# Exact short-term approximations for the distributions of Lévy processes with bounded variation

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**Abstract:** We study a method to approximate the marginal distributions and densities of a Lévy process of bounded variation using polynomial expansions in time building on our previous results in Figueroa and Houdré [10]. We provide a fast recursive formula to approximate the coefficients of the expansions and estimate the order of the approximation error. Our expansions are shown to be the exact counter part of successive approximations of the Lévy process by compound Poisson processes previously proposed by, for instance, Barndorff-Nielsen and Hubalek [4] and others. The approximations are illustrated in the case of a Variance Gamma Model.

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## 1. Introduction

In this paper we consider a Lévy process $X = \{X_t\}_{t\geq 0}$ with Lévy triplet $(\sigma^2, b, \nu)$ such that $\sigma^2 = 0$ and $\int (|x|\wedge 1)\nu(dx) < \infty$. Under these conditions, $X$ is known to be have a.s. paths of bounded variations. We shall also assume that $\nu$ admits a smooth density function $s : \mathbb{R}\backslash\{0\} \to \mathbb{R}_+$.

During the last decade Lévy processes have regained popularity as the underlying source of randomness driving the dynamics of different physical phenomena such as asset prices and weather measurements. One drawback in working with a Lévy process $\{X_t\}_{t\geq 0}$ is that in general neither their marginal distribution functions $F_t(x) = \mathbb{P}(X_t \leq x)$ nor their marginal densities $p_t(x) = F_t'(x)$ are easily computable. In some cases there do exist closed formulas for the marginal densities in terms of special functions (e.g. the *generalized hyperbolic motion* of [1, 9]), but in general, one needs to rely on numerical approximations to find these quantities. For instance, in the so-called CGMY model of [6] (sometimes also called tempered stable process or Kopone/Mantegna model; see [8] for more information), the Lévy triplet $(\sigma^2, b, \nu)$ of the process is such that

$$b \in \mathbb{R}, \quad \sigma = 0, \quad \text{and} \quad s(x) = \begin{cases} \frac{C}{x^{1+Y}} e^{-Gx} & x > 0, \\ \frac{C}{|x|^{1+Y}} e^{-M|x|} & x < 0, \end{cases} \tag{1.1}$$

with $0 \leq Y < 2$, and positive $C$, $G$, and $M$. In the case of $Y = 0$, the model is the well-known variance Gamma process, whose marginal densities admit a closed formula of the form:

$$p_t(x) = \frac{2e^{\theta(x-c)/\sigma^2}}{\sigma\sqrt{2\pi}\nu^{1/\nu}\Gamma(\frac{1}{\nu})}\left(\frac{|x-c|}{\sqrt{\frac{2\theta^2}{\nu}+\sigma^2}}\right)^{\frac{1}{\nu}-\frac{1}{2}} K_{\frac{1}{\nu}-\frac{1}{2}}\left(\frac{|x-c|\sqrt{\frac{2\theta^2}{\nu}+\sigma^2}}{\sigma^2}\right),$$

where $K$ is the modified Bessel function of the second kind (c.f. [7]). However, there is no closed formula for any other value of $Y$ and, in order to compute the marginal density, one would have to use numerical methods such as Fast Fourier Transforms combined with an inversion formula for the characteristic function as proposed in, e.g., Carr et. al. [6, Section 5.1].

A well-known construction method for Lévy processes is via the limit of compound Poisson processes. Indeed, any infinitely-divisible distribution (hence, any marginal distribution of a Lévy process) is the limit of a sequence of compound Poisson distributions (see [12, Corollary 8.8]). Furthermore, with probability 1, any pure-jump Lévy process $\{X_t\}_{t\geq 0}$ is the limit of compound Poisson processes $\{X_t^\varepsilon\}_{t\geq 0}$ as $\varepsilon \to 0$ up to a drift term $b_\varepsilon t$ (c.f. [12, Theorem 19.2]).

In the case of an infinite-jump activity subordinator ($\nu(\mathbb{R}) = \infty$, $\nu((-\infty, 0)) = 0$, and $\int_0^\infty (x \wedge 1)\nu(dx) < \infty$) such that $\nu(dx) = s(x)dx$, Barndorff-Nielsen and Hubalek [4] (see also [2, 3]) consider the approximations:

$$1 - F_{t,}(y) = \lim_{\varepsilon\to 0}\sum_{k=1}^{\infty} A_{k,\varepsilon}(y)\frac{t^k}{k!}, \tag{1.2}$$

$$p_t(y) = \lim_{\varepsilon\to 0}\sum_{k=1}^{\infty} a_{k,\varepsilon}(y)\frac{t^k}{k!}, \tag{1.3}$$

for certain functions $A_{k,\varepsilon}, a_{k,\varepsilon}$ depending on a Lévy density $s_\varepsilon$ of the compound Poisson type that approximates, in some sense, the Lévy density $s$. Concretely, $s_\varepsilon$ is assumed to be such that

$$\lambda_\varepsilon := \int s_\varepsilon(x)dx < \infty, \quad \text{and} \quad \lim_{\varepsilon\to 0}\int_0^\infty (1\wedge x)|s_\varepsilon(x) - s(x)|dx = 0. \tag{1.4}$$

Two typical cases for $s_\varepsilon$ satisfying (1.4) are

$$s_\varepsilon(x) := \mathbf{1}_{|x|>\varepsilon} s(x), \quad \text{and} \quad s_\varepsilon(x) := e^{-\varepsilon/|x|}s(x).$$

By considering the marginal distributions of the compound Poisson process associated with $s_\varepsilon$, Barndorff-Nielsen [2] proposes the following coefficients:

$$A_{k,\varepsilon}(y) := \sum_{q=1}^{k} C_q^k\,(-\lambda_\varepsilon)^{k-q}\int \mathbf{1}_{\{\sum_{i=1}^q u_i \geq y\}}\prod_{i=1}^{q} s_\varepsilon(u_i)du_i, \tag{1.5}$$

$$a_{k,\varepsilon}(y) := \sum_{q=1}^{k} C_q^k\,(-\lambda_\varepsilon)^{k-q}\, s_\varepsilon^{*q}(y), \tag{1.6}$$

where $C_q^k = k!/(k-q)!q!$. Under the above choice of coefficients and under the condition (1.4), Barndorff-Nielsen [2] (see also [3]) proves the pointwise convergence in (1.2) for any $y > 0$. Woerner [14] considers the case of an infinite-jump activity Lévy process of bounded variation ($\sigma = 0$, $\nu(\mathbb{R}) = \infty$, and $\int(|x| \wedge 1)\nu(dx) < \infty$) and obtains pointwise convergence in (1.2) under similar mild conditions. Woerner also proves the convergence (1.3), but assuming that the series on the right-hand side of (1.2) is known to converge uniformly in $\mathbb{R}\backslash\{0\}$ as $\varepsilon \to 0$ (a quite strong condition).

Barndorff-Nielsen [2] posed the following more general problems:

(1) Do the limits $A_k(y) := \lim_{\varepsilon\to 0} A_{k,\varepsilon}(y)$ and $a_k(y) := \lim_{\varepsilon\to 0} a_{k,\varepsilon}(y)$ exist?
(2) If they do exist, is it possible to pass the limit into the summation in (1.2-1.3) and thus get the series expansions:

$$1 - F_t(y) = \sum_{k=1}^{\infty} A_k(y)\frac{t^k}{k!}, \quad p_t(y) = \sum_{k=1}^{\infty} a_k(y)\frac{t^k}{k!}? \tag{1.7}$$

In the case of a subordinator, affirmative answers to both issues were obtained for $p_t$ in [4] under fairly strong assumptions.

The seemly natural convergence of $A_{k,\varepsilon}$ and $a_{k,\varepsilon}$ is not trivial since their corresponding expressions are alternating summations of terms tending to $\pm\infty$ as $\varepsilon \to 0$. In the words of Barndorff-Nielsen, "such a convergence implies a subtle cancellation of singularities". In this paper, we will address this issue and prove the existence of the corresponding limits in the case of a Lévy process of bounded variation under mild smoothness conditions on the Lévy density $s$. Our approach is along the lines of Figueroa and Houdré [10], where finite polynomial expansions of the form

$$\mathbb{P}(X_t \geq y) = \sum_{k=1}^{n} A_k(y)\,\frac{t^k}{k!} + \frac{t^{n+1}}{n!}\mathcal{R}_n(t,y), \tag{1.8}$$

$$p_t(y) = \sum_{k=1}^{n} a_k(y)\,\frac{t^k}{k!} + \frac{t^{n+1}}{n!}\mathcal{R}'_n(t,y), \tag{1.9}$$

were obtained for general Lévy processes, motivated by the work of Rüschendorf and Woerner [11]. Given $\underline{y} > 0$, the remainder functions $\mathcal{R}_n(t,y)$ and $\mathcal{R}'_n(t,y)$ are known to be uniformly bounded on $y > \underline{y}$ for $t$ small enough (depending on $\underline{y}$). This fact opens the door to devise approximations for $1 - F_t(y)$ and $p_t(y)$ uniformly on $y > \underline{y}$ for $t$ small enough. Note that if (1.7) were to exist, the coefficients should coincide with those in (1.8-1.9). One key difference between the expansions (1.2-1.3) and (1.8-1.9) is that there is no limit $\varepsilon \to 0$ involved in the later expansions and hence, we can think of them as "exact" approximation, in the sense that the coefficients can in principle be computed exactly.

Figueroa and Houdré [10] gives "formal" expressions for the coefficients $A_k(y)$ and $a_k(y)$, which are in general hard to use for computational purposes. This computational issue was our original motivation for the current paper. Hence, a second objective here is to provide formulations of the coefficients in (1.8-1.9) that allow us to feasibly compute or approximate the expansions there. This is indeed achievable for Lévy processes of bounded variation. We find that the expressions in (1.5-1.6) are the first order approximations for the coefficients $A_k(y)$ and $a_k(y)$ proposed in [10]. Concretely, taking

$$s_\varepsilon(x) = c_\varepsilon(x)s(x), \tag{1.10}$$

with $c_\varepsilon \in C^\infty$ such that $\mathbf{1}_{|x|\geq\varepsilon} \leq c_\varepsilon(x) \leq \mathbf{1}_{|x|\geq\varepsilon/2}$ and using the notation

$$\nu_\varepsilon(g(x)) := \int g(x)(1 - c_\varepsilon(x))\nu(dx),$$

it turns out that

$$a_k(y) = a_{k,\varepsilon}(y) + O\left(\nu_\varepsilon(|x|)\right), \tag{1.11}$$

as $\varepsilon \to 0$ (similarly for $A_k(y)$) provided that the Lévy density $s$ is smooth away from the origin. In the case of a symmetric Lévy density $s$, one can strengthen the approximation and prove that

$$a_k(y) = a_{k,\varepsilon}(y) + O\left(\nu_\varepsilon(|x|^2)\right),$$

as $\varepsilon \to 0$.

On one hand, (1.11) gives us a method to approximate the coefficients $a_k(y)$ and $A_k(y)$ of [10] by $a_{k,\varepsilon}(y)$ and $A_{k,\varepsilon}(y)$ with $\varepsilon$ small enough. On the other hand, (1.11) gives an answer to the problem (1) described above and raised by Barndorff-Nielsen [2] since, under (1.11),

$$\lim_{\varepsilon\to 0} a_{k,\varepsilon}(y) = a_k(y), \quad \text{and} \quad \lim_{\varepsilon\to 0} A_{k,\varepsilon}(y) = A_k(y).$$

Let us remark that even though we focus on the approximating densities (1.10), our results will hold for more general approximating smooth densities $s_\varepsilon$ of the compound Poisson type (see Proposition 3.2 below).

Using the derivatives $a_{k,\varepsilon}^{(j)}$ (which as expected turns out to be $-A_{k,\varepsilon}^{(j+1)}$), one can improve the order of the approximation (1.11). For instance,

$$a_k(y) = a_{k,\varepsilon}(y) - k a'_{k-1,\varepsilon}(y)\,\nu_\varepsilon(x) + O\left(\nu_\varepsilon(|x|)^2 + \nu_\varepsilon(|x|^2)\right),$$

for $k \geq 2$, with a similar expression holding for $A_k$.

For the computation of $a_{k,\varepsilon}$ and, if necessary, its derivatives, we propose a recursive formula. We show that, for any $y \in \mathbb{R}\backslash\{0\}$,

$$a_{1,\varepsilon}(y) = s_\varepsilon(y), \quad \text{and} \tag{1.12}$$

$$a_{k,\varepsilon}(y) = s_\varepsilon(y)\left(-\lambda_\varepsilon\right)^{k-1} - \lambda_\varepsilon a_{k-1,\varepsilon}(y) + \int_{-\infty}^{\infty} a_{k-1,\varepsilon}(u)s_\varepsilon(y-u)du, \tag{1.13}$$

for $k \geq 2$. A similar recursive formula holds for $A_k(y)$. We also have that

$$a'_{1,\varepsilon}(y) = s'_\varepsilon(y),$$

$$a'_{k,\varepsilon}(y) = s'_\varepsilon(y)\left(-\lambda_\varepsilon\right)^{k-1} - \lambda_\varepsilon a'_{k-1,\varepsilon}(y) + \int_{-\infty}^{\infty} a'_{k-1,\varepsilon}(u)s_\varepsilon(y-u)du.$$

Upper order derivatives of $a_{k,\varepsilon}$ can be seen to satisfy similar recursive formulas as well.

The recursive formulation (1.12-1.13) provides a fast and accurate way to approximate the coefficients of the expansion (1.8-1.9) using fast algorithms for the convolution operator $g * h(y) = \int_{-\infty}^{\infty} g(u)h(y-u)du$, which are built in most of the standard mathematical software. We do not pursue, at this point, the second theoretical issue raised by Barndorff-Nielsen since for applications the approximations (1.8-1.9) seem to be more relevant than (1.7).

The paper is structured as follows. In Section 2, we introduced the formulas in [10] for the approximation of the tail distribution $\mathbb{P}(X_t \geq y)$ with $y > 0$. Also, it is proved the convergence of (1.5) as $\varepsilon \to 0$ towards the coefficient $A_k(y)$ proposed in [10]. We also derive a recursive formula for $A_{k,\varepsilon}(x)$ analog to (1.13). In Section 3, we tackle the same questions for the case of the marginal density functions $p_t$. We finish with some numerical examples in Section 4 for the variance Gamma and the CGMY models.

## 2. Approximation of the marginal right-tail distribution

Let $X = (X_t)_{t\geq 0}$ be a Lévy process with generating triplet $(\sigma^2, b, \nu)$ (see [12] for terminology and background). Throughout, we assume that $\nu$ is determined by a smooth function $s : \mathbb{R}\backslash\{0\} \to [0, \infty)$, called the *Lévy density*, such that

$$\nu(A) := \int_A s(x)dx, \quad \forall A \in \mathcal{B}(\mathbb{R}\backslash\{0\}).$$

In this part we consider polynomial expansions for $\mathbb{P}(X_t \geq y)$ $(y > 0)$ of the form:

$$\mathbb{P}\left(X_t \geq y\right) = \sum_{k=1}^{n} A_k(y)\,\frac{t^k}{k!} + \frac{t^{n+1}}{n!}\mathcal{R}_n(t, y), \tag{2.1}$$

for certain functions $A_k(y)$ and a remainder term $\mathcal{R}_n(t, y)$ that remains bounded for $t$ small enough. An expansion of the form (2.1) was obtained in [10] under the following standing assumption:

**Conditions 2.1.** *For any $\delta > 0$ and any $j \geq 0$,*

$$k_\delta^{(j)} := \sup_{|x|>\delta} |s^{(j)}(x)| < \infty, \quad \text{and} \quad s^{(j)}(x) \stackrel{|x|\to\infty}{\longrightarrow} 0. \tag{2.2}$$

In this paper, we shed some light on the computation of the coefficients $A_k(y)$ proposed in [10], in the case of a drift-less pure-jump Lévy process of bounded variation. The later condition can be expressed in terms of the Lévy triplet of $X$ as

$$(i)\ \sigma = 0, \quad (ii)\ \int_{|x|\leq 1} |x|s(x)dx < \infty, \quad (iii)\ b_0 = b - \int_{x\leq 1} xs(x)dx = 0. \tag{2.3}$$

The driftless condition is very mild since the drift can always be taken into account by shifting the functions $A_k$; indeed, $\tilde{X}_t = X_t + b_0 t$ will have drift $b_0$ and

$$\mathbb{P}\left(\tilde{X}_t \geq y\right) = \sum_{k=1}^{n} A_k(y - b_0 t)\,\frac{t^k}{k!} + \frac{t^{n+1}}{n!}\mathcal{R}_n(t, y - b_0 t).$$

Let us introduce some further needed notation. Throughout the paper, $c_\varepsilon \in C_\infty$ stands for a symmetric smooth truncation function such that $\mathbf{1}_{|x|\geq\varepsilon} \leq c_\varepsilon(x) \leq \mathbf{1}_{|x|\geq\varepsilon/2}$. Also, we let

$$\bar{c}_\varepsilon := 1 - c_\varepsilon, \quad \lambda_{0,\varepsilon} := \int c_\varepsilon(x)s(x)dx, \quad \lambda_{1,\varepsilon} := \int \bar{c}_\varepsilon(x)xs(x)dx, \tag{2.4}$$

$$s_\varepsilon(x) := c_\varepsilon(x)s(x), \quad \bar{s}_\varepsilon(x) := \bar{c}_\varepsilon(x)s(x). \tag{2.5}$$

Note that $s_\varepsilon(x) \to s(x)$, as $\varepsilon \to 0$, and hence, one can think of $s_\varepsilon$ as an approximation of the Lévy density $s$.

Below, $\mathcal{K}_k := \{\mathbf{k} = (p,q,r) \in \mathbb{N}^3 : p+q+r = k\}$ and for each triplet $\mathbf{k} = (p,q,r) \in \mathbb{N}^3$, we let

$$d\pi_{\mathbf{k},\varepsilon} := \prod_{i=1}^{q} \frac{1}{\lambda_{0,\varepsilon}} s_\varepsilon(u_i)du_i \prod_{j=1}^{r} \frac{1}{\lambda_{1,\varepsilon}} d\beta_j w_j \bar{s}_\varepsilon(w_j)dw_j,$$

a probability measure on $\mathbb{R}^q \times [0,1]^r \times \mathbb{R}^r$. We often omit $\varepsilon$ in the subscripts of $\pi$, $\lambda_0$, and $\lambda_1$.

We are now ready to write a formal expression for the coefficients in (2.1), following the approach in [10][1]. Under the conditions (2.3) and assuming that $s$ satisfies (2.2) for any $j \geq 0$ and any $\delta > 0$, the coefficient $A_k(y)$ takes the following form

$$A_k(y) = \sum_{\mathbf{k}\in\mathcal{K}_k} \hat{c}_{\mathbf{k},\varepsilon}\binom{k}{\mathbf{k}} d_{\mathbf{k},\varepsilon}(y), \tag{2.6}$$

where $\varepsilon$ can be chosen arbitrarily in $(0, y/(n+1) \wedge 1)$, and where we have used the following further terminology:

$$\begin{aligned}
&\binom{k}{\mathbf{k}} := \frac{k!}{p!q!r!}, \quad \hat{c}_{\mathbf{k},\varepsilon} := (-\lambda_{0,\varepsilon})^p \lambda_{0,\varepsilon}^{q-\mathbf{1}_{\{r>0,q>0\}}} \lambda_{1,\varepsilon}^r, \\
&d_{\mathbf{k},\varepsilon}(y) := \begin{cases}
\int \mathbf{1}\{\sum_{i=1}^{q} u_i \geq y\}\, d\pi_{\mathbf{k},\varepsilon}, & q \geq 0, r = 0, \\
(-1)^{r-1} \int s_\varepsilon^{(r-1)}\left(y - \sum_{i=2}^{q} u_i - \sum_{j=1}^{r}\beta_j w_j\right) d\pi_{\mathbf{k},\varepsilon}, & q > 0, r > 0, \\
0, & \text{otherwise.}
\end{cases}
\end{aligned} \tag{2.7}$$

[1] For consistency with the paper of Barndorff-Nielsen and Hubalek [4], we don't follow the notation in [10]. Our truncation function $c_\varepsilon$ here is the the truncation function $\bar{c}_\varepsilon$ in [10].

More precisely, it is is proved in [10] (see also Theorem 3.1 below) that, for any fixed $\underline{y} > 0$ and $0 < \varepsilon < \underline{y}/(n+1) \wedge 1$, there exists a $t_0 := t_0(\varepsilon, \underline{y}) > 0$ such that (2.1) holds true for any $y > \underline{y}$ and any $0 < t < t_0$ with $A_k(y)$ given as in (2.6) and $\mathcal{R}_n(t, y) = O_{\varepsilon,\underline{y}}(1)$ as $t \to 0$.

Let us illustrate how (2.6) is used. For $k = 1$,

$$A_1(y) = \int \mathbf{1}_{\{u_1 \geq y\}} s_\varepsilon(u_1) du_1, \tag{2.8}$$

which for any $0 < \varepsilon < y$, becomes $A_1(y) = \int_y^\infty s(u)du = \nu((y, \infty))$. This is the well-known first order approximation for $\mathbb{P}(X_t \geq y)$ (see e.g. [5]). For $k = 2$, we have

$$A_2 = \underbrace{-2\lambda_{0,\varepsilon} \int \mathbf{1}_{\{u_1 \geq y\}} s_\varepsilon(u_1) du_1}_{p=1,q=1,r=0} + \underbrace{\iint \mathbf{1}_{\{u_1+u_2 \geq y\}} s_\varepsilon(u_1) s_\varepsilon(u_2) du_1 du_2}_{p=0,q=2,r=0} \tag{2.9}$$

$$\underbrace{+2 \int \int_0^1 s_\varepsilon(y - \beta w) d\beta w \bar{s}_\varepsilon(w) dw}_{p=0,q=1,r=1}. \tag{2.10}$$

Even though the previous formula for $A_k(y)$ apparently depends on the parameter $\varepsilon$, the validity of the expansion (2.1) for $t$ in some interval $(0, t_0(\varepsilon))$ and $\mathcal{R}_n(t, y) = O(1)$ as $t \to 0$ will imply that $A_k(y)$ should be independent of $\varepsilon$ (at least for those values $\varepsilon$ for which such a $t_0 > 0$ exists). Furthermore, it was proved in [10] (see Remark 4.4 therein) that under the standing Assumption 2.1, $A_k(y)$ is constant for any $\varepsilon < y/(k+1)$. A natural approach to deduce an expression for $A_k$ that is independent of $\varepsilon$ is to make $\varepsilon \to 0$. For instance, in the case of $A_2$, the term in (2.10) vanishes when $\varepsilon \to 0$, while the first two terms converge to

$$A_2 = \int_0^y \int_{y-u}^y s(v)dvs(u)du - 2\int_y^\infty \int_{-\infty}^{y-u} s(v)dvs(u)du - (\nu((y,\infty)))^2;$$

see [10] for formulas for $A_1$ and $A_2$ in the case of a general Lévy process.

The formula (2.7) suggests a decomposition of the terms in (2.6) into two kind of terms: when $r = 0$ and when $r > 0$. The following quantity collects the terms corresponding to $r = 0$:

$$A_{k,\varepsilon}(y; s) := \sum_{\mathbf{k}:\, q>0, r=0} \hat{c}_{\mathbf{k},\varepsilon} \binom{k}{\mathbf{k}} d_{\mathbf{k},\varepsilon}(y) \tag{2.11}$$

$$= \sum_{q=1}^k C_q^k \left(-\int s_\varepsilon(u)du\right)^{k-q} \int \mathbf{1}_{\left\{\sum_{i=1}^q u_i \geq y\right\}} \prod_{i=1}^q s_\varepsilon(u_i) du_i. \tag{2.12}$$

Note that this quantity coincides with the coefficients (1.5) proposed by Barndorff-Nielsen [2]. Since the above quantity depends only on the behavior of the Lévy density away from the origin, we refer to these terms as the "big jumps" terms. We will see in the following two subsections that the limit of $A_{k,\varepsilon}(y;s)$ exists when $\varepsilon \to 0$ and it is given by $A_k(y)$. For now, we obtain a convenient recursive formula for its computation.

**Lemma 2.1.** *$A_{1,\varepsilon}(y;s) = \int_y^\infty s_\varepsilon(u)du$ and for $k \geq 2$,*

$$\begin{aligned} A_{k,\varepsilon}(y;s) &= -A_{k-1,\varepsilon}(y;s)\int s_\varepsilon(u)du \\ &+ \int_0^\infty A_{k-1,\varepsilon}(u;s)s_\varepsilon(y-u)du - \int_0^\infty A_{k-1,\varepsilon}(u;s^-)s_\varepsilon^-(-y-u)du, \quad y \geq 0, \end{aligned} \tag{2.13}$$

*where $s^-(x) := s(-x)$.*

*Proof.* We need some notation. Below,

$$H_k(u_1,\dots,u_k;y) := \sum_{I\subset\{1,\dots,k\}} (-1)^{k-|I|}\mathbf{1}_{\left\{\sum_{i\in I} u_i \geq y\right\}}, \tag{2.14}$$

where $|I|$ is the cardinality of the set $I$ and $\sum_{i\in I} u_i = 0$ if $I=\emptyset$. Clearly,

$$A_{k,\varepsilon}(y;s) = \int H_k(u_1,\dots,u_k;y)\prod_{i=1}^k s_\varepsilon(u_i)du_i.$$

Note that $H_k(u_1,\dots,u_k;y) = H_{k-1}(u_1,\dots,u_{k-1};y-u_k) - H_{k-1}(u_1,\dots,u_{k-1};y)$, and hence,

$$\begin{aligned} &\int_{-\infty}^y s_\varepsilon(u_k)du_k \int H_k(u_1,\dots,u_k;y)\prod_{i=1}^{k-1} s_\varepsilon(u_i)du_i \\ &\quad = \int_{-\infty}^y \left[A_{k-1,\varepsilon}(y-u_k;s) - A_{k-1,\varepsilon}(y;s)\right] s_\varepsilon(u_k)du_k. \end{aligned}$$

If $u_k \geq y$, then

$$\begin{aligned} &\int H_k(u_1,\dots,u_k;y)\prod_{i=1}^{k-1} s_\varepsilon(u_i)du_i = -A_{k-1,\varepsilon}(y;s) \\ &+ \int [\sum_{I\subset\{1,\dots,k-1\},|I|>0} (-1)^{k-1-|I|}\,\mathbf{1}_{\left\{\sum_{i\in I} u_i \geq y-u_k\right\}} + (-1)^{k-1}]\prod_{i=1}^{k-1} s_\varepsilon(u_i)du_i. \end{aligned}$$

Note that, by writing $\mathbf{1}\left\{\sum_{i\in I} u_i \geq y-u_k\right\} = 1 - \mathbf{1}\left\{\sum_{i\in I}(-u_i) > u_k - y\right\}$, the expression in brackets can be written as $-H_{k-1}\left(-u_1,\dots,-u_{k-1};u_k-y\right)$. Then, changing variables $u_i \to -u_i$, $i=1,\dots,k-1$,

$$\begin{aligned} &\int_y^\infty s_\varepsilon(u_k)du_k \int H_k(u_1,\dots,u_k;y)\prod_{i=1}^{k-1} s_\varepsilon(u_i)du_i \\ &= -\int_y^\infty A_{k-1,\varepsilon}(u_k-y;s^-)s_\varepsilon(u_k)du_k - A_{k-1,\varepsilon}(y;s)A_{1,\varepsilon}(y;s), \end{aligned}$$

where we used that $s_\varepsilon^-(u) = s_\varepsilon(-u) = (s^-)_\varepsilon(u)$, because $c_\varepsilon$ is taken to be symmetric. After some extra algebraic manipulation, we get (2.13). □

### *2.1. The limit of the "big jumps" terms*

In this part we analyze the limit of $A_{k,\varepsilon}(y)$ as $\varepsilon \to 0$. This problem has been considered in the literature as early as Barndorff-Nielsen [2], but to the best of our knowledge, its solution has only been obtained in particular cases or for $k$ small.

In terms of the notation of the proof of Lemma 2.1, let us recall that

$$A_{k,\varepsilon}(y;s) = \int H_k(u_1,\dots,u_k;y)\prod_{i=1}^k s_\varepsilon(u_i)du_i, \tag{2.15}$$

with $H_k$ given as in (2.14). Since $s_\varepsilon \to s$, it is natural to assume that $A_{k,\varepsilon}(y)$ will converge to the integral

$$A_k(y;s) := \int H_k(u_1,\dots,u_k;y)\prod_{i=1}^k s(u_i)du_i, \tag{2.16}$$

as $\varepsilon \to 0$. This will indeed be the case if

$$\int |H_k(u_1,\dots,u_k;y)|\prod_{i=1}^k s(u_i)du_i < \infty, \tag{2.17}$$

because $s_\varepsilon \le s$. When e.g. $k = 2$, (2.17) holds for any density of bounded variation since

$$\int \left|\mathbf{1}_{\{u_1+u_2\ge y\}} - \mathbf{1}_{\{u_1\ge y\}} - \mathbf{1}_{\{u_2\ge y\}}\right| s(u_1)s(u_2)du_1du_2 =$$
$$\int_0^y\int_{y-u_1}^y s(u_2)du_2 s(u_1)du_1 + 2\int_y^\infty\int_{-\infty}^{y-u_1} s(u_2)du_2 s(u_1)du_1 + \left(\nu((y,\infty))\right)^2,$$

which can be seen to be finite by splitting the integrals of $du_1$ and using Fubini's Theorem. When $k = 3$, (2.17) does not necessarily hold for a general Lévy density of bounded variation. For instance, if $s(u) = u^{-(1+\alpha)}$ with $1/2 < \alpha < 1$, then

$$\int_0^\delta\int_0^\delta\int |H_k(u_1,u_2,u_3;y)|\prod_{i=1}^3 s(u_i)du_1du_2du_3 = \infty,$$

for $\delta > 0$ small. We will prove that (2.16) is well-defined as a "conditional integral" in that we will be able to partition the space $\mathbb{R}^3$ into disjoint regions, where on each one the triple integral can be defined as an iterated integral.

The following result will be needed to make this idea concrete. Below, $c_\delta$ and $\bar{c}_\delta$ are as in (2.4) and $s\cdot\bar{c}_\delta(u) = s(u)\bar{c}_\delta(u)$ (similarly for $s\cdot c_\delta$). Note that the result is not trivial since $s\cdot\bar{c}_\delta$ truncates $s$ only outside $[-\delta,\delta]$ and $s\cdot\bar{c}_\delta(x)\to+\infty$ as $x\to 0$.

**Lemma 2.2.** *Suppose that $s$ is a Lévy density satisfying (2.3-ii) and (2.2) for any $j \geq 0$. Then, for any $1 \leq j \leq k-1$,*

$$\int \left| \int H_k(u_1,\ldots,u_k;y) \prod_{i=j+1}^{k} s \cdot c_\delta(u_i) du_i \right| \prod_{i=1}^{j} s \cdot \bar{c}_\delta(u_i) du_i < \infty. \tag{2.18}$$

*Proof.* We adopt the notation in (2.5). Fix a non-empty subset $I \subset \{j+1,\ldots,k\}$ and let

$$F_I(x) := F_{I,\delta,j}(x;s) := \int \mathbf{1}_{\left\{\sum_{i\in I} u_i \geq x\right\}} \prod_{i=j+1}^{k} s(u_i) c_\delta(u_i) du_i. \tag{2.19}$$

Under the condition (2.2), $F_I$ is $C_b^\infty$ on $(0,\infty)$. Indeed, if $k'$ is an arbitrary index in $I$, for any $\ell \geq 0$,

$$\begin{aligned} |F_I^{(\ell)}(x)| &= \left| \int s_\delta^{(\ell-1)}\left(x - \sum_{i\in I\setminus\{k'\}} u_i\right) \prod_{i=j+1, i\neq k'}^{k} s_\delta(u_i) du_i \right| \\ &\leq \sup_u |s_\delta^{(\ell-1)}(u)| \left(\int s_\delta(u) du\right)^{k-j-1}, \quad \forall x > 0. \end{aligned} \tag{2.20}$$

Next, since

$$H_k = \sum_{I\subset\{j+1,\ldots,k\}} (-1)^{k-|I|} \sum_{I'\subset\{1,\ldots,j\}} (-1)^{|I'|} \mathbf{1}_{\left\{\sum_{i\in I} u_i \geq y - \sum_{i\in I'} u_i\right\}}, \tag{2.21}$$

for (2.18) to hold it suffices to show that, for each $I \subset \{j+1,\ldots,k\}$,

$$\int \left| \int G_I(u_1,\ldots,u_k) \prod_{i=j+1}^{k} s_\delta(u_i) du_i \right| \prod_{i=1}^{j} \bar{s}_\delta(u_i) du_i < \infty. \tag{2.22}$$

where

$$G_I(u_1,\ldots,u_k;y) := \sum_{I'\subset\{1,\ldots,j\}} (-1)^{|I'|} \mathbf{1}_{\left\{\sum_{i\in I} u_i \geq y - \sum_{i\in I'} u_i\right\}}.$$

Note that

$$G_I(u_1,\ldots,u_k;y) = \sum_{\alpha_1,\ldots,\alpha_j\in\{0,1\}} (-1)^{\sum_{i=1}^{j}\alpha_i} \mathbf{1}_{\left\{\sum_{i\in I} u_i \geq y - \sum_{i=1}^{j} u_i\alpha_i\right\}}.$$

Therefore,

$$\bar{G}_I(u_1,\ldots,u_j;y) := \int G_I(u_1,\ldots,u_k;y) \prod_{i=j+1}^{k} s_\delta(u_i) du_i,$$

is such that

$$\bar{G}_I = \sum_{\alpha_2,\dots,\alpha_j} (-1)^{\sum_{i=2}^{j}\alpha_i} \left( F_I \left( y - \sum_{i=2}^{j} u_i\alpha_i \right) - F_I \left( y - \sum_{i=2}^{j} u_i\alpha_i - u_1 \right) \right)$$

$$= \sum_{\alpha_2,\dots,\alpha_j} (-1)^{\sum_{i=2}^{j}\alpha_i} u_1 \int_0^1 F_I' \left( y - \sum_{i=2}^{j} u_i\alpha_i - \beta_1 u_1 \right) d\beta_1.$$

Proceeding by induction, we obtain

$$\bar{G}_I(u_1,\dots,u_j;y) = \prod_{i=1}^{j} u_i \int_0^1 \dots \int_0^1 F_I^{(j)} \left( y - \sum_{i=1}^{j} u_i\beta_i \right) d\beta_1 \dots d\beta_j.$$

In particular, with the notation

$$K_j(s_\delta) := \sup_u |s_\delta^{(j-1)}(u)| \left( \int s_\delta(u)du \right)^{k-j-1},$$

we have that

$$\left| \int H_k(u_1,\dots,u_k) \prod_{i=j+1}^{k} s_\delta(u_i)du_i \right| \le 2^{k-j} K_j(s_\delta) \prod_{i=1}^{j} u_i, \tag{2.23}$$

and (2.18) follows in light of (2.3-ii). □

The following is a useful corollary of the previous proof.

**Corollary 2.3.** *Under the conditions of Lemma 2.2, the following representation holds true*

$$\int H_k(u_1,\dots,u_k;y) \prod_{i=j+1}^{k} s(u_i)c_\delta(u_i)du_i \tag{2.24}$$

$$= \sum_{I\subset\{j+1,\dots,k\}} (-1)^{k-|I|} \int_{[0,1]^j} F_I^{(j)} \left( y - \sum_{i=1}^{j} u_i\beta_i \right) \prod_{i=1}^{j} d\beta_i\, u_i,$$

*where $F_I$ is given by (2.19).*

Let us finish this part with a couple of important consequences. First, we note that, under the conditions of Lemma 2.2, the operator

$$\widehat{A}_{k,\delta}(y;s) := \sum_{j=0}^{k-1} C_j^k \int \left\{ \int H_k(u_1,\dots,u_k;y) \prod_{i=j+1}^{k} s\cdot c_\delta(u_i)du_i \right\} \prod_{i=1}^{j} s\cdot \bar{c}_\delta(u_i)du_i, \tag{2.25}$$

is well defined for any $\delta > 0$. Furthermore, if (2.17) holds, then $A_k(y;s)$ in (2.16) is well-defined and

$$A_k(y;s) = \widehat{A}_{k,\delta}(y;s),$$

for any $y > 0$ and $0 < \delta < y/k$.

Second, we can deduce that the limit, as $\varepsilon \to 0$, of the quantity $A_{k,\varepsilon}(y;s)$ in (2.12) exists and is given by (2.25).

**Corollary 2.4.** *Let $y > 0$ and $k \geq 1$ and suppose that $s$ satisfies (2.2-2.3). Then,*

$$A_k(y;s) := \lim_{\varepsilon \to 0} A_{k,\varepsilon}(y;s) = \widehat{A}_{k,\delta}(y;s),$$

*for any $0 < \delta < y/k$.*

*Proof.* First note that $A_k(y;s_\varepsilon)$ is well-defined because $\int s_\varepsilon(x)dx < \infty$, and $A_{k,\varepsilon}(y;s)$ in (2.15) is such that

$$A_{k,\varepsilon}(y;s) = A_k(y;s_\varepsilon) = \widehat{A}_{k,\delta}(y;s_\varepsilon), \tag{2.26}$$

whenever $0 < \delta < y/k$. Also, if $0 < \varepsilon < \delta/2$,

$$\begin{aligned} &\int \left\{ \int H_k(u_1,\ldots,u_k;y) \prod_{i=j+1}^{k} s_\varepsilon(u_i)c_\delta(u_i)du_i \right\} \prod_{i=1}^{j} s_\varepsilon(u_i)\bar{c}_\delta(u_i)du_i \\ &\quad = \int \left\{ \int H_k(u_1,\ldots,u_k;y) \prod_{i=j+1}^{k} s_\delta(u_i)du_i \right\} \prod_{i=1}^{j} s_\varepsilon(u_i)\bar{c}_\delta(u_i)du_i. \end{aligned} \tag{2.27}$$

It was proved in Lemma 2.2 (see (2.23)) that

$$\left| \int H_k(u_1,\ldots,u_k;y) \prod_{i=j+1}^{k} s_\delta(u_i)du_i \right| \leq 2^{k-j} K_j(s_\delta) \prod_{i=1}^{j} u_i.$$

Hence, we can apply the dominated convergence theorem to obtain the limit of the right hand side in (2.27) as $\varepsilon \to 0$. This clearly turns out to be

$$\int \left\{ \int H_k(u_1,\ldots,u_k;y) \prod_{i=j+1}^{k} s_\delta(u_i)du_i \right\} \prod_{i=1}^{j} s(u_i)\bar{c}_\delta(u_i)du_i.$$

Therefore,

$$\lim_{\varepsilon \to 0} A_{k,\varepsilon}(y;s) = \lim_{\varepsilon \to 0} \hat{A}_{k,\delta}(y;s_\varepsilon) = \widehat{A}_{k,\delta}(y;s). \tag{2.28}$$

□

The following representation is also an easy consequence of (2.26-2.27) and the representation of Corollary 2.3:

**Lemma 2.5.** *Suppose that the conditions of Lemma 2.2 are satisfied. Then, for any $0 < \delta < y/k$ and $0 < \varepsilon < \delta/2$, $A_{k,\varepsilon}(y;s)$ admits the representation:*

$$\sum_{j=0}^{k-1} C_j^k \sum_I (-1)^{k-|I|} \int_{\mathbb{R}^j \times [0,1]^j} F_{I,\delta,j}^{(j)} \left( y - \sum_{i=1}^{j} u_i\beta_i \right) \prod_{i=1}^{j} d\beta_i s_\varepsilon(u_i)\bar{c}_\delta(u_i)u_i du_i, \tag{2.29}$$

*where the second summation is over all the subsets $I$ of $\{j+1,\ldots,k\}$, and $F_{I,\delta,j}$ is given by (2.19).*

### 2.2. The “small jump” terms

In this part, we shall prove that the sum of all the terms in (2.6) corresponding to $r > 0$,

$$B_{k,\varepsilon}(y;s) := \sum_{\mathbf{k}\in\mathcal{K}_k\; q>0, r>0} \hat{c}_{\mathbf{k},\varepsilon} \binom{k}{\mathbf{k}} d_{\mathbf{k},\varepsilon}(y),$$

vanishes as $\varepsilon \to 0$. This fact will imply that $A_k(y)$, defined by (2.6) for any $0 < \varepsilon < y/(k+1)$, is the limiting value of $\lim_{\varepsilon\to 0} A_{k,\varepsilon}(y;s)$ and can be expressed as (2.25) for any $y > 0$ and $0 < \delta < y/k$.

**Lemma 2.6.** *Suppose that (2.3-ii) and (2.2) are satisfied for any $k \geq 0$. Then,*

$$B_k(y;s) := \lim_{\varepsilon\to 0} B_{k,\varepsilon}(y;s) = 0. \tag{2.30}$$

*Proof.* Note that $B_{k,\varepsilon}$ can be written as

$$\begin{aligned} B_{k,\varepsilon} = & \sum_{r=1}^{k} (-1)^{r-1} C_r^k \sum_{q=1}^{k-r} C_q^{k-r} \left( -\int s_\varepsilon(u) du \right)^{k-r-q} \\ & \times \int s_\varepsilon^{(r-1)} \left( y - \sum_{j=1}^{r} \beta_j w_j - \sum_{i=2}^{q} u_i \right) \prod_{i=2}^{q} s_\varepsilon(u_i) du_i \prod_{j=1}^{r} d\beta_j w_j \bar{s}_\varepsilon(w_i) dw_j. \end{aligned}$$

Hence, with the notation

$$A_{k,\varepsilon}^{(r)}(x) := -\sum_{q=1}^{k} C_q^k \left( -\int s_\varepsilon(u) du \right)^{k-q} \int s_\varepsilon^{(r-1)} \left( x - \sum_{i=2}^{q} u_i \right) \prod_{i=2}^{q} s_\varepsilon(u_i) du_i,$$

it follows that

$$B_{k,\varepsilon}(y) = -\sum_{r=1}^{k} (-1)^{r-1} C_r^k \int A_{k-r,\varepsilon}^{(r)} \left( y - \sum_{j=1}^{r} \beta_j w_j \right) \prod_{j=1}^{r} d\beta_j w_j \bar{s}_\varepsilon(w_i) dw_j. \tag{2.31}$$

Note that $A_{k,\varepsilon}^{(r)}(x)$ is indeed the $r^{th}$ derivative of $A_{k,\varepsilon}(y)$ defined in (2.12) and hence, in light of the representation (2.29),

$$A_{k,\varepsilon}^{(r)}(x) = \sum_{j=0}^{k-1} C_j^k \sum_{I} (-1)^{k-|I|} \int F_I^{(j+r)} \left( x - \sum_{i=1}^{j} u_i \beta_i \right) \prod_{i=1}^{j} d\beta_i s_\varepsilon(u_i) \bar{c}_\delta(u_i) u_i du_i.$$

From (2.20), for any $0 < \delta < y/k$,

$$\|A_{k,\varepsilon}^{(r)}\|_\infty \leq \sum_{j=0}^{k-1} 2^{k-j} C_j^k \|s_\delta^{(j+r-1)}\|_\infty \left( \int_{-\delta}^{\delta} s(u)|u| du \right)^j \left( \int_{|u|>\delta/2} s(u) du \right)^{k-j-1}, \tag{2.32}$$

whenever $0<\varepsilon<\delta/2$. Therefore,

$$|B_{k,\varepsilon}| \leq \sum_{r=1}^{k} \binom{k}{r} K_{k-r,\delta}^{(r)} \left( \int \bar{s}_{\varepsilon}(u)|u|du \right)^{r}, \tag{2.33}$$

where $K_{k,\delta}^{(r)}$ is the right-hand side of (2.32). This implies (2.30). □

The following characterization is a consequence of (2.28) and Lemma 2.6.

**Theorem 2.7.** *Suppose that (2.3-ii) and (2.2) are satisfied for any $k \geq 0$. Then, for any fixed $n \geq 1$, $\underline{y}>0$, and $0<\varepsilon<\underline{y}/(n+1)\wedge 1$, there exists a $t_0 := t_0(\varepsilon,\underline{y})>0$ such that (2.1) holds true for any $y>\underline{y}$ and any $0<t<t_0$ with $\mathcal{R}_n(t,y) = O_{\varepsilon,\underline{y}}(1)$ as $t\to 0$ and*

$$A_k(y) = \sum_{j=0}^{k-1} C_j^k \int \left\{ \int H_k(u_1,\ldots,u_k;y) \prod_{i=j+1}^{k} s_{\varepsilon}(u_i)du_i \right\} \prod_{i=1}^{j} \bar{s}_{\varepsilon}(u_i)du_i. \tag{2.34}$$

*Moreover, the expression (2.34) remains constant for any $0<\varepsilon<\underline{y}/(n+1)\wedge 1$.*

### *2.3. Computational issues and the "left-tail" distribution*

The previous formulas provide two (apparently different) methods to compute the $k^{th}$-order coefficient $A_k(y)$. The first method is via the decomposition

$$A_k(y) := A_{k,\varepsilon}(y;s) + B_{k,\varepsilon}(y;s).$$

The first term above, which can be considered the leading term, can be computed recursively by (2.13). The term $B_{k,\varepsilon}$ can be considered a higher order approximation for $A_k$ since $B_{k,\varepsilon}(y;s) = O(\int_{|u|<\varepsilon}|u|s(u)du)$, as $\varepsilon\to 0$. When $s$ is a symmetric function, we will have that

$$B_{k,\varepsilon}(y;s) = O\left(\int_{|u|<\varepsilon} |u|^2 s(u)du\right),$$

as $\varepsilon\to 0$. Indeed, each term in (2.31) is such that

$$\begin{aligned}
&\int F_I^{(j+r)}\left(x-\sum_{i=1}^{j}u_i\beta_i\right)\prod_{i=1}^{j} d\beta_i s_{\varepsilon}(u_i)\bar{c}_{\delta}(u_i)u_i du_i \\
&= \int \left(F_I^{(j+r)}\left(x-\sum_{i=1}^{j}u_i\beta_i\right) - F_I^{(j+r)}(x)\right)\prod_{i=1}^{j} d\beta_i s_{\varepsilon}(u_i)\bar{c}_{\delta}(u_i)u_i du_i \\
&= \int\int_0^1 F_I^{(j+r+1)}\left(x-\beta_1'\sum_{i=1}^{j}u_i\beta_i\right)d\beta_1'\left(\sum_{i=1}^{j}u_i\beta_i\right)\prod_{i=1}^{j} d\beta_i s_{\varepsilon}(u_i)\bar{c}_{\delta}(u_i)u_i du_i,
\end{aligned}$$

which is $O\left(\int_{|u|<\varepsilon}|u|^2 s(u)du\right)$ in light of the boundedness of $F_I^{(\ell)}$ for any $\ell \geq 0$.

In general, higher order approximations for $B_{k,\varepsilon}(y;s)$ can be obtained from (2.31) using the derivatives of $A_{k-r,\varepsilon}$. Indeed, one can consider successive approximations for $B_{k,\varepsilon}$ as follows:

$$B_{k,\varepsilon}(y;s) = -C_1^k A^{(1)}_{k-1,\varepsilon}(y;s)\,\nu_\varepsilon(x) + O\left(\nu_\varepsilon(|x|)^2 + \nu_\varepsilon(|x|^2)\right), \tag{2.35}$$

$$\begin{aligned} B_{k,\varepsilon}(y;s) = &-C_1^k A^{(1)}_{k-1,\varepsilon}(y;s)\,\nu_\varepsilon(x) \\ &+ \frac{1}{2}C_1^k A^{(2)}_{k-1,\varepsilon}(y;s)\,\nu_\varepsilon(|x|^2) + C_2^k A^{(2)}_{k-2,\varepsilon}(y;s)\,\nu_\varepsilon(x)^2 \\ &+ O\left(\nu_\varepsilon(|x|)^3 + \nu_\varepsilon(|x|^3) + \nu_\varepsilon(|x|^2)\nu_\varepsilon(|x|)\right), \end{aligned} \tag{2.36}$$

as $\varepsilon \to 0$, where we used the notation

$$\nu_\varepsilon(g(x)) = \int g(x)s(x)\bar{c}_\varepsilon(x)dx.$$

The second method to compute $A_k(y)$ is via the formula (2.34). However, a careful look at this expression shows that this method boils down to the first method. Indeed, the leading term of (2.34) (when $j=0$) corresponds to $A_{k,\varepsilon}(y;s)$. The higher order approximations ($j=1,\dots,k-1$) will coincide with each of the terms in $B_{k,\varepsilon}(y;s)$ and in practice will be computed numerically via (2.35).

Another point to consider is the computation of the left tail distribution, $\mathbb{P}(X_t \leq y)$, for $y<0$. Its polynomial approximations can be easily obtained in terms of the Lévy process $X_t^- = -X_t$, which has Lévy triple:

$$\nu^-(dx) := \nu(-dx), \quad b^- := -b, \quad \sigma^- := \sigma.$$

Under the constraints (2.3), $X^-$ is also of bounded variation and driftless with Lévy density $s^-(x) := s(-x)$. Thus, writing $y^- = -y$,

$$\mathbb{P}\left(X_t \leq y\right) = \mathbb{P}\left(X_t^- \geq y^-\right) = \sum_{k=1}^n A_k^-(y^-)\frac{t^k}{k!} + \frac{t^{n+1}}{n!}\mathcal{R}_n^-(t,y^-), \tag{2.37}$$

where $A_k^-$ is obtained from (2.4-2.7) by replacing $s$ with $s^-$. Thus,

$$A_k^-(y^-) := A_{k,\varepsilon}(y^-;s^-) + B_{k,\varepsilon}(y^-;s^-),$$

where $A_{1,\varepsilon}(y^-;s^-) = \int_{y^-}^\infty s_\varepsilon^-(u)du$ and for $k \geq 2$ and $y^- \geq 0$,

$$\begin{aligned} A_{k,\varepsilon}(y^-;s^-) &= -A_{k-1,\varepsilon}(y^-;s^-)\int s_\varepsilon(u)du \\ &+ \int_0^\infty A_{k-1,\varepsilon}(u;s^-)s_\varepsilon^-(y^- - u)du - \int_0^\infty A_{k-1,\varepsilon}(u;s)s_\varepsilon(-y^- - u)du. \end{aligned} \tag{2.38}$$

The higher order terms $B_{k,\varepsilon}(y^-;s^-)$ can similarly be obtained from the approximation (2.35).

Note that (2.13) and (2.38) should be computed simultaneously. Indeed, we can combine both computations in only one as follows. Define the modified spectral function:

$$P_t(y) = \begin{cases} \mathbb{P}\left(X_t \geq y\right), & \text{if } y > 0, \\ \\ -\mathbb{P}\left(X_t \leq y\right), & \text{if } y < 0. \end{cases}$$

Define also

$$\lambda_\varepsilon := \int s_\varepsilon(u)du, \quad \text{and} \quad A_{k,\varepsilon}(y) := -A_{k,\varepsilon}(-y; s^-), \qquad \text{for } y < 0.$$

Then, for any $y \in \mathbb{R}\backslash\{0\}$,

$$P_t(y) = \sum_{k=1}^{n} A_{k,\varepsilon}(y)\frac{t^k}{k!} + O(t^{n+1}) + O\left(\int_{|u|<\varepsilon} |u|s(u)du\right),$$

where $A_{k,\varepsilon}(y)$ can be computed recursively as

$$\begin{aligned} A_{1,\varepsilon}(y) &= \int_y^\infty s(u)du\mathbf{1}_{y>0} - \int_{-\infty}^y s(u)du\mathbf{1}_{y<0}, \\ A_{k,\varepsilon}(y) &= \lambda_\varepsilon A_{k-1,\varepsilon}(y) - \int_{-\infty}^\infty A_{k-1,\varepsilon}(u)s_\varepsilon(y-u)du, \end{aligned}$$

for any $y \in \mathbb{R}\backslash\{0\}$.

## 3. Approximation of the marginal densities

For a general Lévy process, a polynomial expansions for the marginal density $p_t$ of $X_t$ of the form

$$p_t(y) = \sum_{k=1}^{n} -d'_k(y)\,\frac{t^k}{k!} + O(t^{n+1}), \quad y > 0, \tag{3.1}$$

was obtained in [10], under the smoothness condition 3.1 and under the following technical condition on the marginal density $p_t$ of $X_t$:

**Conditions 3.1.** *For any $\delta > 0$ and $j \geq 0$, there exists $\hat{t}_0 := \hat{t}_0(\delta, j) \in (0, \infty]$ such that*

$$m_\delta^{(j)} := \sup_{0<u<\hat{t}_0} \sup_{|x|>\delta} |p_u^{(j)}(x)| < \infty. \tag{3.2}$$

In plain words, the later condition requires that the derivatives $p_t^{(j)}$ remain uniformly bounded away from the origin, as $t \to 0$. It was shown in [10] that such a condition is satisfied with $\hat{t}_0 = \infty$, by symmetric stable Lévy processes and some tempered stable Lévy processes such as the CGMY one. Barndorff-Nielsen [4] (see also Woener [14]) consider the infinite-series form of (3.1) in the case of subordinators (resp. bounded-variation Lévy processes) under other fairly strong conditions.

The following result summarizes the results in [10] for a driftless Lévy process of bounded variation[2]:

**Theorem 3.1.** *Let $\underline{y} > 0$ and $n \geq 1$. Suppose that the standing Assumptions 2.1 and 3.1 are satisfied. Let $t_0 := \min_{j \leq 2n+1} \hat{t}_0(\underline{y}, j) > 0$, where $\hat{t}_0$ is as in (3.2). Then, under the constraints (2.3) and with the notation (2.7), the following statements hold true:*

1. *(2.1) is satisfied for any $0 < t < t_0$ and any $y > \underline{y}$;*
2. *The coefficient $A_k(y)$ admits the representation (2.6) for any $\varepsilon > 0$ and $\mathcal{R}_n(t, y)$ admits the following representation for any $0 < \varepsilon < \underline{y}/(n+2) \wedge 1$:*

$$\mathcal{R}_n(t, y) = \sum_{\mathbf{k}: p+q+r=n+1} \hat{c}_{\mathbf{k}} \binom{n+1}{\mathbf{k}} \int_0^1 (1-\alpha)^n d_{\mathbf{k},\varepsilon}(\alpha t; y) d\alpha, \tag{3.3}$$

*where*

$$d_{\mathbf{k},\varepsilon}(t; y) := \begin{cases} \int p_t^{(r-1)} \left( y - \sum_{j=1}^{r} \beta_j w_j \right) d\pi_{\mathbf{k},\varepsilon}, & \text{if } q = 0, r > 0, \\ \mathbb{E}\, d_{\mathbf{k},\varepsilon}(y - X_t), & \text{otherwise.} \end{cases}$$

3. *For any $0 < t < t_0$ and $y > \underline{y} > 0$, there exist functions $a_k(y)$ such that*

$$p_t(y) = \sum_{k=1}^{n} a_k(y) \frac{t^k}{k!} + \frac{t^{n+1}}{n!} \mathcal{R}'_n(t, y). \tag{3.4}$$

*with $\mathcal{R}'_n(t, y) = O(1)$, as $t \to 0$;*

4. *The coefficient $a_k(y)$ in (3.4) is given by $-A'_k(y)$ and admits the representation*

$$a_k(y) := \sum_{\mathbf{k} \in \mathcal{K}_k} \hat{c}_{\mathbf{k}} \binom{k}{\mathbf{k}} (-d'_{\mathbf{k},\varepsilon}(y)), \tag{3.5}$$

*with*

$$d'_{\mathbf{k},\varepsilon}(y) := \begin{cases} (-1)^{r-1} \int (c_\varepsilon s)^{(r)} \left( y - \sum_{i=2}^{q} u_i - \sum_{j=1}^{r} \beta_j w_j \right) d\pi_{\mathbf{k},\varepsilon}, & q > 0, \\ 0, & o.w. \end{cases} \tag{3.6}$$

As for the expansion for the tail distribution, the coefficient $a_k(y)$ above is constant for any $\varepsilon < y/(k+1)$. It can be divided into two types of terms: when

[2] The result in [10] is weaker, but an analysis of the proofs there shows that the result given here holds true.

$r = 0$ and when $r > 0$. Let $a_{k,\varepsilon}$ be the sum of all the terms such that $r = 0$. Then,

$$a_{k,\varepsilon}(y; s) = \sum_{q=1}^{k} C_q^k \left( -\int s_\varepsilon(u)du \right)^{k-q} \int s_\varepsilon \left( y - \sum_{i=2}^{q} u_i \right) \prod_{i=2}^{q} s_\varepsilon(u_i)du_i. \quad (3.7)$$

which coincides with the coefficients (1.6) proposed by Barndorff-Nielsen [2]. As it was mentioned in the introduction, the convergence of (3.7) when $s_\varepsilon$ represents a general approximation for $s$ has been considered in the literature (see e.g. Barnrdorff-Nielsen and Hubalek [4] and also Woerner [14]). To the best of our knowledge, the existence of the limit and its actual value were not known except for some special cases. Using the results of Section 2.1, we can solve this problem under quite general conditions. To avoid confusion with $s_\varepsilon$, which in this paper is taken to be of the form $s_\varepsilon = c_\varepsilon s$, we consider a general approximating function $\tilde{s}_\varepsilon$ and define

$$\tilde{a}_{k,\varepsilon}(y) = \sum_{q=1}^{k} C_q^k \left( -\int \tilde{s}_\varepsilon(u)du \right)^{k-q} \tilde{s}_\varepsilon^{\,*q}(y), \quad (3.8)$$

where $*k$ indicates the $k^{th}-$fold convolution. Obviously, $a_{k,\varepsilon}$ is obtained from $\tilde{a}_{k,\varepsilon}$ by taking $\tilde{s}_\varepsilon = s_\varepsilon$.

**Proposition 3.2.** *Let $\tilde{s}_\varepsilon$, $\varepsilon > 0$, be non-negative integrable functions such that*

*(i)* $\tilde{s}_\varepsilon(u) \to s(u)$, *as* $\varepsilon \to 0$, *for a function $s$ such that* $\int s(u)(1 \wedge |u|)du < \infty$;
*(ii)* $\tilde{s}_\varepsilon(u) \leq \hat{s}(u)$, *for a function $\hat{s}$ such that* $\int \hat{s}(u)(1 \wedge |u|)du < \infty$;
*(iii)* *For any $j \in \mathbb{N}$ and $\delta > 0$ there exist constants $\varepsilon_0(j, \delta) > 0$ and $k_\delta^{(j)} < \infty$ such that*

$$\sup_{|x|>\delta} |\tilde{s}_\varepsilon^{(j)}(x)| \leq k_\delta^{(j)}, \quad \text{and} \quad \tilde{s}_\varepsilon^{(j)}(x) \stackrel{|x|\to\infty}{\longrightarrow} 0.$$

*for any $0 < \varepsilon < \varepsilon_0(j, \delta)$.*

*Then,*

$$\tilde{a}_k(y) := \lim_{\varepsilon \to 0} \tilde{a}_{k,\varepsilon}(y),$$

*exists, and admits the following representation for any $\delta < y/k$ and any smooth truncation function $c_\delta$ as in (2.4):*

$$\sum_{j=0}^{k-1} C_j^k \sum_I (-1)^{k-|I|} \int_{\mathbb{R}^j \times [0,1]^j} F_I^{(j+1)} \left( y - \sum_{i=1}^{j} u_i \beta_i \right) \prod_{i=1}^{j} d\beta_i s(u_i) \bar{c}_\delta(u_i) u_i du_i, \quad (3.9)$$

*where the second summation above is over all subsets $I$ of $\{j+1, \ldots, k\}$, and $F_I$ is as in (2.19).*

*Proof.* The proof heavily uses Lemma 2.2 and Corollary 2.3 applied to $\tilde{s}_\varepsilon$. Indeed, (3.8) can be written as

$$\tilde{a}'_{k,\varepsilon}(y) = -\frac{d}{dy} \tilde{A}_{k,\varepsilon}(y),$$

where

$$\begin{aligned}\tilde{A}_{k,\varepsilon}(y) &:= \sum_{q=1}^{k} C_q^k (-1)^{k-q} \int \mathbf{1}_{\{\sum_{i=1}^{q} u_i \geq y\}} \prod_{i=1}^{k} \tilde{s}_\varepsilon(u_i) du_i \\ &= \int H_k(u_1, \ldots, u_k; y) \prod_{i=1}^{k} \tilde{s}_\varepsilon(u_i) du_i = \widehat{A}_{k,\delta}(y; \tilde{s}_\varepsilon),\end{aligned}$$

if $y > 0$ and $\delta < y/k$, where $\widehat{A}$ is given by (2.25). Then, from representation (2.24), $\tilde{A}_{k,\varepsilon}(y)$ can be written as

$$\sum_{j=0}^{k-1} C_j^k \sum_{I \subset \{j+1,\ldots,k\}} (-1)^{k-|I|} \int_{\mathbb{R}^j \times [0,1]^j} F_{I,\delta,j,\varepsilon}^{(j)} \left( y - \sum_{i=1}^{j} u_i \beta_i \right) \prod_{i=1}^{j} d\beta_i \tilde{s}_\varepsilon \cdot \bar{c}_\delta(u_i) u_i du_i,$$

where

$$F_{I,\delta,j,\varepsilon}(x) := \int \mathbf{1}_{\left\{\sum_{i \in I} u_i \geq x\right\}} \prod_{i=j+1}^{k} \tilde{s}_\varepsilon(u_i) c_\delta(u_i) du_i.$$

In view of the assumption (iii), there exists an $\varepsilon_0 > 0$ such that $F_{I,\delta,j,\varepsilon}$ has bounded derivatives of order $j \leq k$ and

$$F_{I,\delta,j,\varepsilon}^{(j)}(x) = \int (s_\varepsilon \cdot c_\delta)^{(j-1)} \left( x - \sum_{i=1}^{k-1} u_i \right) \prod_{i=j+1}^{k-1} s_\varepsilon \cdot c_\delta(u_i) du_i,$$

for any $0 < \varepsilon < \varepsilon_0$. Thus,

$$\tilde{A}'_{k,\varepsilon}(y) = \sum_{j=0}^{k-1} C_j^k \sum_{I} (-1)^{k-|I|} \int F_{I,\delta,j,\varepsilon}^{(j+1)} \left( y - \sum_{i=1}^{j} u_i \beta_i \right) \prod_{i=1}^{j} d\beta_i \tilde{s}_\varepsilon \cdot \bar{c}_\delta(u_i) u_i du_i.$$

By the assumptions (ii)-(iii), the dominated convergence theorem can be applied and, by the assumption (i), the limit when $\varepsilon \to 0$ will be (3.9). □

### *3.1. Computational issues and the density for negative values*

**(1)** As it happens with the approximation of the tail distribution, it is more natural to compute the coefficient $a_k(y)$ via the decomposition:

$$a_k(y) = a_{k,\varepsilon}(y; s) + b_{k,\varepsilon}(y; s),$$

where $a_{k,\varepsilon} = -A'_{k,\varepsilon}$ and $b_{k,\varepsilon} := -B'_{k,\varepsilon}$, instead of using the expression for the limit of $a_{k,\varepsilon}(y; s)$ described in the Proposition 3.2. Using the formula (2.13), one can compute $A'_{k,\varepsilon}(y; s)$ as follows:

$$\begin{aligned}&A'_{1,\varepsilon}(y; s) = -s_\varepsilon(y), \\ &A'_{k,\varepsilon}(y; s) = -A'_{k-1,\varepsilon}(y; s) \int s_\varepsilon(u) du \\ &\quad + \int_0^\infty A_{k-1,\varepsilon}(u; s) s'_\varepsilon(y - u) du + \int_0^\infty A_{k-1,\varepsilon}(u; s^-) \left(s_\varepsilon^-\right)' (-y - u) du.\end{aligned}$$

An integration by parts gives,

$$A'_{k,\varepsilon}(y;s) = -A'_{k-1,\varepsilon}(y;s)\int s_\varepsilon(u)du + s_\varepsilon(y)\left(A_{k-1,\varepsilon}(0;s) + A_{k-1,\varepsilon}(0;s^-)\right)$$
$$+ \int_0^\infty A'_{k-1,\varepsilon}(u;s)s_\varepsilon(y-u)du + \int_0^\infty A'_{k-1,\varepsilon}(u;s^-)s_\varepsilon^-(-y-u)du.$$

But, using (2.12), it turns out that

$$A_{k,\varepsilon}(0;s) + A_{k,\varepsilon}(0;s^-) = -\left(-\int s_\varepsilon(u)du\right)^k.$$

Thus, we get the following recursive formulas:

$$\begin{aligned}
&A'_{1,\varepsilon}(y;s) = -s_\varepsilon(y), \\
&A'_{k,\varepsilon}(y;s) = -s_\varepsilon(y)\left(-\int s_\varepsilon(u)du\right)^{k-1} - A'_{k-1,\varepsilon}(y;s)\int s_\varepsilon(u)du \qquad (3.10)\\
&\quad + \int_0^\infty A'_{k-1,\varepsilon}(u;s)s_\varepsilon(y-u)du + \int_0^\infty A'_{k-1,\varepsilon}(u;s^-)s_\varepsilon^-(-y-u)du.
\end{aligned}$$

**(2)** To obtain higher order approximations for $a_k(y)$, one can compute the successive terms of $B'_{k,\varepsilon}(y;s)$ as it was done in (2.35-2.36). Hence, for instance,

$$B'_{k,\varepsilon}(y;s) = -C_1^k A^{(2)}_{k-1,\varepsilon}(y;s)\,\nu_\varepsilon(x) + O\left(\nu_\varepsilon(|x|)^2 + \nu_\varepsilon(|x|^2)\right).$$

**(3)** To approximate the marginal density $p_t(y)$ for values of $y<0$, one need to consider $s^-(u) = s(-u)$ in the previous formulas similar to (2.37). Concretely,

$$p_t(y) = \sum_{k=1}^n -\left(A_k^-\right)'(-y)\,\frac{t^k}{k!} + \frac{t^{n+1}}{n!}\left(\mathcal{R}_n^-\right)'(t,-y),$$

for $\varepsilon < y/(k+1)$, where $A_k^-$ is obtained from (2.4-2.7) by replacing $s$ with $s^-$. Thus, we have that $-\left(A_k^-\right)'(-y) = -A'_{k,\varepsilon}(-y;s^-) - B'_{k,\varepsilon}(-y;s^-)$ with

$$\begin{aligned}
&A'_{1,\varepsilon}(-y;s^-) = -s_\varepsilon(y), \qquad (3.11)\\
&A'_{k,\varepsilon}(-y;s^-) = -s_\varepsilon(y)\left(-\int s_\varepsilon(u)du\right)^{k-1} - A'_{k-1,\varepsilon}(-y;s^-)\int s_\varepsilon(u)du\\
&\quad + \int_0^\infty A'_{k-1,\varepsilon}(u;s^-)s_\varepsilon^-(-y-u)du + \int_0^\infty A'_{k-1,\varepsilon}(u;s)s_\varepsilon(y-u)du.
\end{aligned}$$

**(4)** One can combine the previous formulas in one simple expression. Indeed, define

$$\lambda_\varepsilon := \int s_\varepsilon(u)du, \quad a_{k,\varepsilon}(y) := -A'_{k,\varepsilon}(-y;s^-), \quad \text{for } y<0.$$

Then, we note that, for any $y \in \mathbb{R}\backslash\{0\}$,

$$a_{1,\varepsilon}(y) = s_\varepsilon(y), \tag{3.12}$$

$$a_{k,\varepsilon}(y) = s_\varepsilon(y)\left(-\lambda_\varepsilon\right)^{k-1} - \lambda_\varepsilon a_{k-1,\varepsilon}(y) + \int_{-\infty}^{\infty} a_{k-1,\varepsilon}(u) s_\varepsilon(y-u)du. \tag{3.13}$$

Moreover, from the representation (2.31) and the boundedness of the derivatives of $A_{k,\varepsilon}$ (similar to (2.33)),

$$p_t(y) = \sum_{k=1}^{n} a_{k,\varepsilon}(y)\frac{t^k}{k!} + O(t^{n+1}) + O\left(\int_{|u|<\varepsilon} |u| s(u)du\right).$$

**(5)** Using the same arguments as above, one can obtain recursive formulas for higher order derivatives of $a_{k,\varepsilon}$. For instance, it turns out that

$$a'_{k,\varepsilon}(y) := \begin{cases} -A^{(2)}_{k,\varepsilon}(y;s), & \text{if } y>0, \\ A^{(2)}_{k,\varepsilon}(-y;s^-), & \text{if } y<0, \end{cases}$$

satisfies the recursive formulas

$$a'_{1,\varepsilon}(y) = s'_\varepsilon(y),$$

$$a'_{k,\varepsilon}(y) = s'_\varepsilon(y)\left(-\lambda_\varepsilon\right)^{k-1} - \lambda_\varepsilon a'_{k-1,\varepsilon}(y) + \int_{-\infty}^{\infty} a'_{k-1,\varepsilon}(u) s_\varepsilon(y-u)du,$$

which can subsequently be used to improve the approximation of $p_t$ as follows:

$$p_t(y) = \sum_{k=1}^{n} \left(a_{k,\varepsilon}(y) - k a'_{k-1,\varepsilon}(y)\nu_\varepsilon(x)\right)\frac{t^k}{k!}$$
$$+ O(t^{n+1}) + O\left(\left(\int_{|u|<\varepsilon}|u|s(u)du\right)^2 + \int_{|u|<\varepsilon}|u|^2 s(u)du\right).$$

## 4. Numerical examples

In this part we illustrate our approximations for a Variance Gamma Lévy process. We also touch upon, only briefly, the CGMY model of [6]. The values of the model parameters were motivated by previous empirical studies based on maximum likelihood estimation.

### 4.1. Variance Gamma process

A variance Gamma process is a pure-jump Lévy process with Lévy density

$$s(x) = \begin{cases} \frac{\alpha}{|x|}\exp\left(-\frac{|x|}{\beta^-}\right), & \text{if } x<0, \\ \frac{\alpha}{x}\exp\left(-\frac{x}{\beta^+}\right), & \text{if } x>0. \end{cases}$$

Among the many Lévy-based financial models, this is considered one of the most popular even in the financial industry. Multiple empirical studies have been performed (see Seneta [13] for a nice review). For instance, using the maximum likelihood method, Carr et. al. [7] reports (annualized) values of $\hat{\alpha} = 500$, $\hat{\beta}^{+} = 0.0037056$, and $\hat{\beta}^{-} = 0.0037067$ based on daily returns of the S&P stock index from January 1992 to September 1994. For simplicity we assume that the process is driftless and also that $\hat{\beta}^{+} = \hat{\beta}^{-} = 0.0037$.

We consider the polynomial approximations (1.9), approximating the coefficients $a_k(y)$ by (1.6) with

$$s_\varepsilon = s(x)\mathbf{1}_{|x|\geq\varepsilon}.$$

The expansions (1.6) are computed using the recursive formula (1.13). The implementation is done in MatLab. In general, especially for large values of $t$ (say $t \geq 1/365$) and for even degrees of the polynomial, the polynomial expansions can go negative for small values of $x$. Due to this, we consider the following natural modification of the approximating density that is monotone increasing on $(-\infty, 0)$ and monotone decreasing on $(0, \infty)$. Concretely, let $\tilde{p}_t$ be the approximating polynomial function on the right-hand side of (1.9) and suppose that we obtain the values of this density on a partition $x_{-N/2} < \cdots < x_0 = 0 < x_1 < \cdots < x_{N/2}$. Then, we consider

$$\hat{p}_t(x_i) = \begin{cases} \max\{\tilde{p}_t(x_i), \tilde{p}_t(x_{i-1})\} & \text{if } -N/2 < i \leq 0, \\ \min\{\tilde{p}_t(x_i), \tilde{p}_t(x_{i+1})\} & \text{if } 0 < i < N/2, \\ \max\{\tilde{p}_t(x_i), 0\} & \text{if } i = N/2, -N/2. \end{cases}$$

Figure 1 shows, for $x \in [-.012, 0.012]$, the "monotone" polynomial approximations $\hat{p}_t(x)$ of order $n = 1, 5, 11, \ldots, 35$ corresponding to $t = 1/365$ years. We take $\varepsilon = .0001$ and $N = 40,000$ points $x_i$ equally spaced on $[-.05, .05]$ for the computation of the convolutions. The implementation runs relatively fast (about half a minute to generate the 35 approximations, but only a few seconds if one takes $N = 10,000$ points for which the approximations are quite good too). The odd degree approximations always outperform the even degree approximations.

Figure 2 shows, for $x \in [-.0015, 0.0015]$, the polynomial approximations $\tilde{p}_t(x)$ of order $n = 1, 3, 7, \ldots, 19$ corresponding to $t = 1/(3 * 365)$ years. Again, $\varepsilon = .0001$ and $N = 40,000$ regular partition points on $[-.05, .05]$. Note that in this case the approximations are always monotone increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Figure 3 shows also the results taking $N = 15,000$ points on $[-.01, .01]$ with $\varepsilon = .00001$.

We now consider briefly the CGMY model (1.1) with parameters $C = 280.11$, $G = M = 1/102.84$, and $Y = .1191$. These parameters were obtained in [6] from daily returns of Microsoft over the period January 1, 1994 to December 31, 1998 using maximum likelihood estimation. Here, we consider the approximating density

$$s_\varepsilon(x) = s(x)e^{-\varepsilon/|x|},$$

with $\varepsilon = .001$ and $N = 10,000$ partition points on $[-.1, .1]$. Figure 4 shows the monotone polynomial approximations $\hat{p}_t(x)$ for $x \in [-.05, 0.05]$ of order

$n = 1, 3, 7, \dots, 23$ corresponding to $t = 1/365$ years. This seems close to the "true" Lévy density which is shown in [6, Figure 4] using the Fast Fourier Transform.

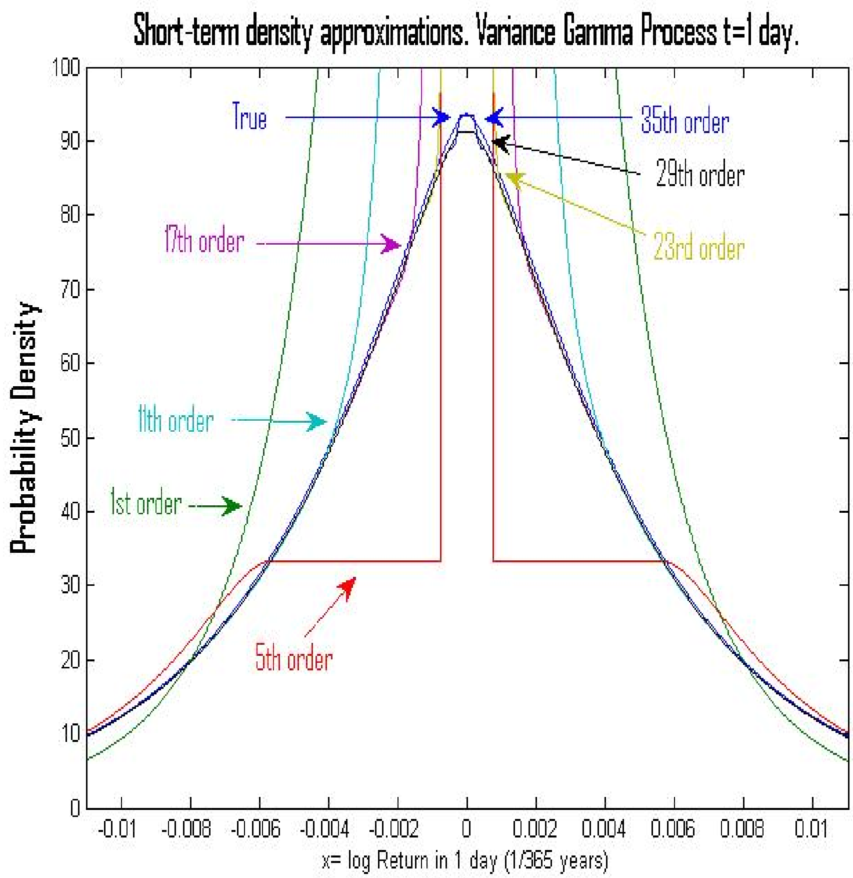


FIG 1. *Polynomial approximations of the marginal density of a Variance Gamma process with* $t = 1/365$ *years using the approximation* $s(x)\mathbf{1}_{|x|\geq\varepsilon}$ *with* $\varepsilon = .0001$.

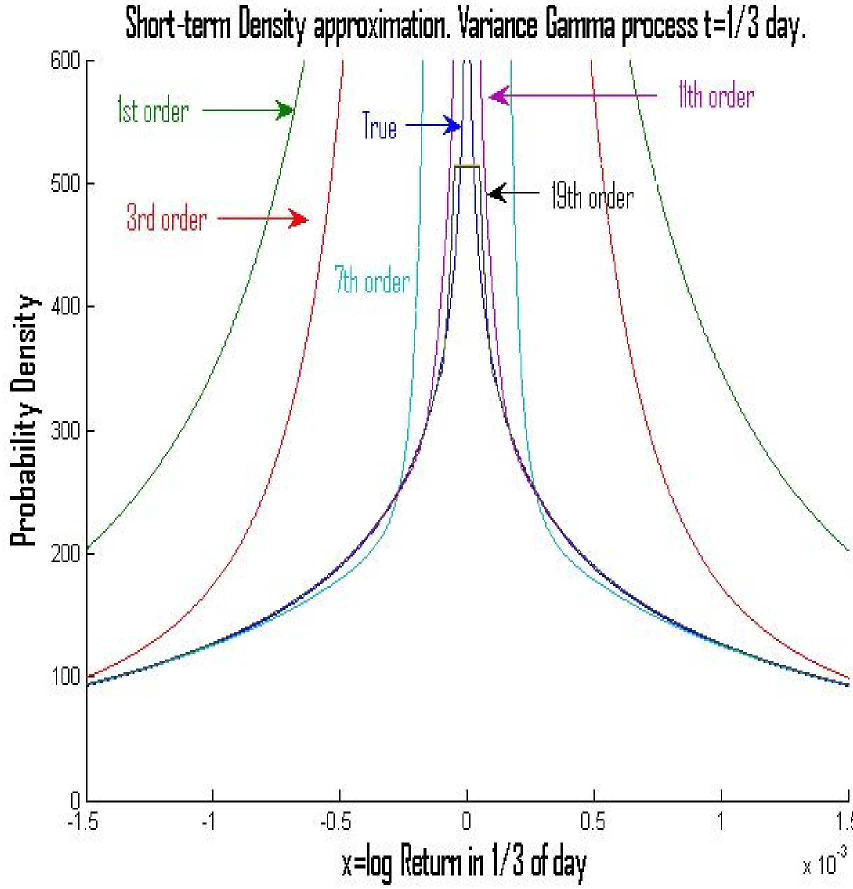


FIG 2. *Polynomial approximations of the marginal density of a Variance Gamma process with* $t = 1/(3 * 365)$ *years using the approximation* $s(x)\mathbf{1}_{|x|\geq\varepsilon}$ *with* $\varepsilon = .0001$.

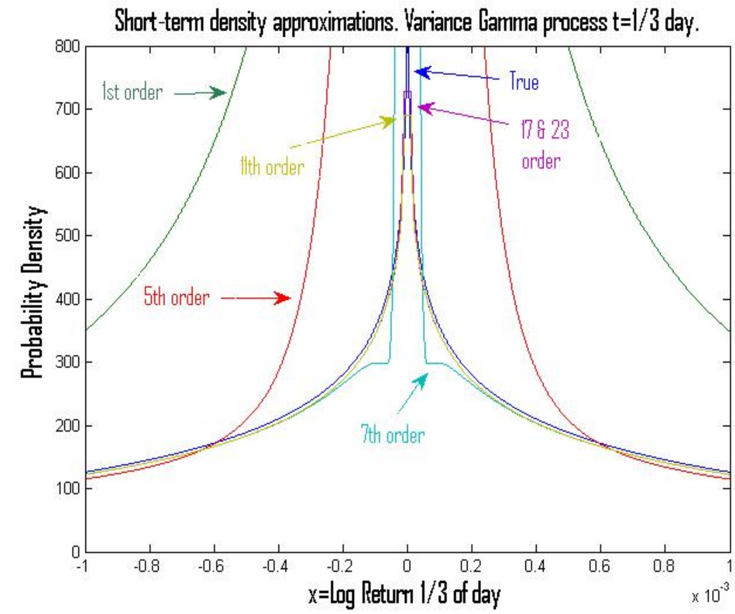


FIG 3. *Polynomial approximations of the marginal density of a Variance Gamma process with* $t = 1/(3 * 365)$ *years using the approximation* $s(x)\mathbf{1}_{|x|\geq\varepsilon}$ *with* $\varepsilon = .00001$.

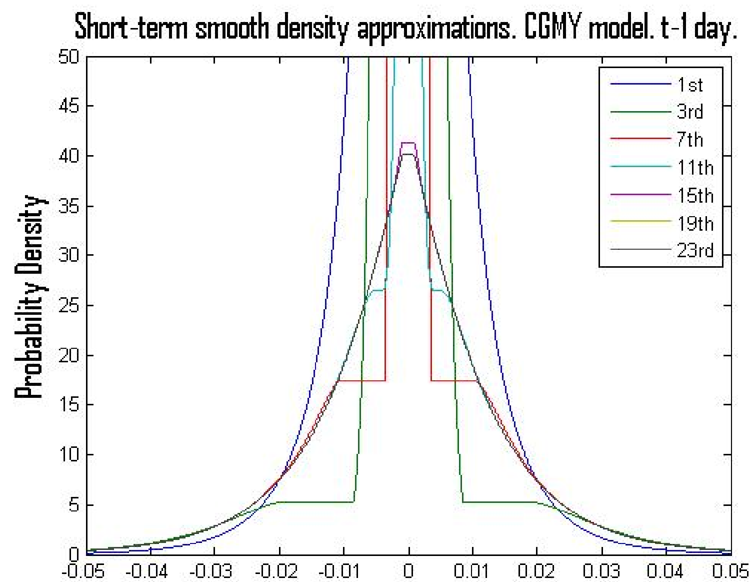


FIG 4. *Polynomial approximations of the marginal density of the CGMY model with* $t = 1/(3 * 365)$ *years using the approximation* $s(x)e^{-\varepsilon/|x|}$ *with* $\varepsilon = .001$.

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