1) + \frac{2}{n-1} \var(X_1)^2.
\]

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where the sampling horizon is $T$ and the sampling time mesh is $\delta_n = \frac{T}{n}$.

2. Bias and variance for $\hat{\sigma}_n$:

$$\mathbb{E} \hat{\sigma}_n^2 = \frac{1}{\delta_n} \text{Var}(X_{\delta_n}) = \sigma^2; \quad \text{Var}(\hat{\sigma}_n^2) = \frac{1}{T} c_4(X_1) + \frac{2}{n-1} \text{Var}(X_1)^2.$$

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Conclusions

1. We analyze the capability of the VG and NIG models to describe the features of intraday data at different sampling frequencies.
2. Study the performance of the MME and the MLE using intraday data.
3. The estimation of the volatility parameter is found to be stable at different frequencies and that neither high-frequency sampling nor MLE reduce significantly the estimation error for $\sigma$.
4. The estimation error of the parameter controlling the kurtosis can be reduced significantly by using MLE and intraday data, up to some critical frequency where noise and numerical stability become an issue.
5. The VG MLE is shown to exhibit certain numerical instability when working with high-frequency data in a sharp contrast with the NIG MLE.
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Ongoing and future work

1. Determine the infill asymptotic behavior of the MLE; In particular, can we attain consistency and efficiency?
2. Characterizing theoretically the effects of microstructure noise in the estimation results;
3. Propose estimators that are robust against microstructure noise;
Stock prices at different resolutions

INTC Daily Prices
Stock prices at different resolutions

INTC 5 Sec. Prices
Stock prices at different resolutions

INTC Prices in 5 Sec. freq during Jan 9, 2003 (4681 prices)
Empirical distribution of returns

Graph taken from “Time consistency of Lévy models” by Eberlein and Özkan, 2002

Log return during a given time period = log \( \frac{\text{Final price}}{\text{Initial price}} \).

Figure 1: Empirical density of one-hour returns (Bayer) vs. density of fitted hyperbolic (blue) and fitted normal distribution (red).
Dynamics of the price process

Graph taken from “Financial Modeling with Jump Processes” by Cont and Tankov, 2004

**FIGURE 1.2:** Evolution of SLM (NYSE), January-March 1993, compared with a scenario simulated from a Black-Scholes model with same annualized return and volatility.
Times series of returns

Graph taken from “Financial Modeling with Jump Processes” by Cont and Tankov, 2004

Five-minute log-return for Yen/Deutschemark exchange rate, 1992-1995

BMW daily log-returns
Empirical performance of NIG and GH models


**Figure 1.5:** Densities and log-densities of high frequency data.
Empirical performance of the CGMY Lévy model