Short-time asymptotics for the implied volatility skew under a stochastic volatility model with Lévy jumps

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Abstract

The implied volatility slope has received relatively little attention in the literature on short-time asymptotics for financial models with jumps, despite its importance in model selection and calibration. In this paper, we fill this gap by providing high-order asymptotic expansions for the at-the-money implied volatility slope of a rich class of stochastic volatility models with independent stable-like jumps of infinite variation. The case of a pure-jump stable-like Lévy model is also considered under the minimal possible conditions for the resulting expansion to be well defined. As an intermediary result, we also obtain high-order expansions for at-the-money digital call option prices. The results obtained herein are markedly different from those obtained in recent papers ([10, 12]) for “close-to-the-money” option prices and implied volatility, and aid in understanding how the behavior of implied volatility near expiry is affected by important model parameters, such as the leverage and vol vol parameters, that were not present in the aforementioned earlier results. Our simulation results also indicate that for parameter values of relevance in finance, the asymptotic expansions give a good fit for maturities up to one month.

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1 Introduction

A number of recent papers have shed light on the asymptotic behavior of option prices and implied volatility in models with jumps (see [1], [7], [9], [10], [12], [17], [21], and references therein). In this paper we investigate the behavior of the at-the-money (ATM) implied volatility slope (i.e., the strike derivative), as maturity tends to zero. This quantity has received comparatively little attention in the literature, but is highly relevant for model selection, and when calibrating models to observed option prices, especially in FX markets, where it is standard to effectively quote directly on smile slope and convexity (c.f. Section 2 in [1]). It is well known that in the presence of jumps, the implied volatility of out-of-the-money (OTM) options explodes as maturity becomes small, while for ATM options it converges to the volatility of the continuous component (see, e.g., [9], [12], and [21]). Despite the smile explosion, the limiting behavior of the OTM implied volatility slope can usually be elucidated from the corresponding asymptotic expansion for implied volatility, by taking the derivative with respect to strike (see (1.19) below for further details). However, that is not possible in the ATM case, which is notoriously difficult when it comes to short-time asymptotics and usually requires a separate and more delicate analysis.

In what follows, we assume that the interest rate $r$ is 0 and that the price process $S := (S_t)_{t \geq 0}$ of the underlying asset is a P-martingale, normalized to $S_0 = 1$. We denote the implied volatility by $\hat{\sigma}(\kappa, t)$, where $\kappa := \log(K)$ is the
log-strike (or log-moneyness), and \( t \) is the time-to-maturity. Let \( C(\kappa, t) = \mathbb{E}(S_t - e^\kappa)_+ \) denote the price of a European call option with maturity \( t \) and log-strike \( \kappa \), and \( C^{BS}(\kappa, t, \sigma) \) denote the price of the corresponding option under a Black-Scholes model with volatility \( \sigma \). The following classical formula for the implied volatility slope is obtained from \( C^{BS}(\kappa, t, \hat{\sigma}(\kappa, t)) = C(\kappa, t) \), via the implicit function theorem:

\[
\frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa} = \frac{C_\kappa(\kappa, t) - C^{BS}_\kappa(\kappa, t, \hat{\sigma}(\kappa, t))}{C^{BS}_\sigma(\kappa, t, \hat{\sigma}(\kappa, t))} = -\frac{e^{\kappa}\mathbb{P}(S_t \geq e^\kappa) - e^{\kappa}\Phi\left(-\kappa + \frac{\hat{\sigma}(\kappa, t)}{\sigma}\right)}{\sqrt{2\pi} \phi\left(\frac{-\kappa + \hat{\sigma}(\kappa, t)}{\sigma}\right)} \tag{1.1},
\]

where, hereafter, \( \Phi \) and \( \phi \) are the standard Gaussian cumulative distribution function (CDF) and probability density function (PDF), respectively, and we have used the well known identity \( C_\kappa(\kappa, t) = -e^{\kappa}\mathbb{P}(S_t \geq e^\kappa) \), which holds, e.g., whenever \( S_t \) admits a density function. Hence, the problem of obtaining an expansion for the implied volatility slope in small time, boils down to obtaining asymptotic expansions for the implied volatility \( \hat{\sigma}(\kappa, t) \), and the transition probability \( \mathbb{P}(S_t \geq e^\kappa) \), which, in financial terms, can be thought of as the price of a digital call option with log-strike \( \kappa \).

Due to their high liquidity and practical relevance, we will mainly be concerned with ATM options, i.e., when \( \kappa = 0 \). In that case, by letting \( \hat{\sigma}(t) := \hat{\sigma}(0, t) \) and using the fact that \( \hat{\sigma}(\kappa, t) \sqrt{t} \to 0 \) as \( t \to 0 \) (which holds under mild conditions), in addition to the standard approximations \( \Phi(x) = \frac{1}{2} + \frac{x}{\sqrt{2\pi}} + O(x^3) \) and \( 1/(\sqrt{2\pi} \phi(x)) = 1 + x^2/2 + O(x^4) \), as \( x \to 0 \), (1.1) can be written as

\[
\frac{\partial \hat{\sigma}(0, t)}{\partial \kappa} \bigg|_{\kappa = 0} = \sqrt{\frac{2\pi}{t}} \left( \frac{1}{2} - \mathbb{P}(S_t \geq 1) - \frac{\hat{\sigma}(t)\sqrt{t}}{2\sqrt{2\pi}} + O\left(\left(\hat{\sigma}(t)\sqrt{t}\right)^3\right)\right) \left(1 + \frac{\left(\hat{\sigma}(t)\sqrt{t}\right)^2}{8} + O\left(\left(\hat{\sigma}(t)\sqrt{t}\right)^4\right)\right), \tag{1.2}
\]

as \( t \to 0 \). In [22] it is shown that (1.2) converges to a nonzero value for a Merton type jump-diffusion model, and considerable empirical support is provided for a negative relation between expected stock returns and the ATM implied volatility slope, estimated by the difference between implied volatilities of short-term, near-the-money call and put options. The result in [22] is generalized in [7] to general Lévy jump-diffusion models. Moreover, for pure-jump infinity activity models, the authors show that the leading order term is of order \( t^{-\frac{1}{2}} \) for bounded variation Lévy processes, as well as some specific infinite variation cases such as the Normal Inverse Gaussian (NIG), the Meixner, and the CGMY models. For tempered stable Lévy processes, as defined in [5], Proposition 8.2 in [1] can be interpreted as giving an expression for the ATM strike derivative, but under some extremely restrictive assumptions on the model parameters. Finally, it is worth mentioning a related literature on the relative prices of OTM calls and puts, termed the skewness premium in [2], and used as a measure of the asymmetry in the volatility smile. Sufficient conditions for the skewness premium to be positive or negative, in terms of the model parameters, are derived in [8], for a large class of exponential Lévy models.

In this paper, we consider the case of tempered stable-like Lévy processes, as introduced [10] and [12], with and without an independent continuous component. More concretely, in the pure-jump case, we consider

\[
S_t := S_0 e^{X_t}, \tag{1.3}
\]

where \( X \) stands for a pure-jump Lévy process with a Lévy measure of the form

\[
\nu(dx) = C\left(\frac{x}{|x|}\right)|x|^{-Y-1}\tilde{q}(x)dx, \tag{1.4}
\]

for some constants \( C(1), C(-1) \in [0, \infty) \) such that \( C(1) + C(-1) > 0 \), \( Y \in (1, 2) \), and a bounded function \( \tilde{q} : \mathbb{R}\setminus\{0\} \to [0, \infty) \) such that \( \tilde{q}(x) \to 1 \) as \( x \to 0 \). In the case of \( \tilde{q}(x) \equiv 1 \), \( X \) is the well known stable process (cf. [19]), while for \( \tilde{q}(x) = e^{-Mx}1_{\{x>0\}} + e^{-G|x|}1_{\{x<0\}} \), \( X \) is commonly referred to as a tempered stable Lévy process (cf. [5]). We also point out that the restriction on \( Y \) implies that \( X \) is of infinite variation, and is supported by recent econometric studies of high-frequency financial data sets (see Remark 2.2 in [10]).

For the model described in the previous paragraph, a second order expansion for the ATM implied volatility is given in Theorem 3.1 of [12], under a minimal integrability condition on \( \tilde{q} \) around the origin. Specifically, it is proved that

\[
\hat{\sigma}(t) = \sqrt{2\pi} \sigma_1 t^{\frac{1}{2}} + \sqrt{2\pi} \sigma_2 t^{\frac{1}{2}} + o(t^{\frac{1}{2}}), \quad t \to 0, \tag{1.5}
\]
with \( \sigma_1 := \mathbb{E}(Z_1^+) \), where, under \( \overline{P} \), \( \{Z_t\}_{t \geq 0} \) is a strictly process with Lévy measure \( \nu(dx) = C(x/|x|)|x|^{-Y-1}dx \), and

\[
\sigma_2 := \overline{P}(Z_1 < 0) C(1) \int_0^\infty (e^{s\tilde{q}(x)} - \tilde{q}(x) - x) x^{-Y-1} dx
- \overline{P}(Z_1 \geq 0) C(-1) \int_{-\infty}^0 (e^{s\tilde{q}(x)} - \tilde{q}(x) - x) |x|^{-Y-1} dx.
\] (1.6)

For the transition probability appearing in (1.1), we have \( \mathbb{P}(S_t \geq e^k) = \mathbb{P}(X_t \geq \kappa) \), but while a lot is known about \( \mathbb{P}(X_t \geq \kappa) \) for nonzero \( \kappa \) (cf. [11]), much less has been said about \( \mathbb{P}(X_t \geq 0) \) for processes of infinite jump activity. The leading order term for certain Lévy models (e.g., NIG, Meixner, and CGMY) is obtained in [7], while, for tempered stable-like processes,

\[ \mathbb{P}(X_t \geq 0) \rightarrow \overline{P}(Z_t \geq 0), \quad t \rightarrow 0, \] (1.7)

under a mild regularity condition on \( \tilde{q} \) around the origin. The limit (1.7) is a consequence of the fact that \( t^{-1/Y} X_t \) converges in law to \( Z_1 \), as \( t \rightarrow 0 \) (cf. [18]), and can not be extended to higher order terms. However, procedures similar to the ones used to derive the “close-to-the-money” option price expansions in [10] and [12], allow us to obtain a closer look at the convergence. The initial step in doing so is writing

\[ \mathbb{P}(X_t \geq 0) - \overline{P}(Z_t \geq 0) = \left( \overline{P}(Z_1 \geq -\gamma t^{1-\gamma}) - \overline{P}(Z_1 \geq 0) \right) + \mathbb{E} \left( (e^{-U_t} - 1) \mathbf{1}_{\{Z_t \geq \gamma t^\gamma\}} \right), \] (1.8)

where \( \overline{P} \) is a probability measure equivalent to \( \mathbb{P} \), under which \( X_t \) has the representation \( Z_t + \gamma t \), and \( U_t \) denotes the likelihood ratio, i.e., \( d\overline{P}|_X = e^{-U_t} d\mathbb{P}|_X \). The first part on the right-hand side of (1.8) which relates only to the stable part of \( X \), can be written as \( \int_{-\gamma t^{1-\gamma}} \tilde{f}_Z(z) dz \), and therefore handled using a well known expansion of the strictly \( Y \)-stable density \( \tilde{f}_Z \) around the origin (see (14.34) in [19]). The second part, on the other hand, appears as a result of the discrepancy between \( X \) and a stable process, and can be analyzed using similar yet more intricate tools than those employed in [12]. Combining the two previously mentioned ideas, the following novel higher order asymptotic expansion is obtained:

\[ \mathbb{P}(X_t \geq 0) - \overline{P}(Z_t \geq 0) = \sum_{k=1}^n d_k t^{k(1-\gamma)} + e t^\gamma + f t + o(t), \quad t \rightarrow 0, \] (1.9)

where \( n := \max\{k \geq 3 : k (1 - 1/Y) \leq 1\} \). It is important to point out that the leading order term is \( d_1 t^{1-\gamma} \) for all \( Y \in (1, 2) \). This can be compared to the expansion for ATM option prices given in Theorem 3.1 of [12], where the first and second order terms are of order \( t^{\gamma} \) and \( t \), i.e. the convergence here is slower.

As mentioned above, the implied volatility of OTM options explodes as \( t \rightarrow 0 \), while for ATM options it converges to the volatility of the continuous component. As a by-product of (1.5) and (1.9), together with (1.2), we deduce an expansion for the ATM implied volatility slope, which also exhibits explosive behavior, as \( t \rightarrow 0 \). Furthermore, some qualitative properties, such as its sign and order of convergence, can easily be recovered from the model parameters (see Remark 3.4 for further details). We reiterate that this kind of information is important when it comes to model selection and calibration. For example, in FX markets the smile skew has been observed to be very steep for small maturity options, which contradicts a model where the FX dynamics are driven by a pure diffusion process, as such processes result in a finite limit of the ATM slope.

In order to incorporate a continuous component of diffusive type into the price dynamics, we consider the model

\[ S_t := S_0 e^{X_t + Y_t}, \quad (V_t)_{t \geq 0} \text{ is a stochastic volatility process of the form} \]

\[ dV_t = \mu(Y_t) dt + \sigma(Y_t) \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right), \quad V_0 = 0, \] (1.10)

\[ dY_t = \alpha(Y_t) dt + \gamma(Y_t) dW_t^1, \quad Y_0 = y_0, \] (1.11)

and \( (W_t^1)_{t \geq 0} \) and \( (W_t^2)_{t \geq 0} \) are independent standard Brownian motions. From a practical point of view, such models are more appealing than the ones based only on a Brownian motion, since, among other reasons, homogeneous Lévy
processes do in general not fare well when considering options across maturities, and are not capable of incorporating mean-reverting or clustered volatility effects, as observed in empirical time series of financial returns. Under the model \((S_t)_{t \geq 0}\), Theorem 4.1 in [12] supplies an expansion for the ATM implied volatility,

\[
\hat{\sigma}(t) = \sigma(y_0) + \tilde{\sigma}_1 t^{\frac{3-Y}{2}} + o(t^{\frac{3-Y}{2}}), \quad t \to 0. \tag{1.12}
\]

where

\[
\tilde{\sigma}_1 := \frac{(C(1) + C(-1))2^{2-Y}}{Y(Y-1)} \Gamma \left( 1 - \frac{Y}{2} \right) \sigma(y_0)^{1-Y}. \tag{1.13}
\]

Next, it is easy to show that \(\mathbb{P}(X_t + V_t \geq 0) \to 1/2\), as \(t \to 0\), but the literature is quite sparse beyond that. However, procedures similar to the ones used in [12] can again be used to find a higher order asymptotic expansion, which offers a significant improvement over existing results. Concretely, we show that

\[
\mathbb{P}(X_t + V_t \geq 0) = \frac{1}{2} + \sum_{k=1}^{n} d_k t^{k(1-\frac{Y}{2})} + e t^{\frac{3-Y}{2}} + f t^{\frac{3-Y}{2}} + o(t^{\frac{3-Y}{2}}), \quad t \to 0, \tag{1.14}
\]

where \(n := \max\{k \geq 3 : k (1 - Y/2) \leq (3 - Y)/2\}\). Comparing this to the expansion for ATM option prices given in Theorem 4.1 of [12], where the first and second order terms were of order \(t^{\frac{1}{2}}\) and \(t^{\frac{3-Y}{2}}\), reveals that the convergence here is slower, as in the pure-jump case, unless \(C(1) = C(-1)\), in which case the summation term vanishes.

Another crucial difference is that the expansion (1.14) is more “sensitive” to various key parameters of the underlying model, since, as it turns out, the correlation coefficient \(\rho\) and the volatility of volatility, \(\sigma'(y_0)\gamma(y_0)\), both appear in the expansion. This is in sharp contrast to the expansions for option prices and implied volatility, where the impact of replacing the Brownian component by a stochastic volatility process was merely to replace the volatility of the Brownian component, \(\sigma\), by the spot volatility, \(\sigma(y_0)\). Finally, piecing together the above results gives an expansion for the ATM implied volatility slope and, as in the pure-jump case, qualitative properties such as its sign and order of convergence can easily be recovered from the model parameters (see Remark 4.4). The limiting behavior turns out to be less explosive than in the pure-jump case and, in the symmetric case when \(C(1) = C(-1)\), it converges to a nonzero value, as \(t \to 0\), as in jump-diffusion models.

As mentioned above, several important features of the model’s implied volatility smile can directly be inferred from its parameters, via the asymptotic expansions obtained herein. However, it is also known that, in the presence of jumps, the domain of validity of such expressions can be small, which could potentially limit their usefulness for practical work. Nevertheless, numerical simulations carried out in Section 5 indicate that for the important class of tempered stable processes, with practically relevant parameter values, the expansions for ATM digital call prices and the implied volatility slope give good approximations for maturities up to a month. Hence, the expansions obtained in the present manuscript are of practical relevance in, for example, FX markets, where there are actively traded options with very short maturities (cf. [1]).

For comparison purposes, let us finish this section by briefly considering the case of nonzero log-strikes \((\kappa \neq 0)\) for exponential Lévy models. For that purpose, let \(S_t := S_0 e^{X_t + \sigma W_t}\), where \(S_0 = 1\), \((X_t)_{t \geq 0}\) is a Lévy process with generating triplet \((0, b, \nu)\), and \((W_t)_{t \geq 0}\) is a standard Brownian motion. Equation (1.1) can then be used to obtain an asymptotic expansion for the implied volatility slope. The following result for the transition probability is well known (see, e.g., [19], Corollary 8.9),

\[
\mathbb{P}(X_t + \sigma W_t \geq \kappa) = tv([\kappa, \infty))1_{\{\kappa > 0\}} + (1 - tv((-\infty, \kappa]))1_{\{\kappa < 0\}} + o(t), \quad t \to 0, \tag{1.15}
\]

and, for the implied volatility, Theorem 2.3 in [9] states that

\[
\hat{\sigma}^2(\kappa, t) = \frac{\kappa^2}{2 \log \frac{1}{t}} \left( 1 + V_1(t, \kappa) + o \left( \frac{1}{\log \frac{1}{t}} \right) \right), \quad t \to 0, \tag{1.16}
\]

under some mild conditions on the Lévy measure, where

\[
V_1(t, \kappa) := \frac{1}{\log \frac{1}{t}} \log \left( \frac{4 \sqrt{\pi} \sigma_0(\kappa) e^{-\kappa/2}}{|\kappa|} \left( \frac{\log \frac{1}{t}}{\frac{1}{t}} \right)^{\frac{3}{2}} \right), \tag{1.17}
\]
and \( q_0(\kappa) := \int_{\mathbb{R}_0} (e^x - e^\kappa + \nu(dx)1_{\kappa > 0} + \int_{\mathbb{R}_0} (e^\kappa - e^x)^+ \nu(dx)1_{\kappa < 0}. \) In particular, (1.16) implies that \( \hat{\sigma}^2(\kappa, t) \rightarrow 0 \) and, furthermore,

\[
\phi \left( -\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t) t \right) = \frac{1}{\sqrt{2\pi}} \exp \left( \frac{\kappa}{2} \left( \frac{1}{\hat{\sigma}(\kappa, t) \sqrt{t}} \right)^2 + \frac{\kappa}{2} - \frac{1}{2} \left( \hat{\sigma}(\kappa, t) \sqrt{t} \right)^2 \right) \sim e^{\frac{\kappa^2}{2}} e^{-\frac{1}{2} \left( \frac{\kappa}{\hat{\sigma}(\kappa, t) \sqrt{t}} \right)^2},
\]
as \( t \rightarrow 0. \) Similarly, using \( \Phi(-x) \sim \phi(x)/x, \) as \( x \rightarrow \infty, \) we have, for \( \kappa > 0, \)

\[
\Phi \left( -\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t) t \right) \sim e^{-\frac{\kappa}{2}} e^{-\frac{1}{2} \left( \frac{\kappa}{\hat{\sigma}(\kappa, t) \sqrt{t}} \right)^2}, \quad t \rightarrow 0,
\]

while, for \( \kappa < 0, \)

\[
1 - \Phi \left( -\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t) t \right) \sim e^{-\frac{\kappa}{2}} e^{-\frac{1}{2} \left( \frac{\kappa}{\hat{\sigma}(\kappa, t) \sqrt{t}} \right)^2}, \quad t \rightarrow 0.
\]

Plugging the above relations into (1.1) then yields the leading order term for the OTM implied volatility slope,

\[
\frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa} \sqrt{2t \log \frac{1}{t}} \sim \frac{\kappa}{|\kappa|}, \quad t \rightarrow 0. \tag{1.18}
\]

Not surprisingly, this shows that the implied volatility slope also blows up away from the money, as maturity becomes small. Similarly, one can obtain a higher order expansion

\[
\frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa} \sqrt{2t \log \frac{1}{t}} = \frac{\kappa}{|\kappa|} \left( 1 + \frac{V_1(t, \kappa)}{2} - \frac{\kappa}{|\kappa|} \frac{1}{2 \log \frac{1}{t}} \left( 1 + \frac{\kappa}{2} - \frac{b_0(\kappa)}{a_0(\kappa)} \right) + o \left( \frac{1}{\log \frac{1}{t}} \right) \right), \quad t \rightarrow 0, \tag{1.19}
\]

where \( b_0(\kappa) = -e^\kappa (\nu([\kappa, \infty))1_{\kappa > 0} + \nu((-\infty, \kappa])1_{\kappa < 0}). \) It is interesting to note that (1.19) can also be elucidated (at least formally) by differentiating the expression (1.16) for the implied volatility. However, when analyzing the ATM slope, that is not an option, which further justifies the approach adopted in the present paper.

The rest of this paper is organized as follows. Section 2 introduces the class of tempered stable-like Lévy processes, while Section 3 contains the results for the transition probability and implied volatility slope in the pure-jump case. Section 4 then defines the continuous component and presents the analogous results in the mixed case. Finally, Section 5 contains numerical examples. Proofs of technical results and lemmas are collected in the appendices.

### 2 Framework and auxiliary lemmas

Let \( X := (X_t)_{t \geq 0} \) denote a pure-jump tempered stable-like Lévy process, as introduced [10] and [12], defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) satisfying the usual conditions. That is, \( X \) is a Lévy process with triplet \((0, b, \nu)\) relative to the truncation function \( 1_{|x| \leq 1} \) (see Section 8 in [19]), where the Lévy measure \( \nu \) is given by (1.4), for some constants \( C(1), C(-1) \in (0, \infty) \) such that \( C(1) + C(-1) > 0, \) \( Y \in (1, 2), \) and a bounded function \( \bar{q} : \mathbb{R} \setminus \{0\} \rightarrow (0, \infty) \) such that \( \bar{q}(x) \rightarrow 1 \) as \( x \rightarrow 0. \) Let us also introduce the following additional technical conditions, conveniently selected to facilitate the proofs of some of the results that follow:

\[
\begin{align*}
(i) \quad & \int_{|x| \leq 1} \left| \frac{\bar{q}(x)}{|x|} - 1 - \alpha \frac{x}{|x|} \right| |x|^{-1} \, dx < \infty; \quad (ii) \quad & \limsup_{|x| \rightarrow \infty} \frac{|\ln \bar{q}(x)|}{|x|} < \infty; \quad (iii) \quad & \inf_{|x| < \varepsilon} \bar{q}(x) > 0, \quad \forall \varepsilon > 0. \tag{2.1}
\end{align*}
\]

Here, \( \alpha(1) \) and \( \alpha(-1) \) are real-valued constants. We emphasize that the main results below only require condition (i) to be satisfied, which controls the behavior of the Lévy density around the origin. In particular, a sufficient condition for all \( 1 < Y < 2 \) is given by \( \bar{q}(x) = 1 + \alpha (x/|x|) x + O(x^2), \) as \( x \rightarrow 0. \)
Next, define the measure transformation \( \mathbb{P} \to \tilde{\mathbb{P}} \), under which \( X \) has Lévy triplet \((0, \tilde{\nu}, \tilde{\nu})\), where \( \tilde{\nu} \) is the \( Y \)-stable measure

\[
\tilde{\nu}(dx) = C \left( \frac{x}{|x|} \right)|x|^{-Y-1}dx,
\]

and \( \tilde{b} \) is given by

\[
\tilde{b} := b + \int_{|x| \leq 1} x(\tilde{\nu} - \nu)(dx) = b + C(1) \int_0^1 (1 - \tilde{q}(x)) x^{-Y}dx - C(-1) \int_{-1}^0 (1 - \tilde{q}(x)) |x|^{-Y}dx.
\]

In particular, the centered process \((Z_t)_{t \geq 0}\), defined by

\[
Z_t := X_t - \tilde{\gamma}t,
\]

with \( \tilde{\gamma} := \mathbb{E}(X_1) \), is a strictly \( Y \)-stable process under \( \tilde{\mathbb{P}} \). Note that

\[
\tilde{\gamma} = b + \frac{C(1) - C(-1)}{Y - 1} + C(1) \int_0^1 x^{-Y}(1 - \tilde{q}(x)) dx - C(-1) \int_{-1}^0 |x|^{-Y}(1 - \tilde{q}(x)) dx.
\]

As is well known, a necessary and sufficient condition on the parameter \( b \) for \( S_t := S_0e^{X_t} \) to be a martingale is given by

\[
b = -\int_{\mathbb{R}} (e^x - 1 - x1_{|x| \leq 1}) \nu(dx).
\]

In that case, \( \tilde{\gamma} \) can also be written as

\[
\tilde{\gamma} = -C(1) \int_0^\infty (e^x \tilde{q}(x) - \tilde{q}(x) - x)x^{-Y-1}dx - C(-1) \int_{-\infty}^0 (e^x \tilde{q}(x) - \tilde{q}(x) - x)|x|^{-Y-1}dx.
\]

By virtue of Theorem 33.1 in [19], a necessary and sufficient condition for the measure transformation \( \mathbb{P} \to \tilde{\mathbb{P}} \) to be well defined is given by

\[
\int_{\mathbb{R}_0} \left( e^{\varphi(x)/2} - 1 \right)^2 \nu(dx) < \infty,
\]

where, hereafter,

\[
\varphi(x) := -\ln \tilde{q}(x).
\]

In what follows it will be useful to write the log-density process \( U_t := \log \frac{d\mathbb{P}_{t,x}}{d\mathbb{P}_{0,x}} \) as

\[
U_t = \tilde{U}_t + \eta t := \int_0^t \int_{\mathbb{R}_0} \varphi(x)\tilde{N}(ds, dx) + t \int_{\mathbb{R}_0} \left( e^{-\varphi(x)} - 1 + \varphi(x) \right) \tilde{\nu}(dx),
\]

which follows from Theorem 33.2 in [19], and is valid provided that

\[
\int_{\mathbb{R}_0} \left| e^{-\varphi(x)} - 1 + \varphi(x) \right| \tilde{\nu}(dx) < \infty.
\]

We shall also make use of the following decomposition

\[
Z_t = \int_0^t \int x\tilde{N}(ds, dx) = \int_0^t \int_{-\infty}^\infty x\tilde{N}(ds, dx) + \int_0^t \int_{-\infty}^0 x\tilde{N}(ds, dx) =: Z^{(p)}_t + Z^{(n)}_t,
\]

where, under \( \tilde{\mathbb{P}} \), \( Z^{(p)}_t \) and \( Z^{(n)}_t \) are strictly \( Y \)-stable random variables with respective Lévy measures

\[
\tilde{\nu}^{(p)}(dx) := C(1)|x|^{-Y-1}1_{\{x > 0\}}dx, \quad \tilde{\nu}^{(n)}(dx) := C(-1)|x|^{-Y-1}1_{\{x < 0\}}dx.
\]
For future reference, $L_Z$ denotes the infinitesimal generator of the process $(Z_t)_{t \geq 0}$, which, for a function $g \in C_b^2$, is given by

$$
(L_Z g)(x) = \int_{[0,x]} (g(u + x) - g(x) - u g'(x)) \frac{u}{|u|} |u|^{-Y-1} du.
$$

(2.11)

We end this section by collecting a few lemmas that will be needed in the sequel. The proof of the second one can be found in [12] (see Lemma A.2 therein), while the proofs of the other two are deferred to Appendix A.

Lemma 2.1. Under (2.1), both (2.6) and (2.8) hold true.

Lemma 2.2. Under (2.1), there exist constants $\tilde{k} < \infty$ and $t_0 > 0$ such that

(i) $\frac{1}{t} \mathbb{P} (|\tilde{U}_t| \geq v) \leq \tilde{k} v^{-Y}$,

(ii) $\frac{1}{t} \mathbb{P} (|Z_t| \geq v) \leq \tilde{k} v^{-Y},

for any $0 < t \leq t_0$ and $v > 0$.

Lemma 2.3. Under (2.1),

(i) $\lim_{t \to 0} \frac{1}{t} \mathbb{P} (Z_t + \gamma t \geq 0, \tilde{U}_t \geq v) = \int_0^{\infty} \mathbf{1}_{\{\varphi(x) \geq v\}} \tilde{\nu}(dx)$

(ii) $\lim_{t \to 0} \frac{1}{t} \mathbb{P} (Z_t + \gamma t \geq 0, \tilde{U}_t \leq -v) = \int_0^{\infty} \mathbf{1}_{\{\varphi(x) \leq -v\}} \tilde{\nu}(dx),

for any $v > 0$.

3 Pure-jump Lévy model

In this section, we study the short-time asymptotic behavior of the ATM implied volatility slope under the exponential Lévy model $S_t := S_0 e^{X_t}$, where $X := (X_t)_{t \geq 0}$ is a pure-jump tempered stable-like process as described in the previous section. As explained in the introduction, the ATM slope is related to the probability of the process $X$ being positive. The following theorem gives an asymptotic expansion, in small time, for such a probability, which sometimes is termed the positivity parameter of a process (cf. [3, p. 218]). The expansion is explicit up to a term of order $O(t^{1/Y})$ (see the subsequent Remark 3.5). Below, $\gamma$ is as in Eq. (2.4), $Z_1^{(p)}$ and $Z_1^{(n)}$ are defined as in Eq. (2.9), and, finally, $f_{Z_1^{(p)}}$, $f_{Z_1^{(n)}}$, and $f_Z$, are the probability density functions of $Z_1^{(p)}$, $Z_1^{(n)}$, and $Z_1 = Z_1^{(p)} + Z_1^{(n)}$, respectively.

Theorem 3.1. Let $X$ be a tempered stable-like Lévy process with a Lévy measure as described in (1.4). Furthermore, assume that the condition

$$
\int_{|x| \leq 1} \left| \tilde{q}(x) - 1 - \alpha \left( \frac{x}{|x|} \right) \right| |x|^{-Y-1} dx < \infty,
$$

(3.1)
is satisfied for some constants $\alpha(1), \alpha(-1) \in \mathbb{R}$. Then,

$$
\mathbb{P} (X_t \geq 0) - \mathbb{P} (Z_1 \geq 0) = \sum_{k=1}^{n} d_k t^{k(1-\frac{1}{Y})} + e t^\frac{1}{Y} + f t + o(t), \quad t \to 0,
$$

(3.2)

where $n := \max\{k \geq 3 : k \left( 1 - \frac{1}{Y} \right) \leq 1 \}$, and

$$
d_k := \frac{(-1)^{k-1}}{k!} \tilde{\gamma}^k f_Z^{(k-1)}(0), \quad 1 \leq k \leq n,
$$

(3.3)

$$
e := \alpha(1) \mathbb{E} \left( Z_1^{(p)} \mathbf{1}_{Z_1^{(p)} + Z_1^{(n)} \geq 0} \right) - \alpha(1) \mathbb{E} \left( Z_1^{(n)} \mathbf{1}_{Z_1^{(p)} + Z_1^{(n)} \geq 0} \right),
$$

(3.4)

$$
f := \tilde{\gamma} (\alpha(1) - \alpha(-1)) \mathbb{E} \left( Z_1^{(p)} f_{Z_1^{(p)}} - Z_1^{(p)} \right)
$$

$$
+ \mathbb{P} (Z_1 \leq 0) C(1) \int_{0}^{\infty} (\tilde{q}(x) - 1 - \alpha(1) x) x^{-Y-1} dx
$$

$$
- \mathbb{P} (Z_1 > 0) C(-1) \int_{-\infty}^{0} (\tilde{q}(x) - 1 - \alpha(-1) x) |x|^{-Y-1} dx.
$$

(3.5)
Remark 3.2. The processes covered by Theorem 3.1 include stable processes, where \( \bar{q}(x) \equiv 1 \), and tempered stable processes as defined in [5], where \( \bar{q}(x) = e^{-\alpha(1)x}1_{\{x>0\}} + e^{\alpha(1)x}1_{\{x<0\}} \), with \( \alpha(1), \alpha(-1) > 0 \). As mentioned in the introduction, they are of particular importance for practical applications, and will be studied numerically in Section 5. It is also important to note that condition (3.1) is the minimal condition needed for the expansion to make sense. That is, if (3.1) does not hold, the coefficient \( f \) is not well defined.

For \( b \) satisfying condition (2.5), \((S_t)_{t \geq 0}\) is a martingale, and the previous result can be translated into an asymptotic expansion for ATM digital call prices. Together with (1.2) and (1.5)-(1.6), it then gives the following asymptotic expansion for the ATM implied volatility slope.

**Corollary 3.3.** Let \( X \) be a tempered stable-like Lévy process as in Theorem 3.1, with \( b \) as in (2.5), so that \( S_t := S_0 e^{X_t} \) is a martingale. Then,

\[
\frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa} \bigg|_{\kappa=0} = \sqrt{\frac{2\pi}{t}} \left( \frac{1}{2} - \hat{\mathbb{P}}(Z_1 \geq 0) - \sum_{k=1}^{n} \frac{d_k k(1-\frac{1}{\gamma})}{t^{k+1}} - \left( e + \frac{\sigma_1}{2} \right) t^{\frac{1}{\gamma}} - \left( f + \frac{\sigma_2}{2} \right) t + o(t) \right), \quad t \to 0, \tag{3.6}
\]

with \( \sigma_1 := \hat{\mathbb{E}}(Z_1^+) \) and \( \sigma_2 \) as in (1.6).

**Remark 3.4.** A few comments are in order:

(a) It is important to point out that the leading order term of (3.2) is \( d_1 t^{1-\frac{1}{\gamma}} \) for all \( Y \in (1, 2) \). Furthermore, the coefficients of (3.2) can be ranked as

\[ d_1 t^{1-\frac{1}{\gamma}} \succ \cdots \succ d_m t^{m(1-\frac{1}{\gamma})} \succ et^{1-\frac{1}{\gamma}} \succ d_{m+1} t^{(m+1)(1-\frac{1}{\gamma})} \succ \cdots \succ d_n t^{n(1-\frac{1}{\gamma})} \geq f t, \quad t \to 0, \]

where \( m = \max\{k \geq 2 : k(1-1/Y) \leq 1/Y \} \) and, as usual, \( f(t) \succ g(t) \) (resp. \( f(t) \succeq g(t) \)) if \( \lim_{t \to 0} |f(t)/g(t)| = \infty \) (resp. \( \liminf_{t \to 0} |f(t)/g(t)| > 0 \)). This can be compared to the expansion for ATM option prices given in Theorem 3.1 of [12], where the first and second order terms are of order \( t^{\frac{1}{\gamma}} \) and \( t \), respectively, i.e. the convergence here is slower.

(b) It is informative to note that the summation term in (3.2) comes from expanding the probability of a stable process with drift being positive. Specifically, we have

\[
\hat{\mathbb{P}}(Z_t + \tilde{\gamma} t \geq 0) - \hat{\mathbb{P}}(Z_1 \geq 0) = \hat{\mathbb{P}}(Z_t + \tilde{\gamma} t^{1-\frac{1}{\gamma}} \geq 0) - \hat{\mathbb{P}}(Z_1 \geq 0) = \sum_{k=1}^{n} \frac{d_k k(1-\frac{1}{\gamma})}{t^{k+1}} + O(\gamma^{n+1}(1-\frac{1}{\gamma})) ,
\]

as \( t \to 0 \). The other terms, \( e \) and \( f \), arise as a result of the discrepancy between \( X \) and a stable process. In particular, this implies that the probability of \( X \) being positive at time \( t \) can be approximated, for small \( t \), by the analogous probability for a stable process, up to an error term of order \( O(t^{1-\frac{1}{\gamma}}) \). Similarly, the same probability can be approximated by that of a tempered stable process (as defined in Remark 3.2), up to an error term of order \( O(t) \).

(c) As mentioned in the introduction, the implied volatility of OTM options explodes as \( t \to 0 \), while for ATM options it converges to the volatility of the continuous component. Here we see that the ATM implied volatility slope also blows up as \( t \to 0 \), with a sign that can easily be recovered from the model parameters. Indeed, when \( Z_1 \) is symmetric (i.e. \( C(1) = C(-1) \)), it is of order \( t^{1-\frac{1}{\gamma}} \), with the same sign as the parameter \( \tilde{\gamma} \), i.e. the center of \( X \) under \( \hat{\mathbb{P}} \), but, when \( C(1) > C(-1) \) (resp. \( C(1) < C(-1) \)), it is of order \( t^{1-\frac{1}{\gamma}} \) with a negative (resp. positive) sign.

**Remark 3.5.** There exist explicit expressions for \( \hat{\mathbb{E}}(Z_1^+) \) and \( \hat{\mathbb{P}}(Z_1 \geq 0) \) (see [10] and references therein):

\[
\hat{\mathbb{E}}(Z_1^+) = \frac{A^+}{\pi} \Gamma(-Y)^+ \left| \cos \left( \frac{\pi Y}{2} \right) \right|^+ \cos \left( \frac{1}{Y} \arctan \left( \frac{B}{A} \tan \left( \frac{Y}{2} \right) \right) \right) \Gamma \left( 1 - \frac{1}{Y} \right) \left( 1 + \left( \frac{B}{A} \right)^2 \tan^2 \left( \frac{\pi Y}{2} \right) \right)^{\frac{1}{2Y}},
\]

\[
\hat{\mathbb{P}}(Z_1 \geq 0) = \frac{1}{2} + \frac{1}{\pi Y} \arctan \left( \frac{B}{A} \tan \left( \frac{Y}{2} \right) \right),
\]

8
where $A := C(1) + C(-1)$ and $B := C(1) - C(-1)$. The derivatives $f_{Z}^{(k-1)}(0)$ can also be explicitly computed from the polynomial expansion for the stable density (see, e.g., Eq. (4.2.9) in [23]). Indeed, it follows that

$$f_{Z}^{(k-1)}(0) = (-1)^{k-1} \frac{\Gamma \left( \frac{k}{2} + 1 \right)}{k \pi} \sin \left( \frac{\rho k \pi}{c} \right),$$

where

$$\rho = \frac{\delta + Y}{2Y}, \quad \delta = \frac{2}{\pi} \arctan \left( \beta \tan \left( \frac{Y \pi}{2} \right) \right), \quad c_0 = \cos \left( \arctan \left( \beta \tan \frac{\pi Y}{2} \right) \right),$$

and $\beta = (C(1) - C(-1))/(C(1) + C(-1))$ and $c = -\Gamma(-Y) \cos \left( \frac{\pi Y}{2} \right) (C(1) + C(-1))$ are the skewness and scale parameters of $Z_1$.

**Proof of Theorem 3.1.**

**Step 1:** Let $X$ be a tempered stable-like process as in the statement of the theorem. In this step, we will show that (3.2) holds under the additional assumptions that the $\tilde{q}$-function of $X$ satisfies (2.1-ii) and (2.1-iii), so that Lemmas 2.1-2.3 are valid. Throughout, we use the notation introduced in the previous section. Let us start by noting that (3.2) holds under the additional assumptions that the $\tilde{q}$-function of $X$ satisfies (2.1-ii) and (2.1-iii), so that Lemmas 2.1-2.3 are valid. Throughout, we use the notation introduced in the previous section. Let us start by noting that

$$\mathbb{P}(X_t \geq 0) - \mathbb{P}(Z_1 \geq 0) = \mathbb{E} \left( e^{-U_1} \mathbf{1}_{\{Z_1 \geq -\tilde{q}t\}} \right) - \mathbb{E} \left( \mathbf{1}_{\{Z_1 \geq 0\}} \right),$$

and we look at each of the two terms separately. For the first one, we have

$$I_1(t) = \mathbb{P}(Z_1 \geq -\tilde{q}t^{1-\frac{1}{4}}) - \mathbb{P}(Z_1 \geq 0) = \int_{-\tilde{q}t^{1-\frac{1}{4}}}^{0} f_Z(z) dz,$$

and, since $f_Z(\cdot)$ is a smooth function (see e.g. [19], Prop. 28.3), we can write, for any $N \in \mathbb{N}$,

$$f_Z(z) = \sum_{n=0}^{N} f_{Z}^{(n)}(0) \frac{z^n}{n!} + h_N(z),$$

where $h_N(z) = o(z^N)$ as $z \to 0$, so we can find $t_N > 0$ such that $|h_N(z)| \leq |z|^N$, for $|z| \leq |\tilde{q}| t_N^{1-\frac{1}{4}}$. Plugging this into (3.8) gives

$$I_1(t) = \sum_{n=0}^{N-1} f_{Z}^{(n)}(0) \frac{z^n}{n!} \int_{-\tilde{q}t^{1-\frac{1}{4}}}^{0} z^n dz + \int_{-\tilde{q}t^{1-\frac{1}{4}}}^{0} h_N(z) dz$$

$$= \frac{1}{N!} \sum_{n=0}^{N} (-1)^{n} \frac{\tilde{q}}{n!} f_{Z}^{(n-1)}(0) t^n (1-\frac{1}{4}) + O \left( t^{(N+1)}(1-\frac{1}{4}) \right), \quad t \to 0. \quad (3.9)$$

Next, we further decompose the second part of (3.7) as follows

$$I_2(t) = \mathbb{E} \left( e^{-U_1} - 1 \right) \mathbf{1}_{\{Z_1 \geq -\tilde{q}t\}}$$

$$= \mathbb{E} \left( e^{-\tilde{q}t} - 1 \right) \mathbf{1}_{\{Z_1 \geq -\tilde{q}t\}} + (e^{-\eta t} - 1) \mathbb{E} \left( e^{-\tilde{q}t} - 1 \right) \mathbf{1}_{\{Z_1 \geq -\tilde{q}t\}} + (e^{-\eta t} - 1) \mathbb{E} \left( \mathbf{1}_{\{Z_1 \geq -\tilde{q}t\}} \right)$$

$$=: I_2^1(t) + I_2^2(t) + I_2^3(t), \quad (3.10)$$

where it is clear that

$$I_2^3(t) = o(t), \quad I_2^3(t) = -\eta \mathbb{P}(Z_1 \geq 0) t + o(t), \quad t \to 0. \quad (3.11)$$
We use Fubini’s theorem on the first term to write
\[
I_2^1(t) = \tilde{E} \left( (e^{-\tilde{U}_t} - 1 + \tilde{U}_t) 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) - \tilde{E} \left( \tilde{U}_t 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) \\
= \int_{-\infty}^{0} (e^{-x} - 1) \tilde{P} \left( Z_t \geq -\tilde{\gamma} t, \tilde{U}_t \leq x \right) dx - \int_{0}^{\infty} (e^{-x} - 1) \tilde{P} \left( Z_t \geq -\tilde{\gamma} t, \tilde{U}_t \geq x \right) dx - \tilde{E} \left( \tilde{U}_t 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) \\
=: J_2^1(t) + J_2^2(t) + J_2^3(t),
\]
and consider each term individually. For the first one, we let \( K > 0 \) and split the integral into two parts,
\[
J_2^1(t) = \left( \int_{-\infty}^{-K} + \int_{-K}^{0} \right) (e^{-x} - 1) \tilde{P} \left( Z_t \geq -\tilde{\gamma} t, \tilde{U}_t \leq x \right) dx,
\]
and, by (2.12) and analogous arguments to those in [12, Eqs. (A.12)-(A.14)], we can apply the dominated convergence theorem to obtain
\[
\lim_{t \to 0} \frac{1}{t} J_2^1(t) = \int_{-\infty}^{0} (e^{-x} - 1) \lim_{t \to 0} \frac{1}{t} \tilde{P} \left( Z_t \geq -\tilde{\gamma} t, \tilde{U}_t \leq x \right) dx = \int_{0}^{\infty} (e^{-x} - 1) \int_{0}^{\infty} 1_{\{\varphi(y) \leq x\}} \tilde{P}(dy) dx =: \vartheta_1,
\]
where the last step follows from (2.14). For the second one, we similarly use (2.12) to apply the dominated convergence theorem,
\[
\lim_{t \to 0} \frac{1}{t} J_2^2(t) = - \int_{0}^{\infty} (e^{-x} - 1) \lim_{t \to 0} \frac{1}{t} \tilde{P} \left( Z_t \geq -\tilde{\gamma} t, \tilde{U}_t \geq x \right) dx = - \int_{0}^{\infty} (e^{-x} - 1) \int_{0}^{\infty} 1_{\{\varphi(y) \geq x\}} \tilde{P}(dy) dx =: \vartheta_2,
\]
where the last step follows from (2.13). Finally, to deal with the third term we decompose \( \tilde{U}_t = \int_0^t \int \varphi(x) N(ds, dx) \) as
\[
\tilde{U}_t = \int_0^t \left( \varphi(x) + \left( \frac{x}{|x|} \right) \right) N(ds, dx) - \int_0^t \left( \frac{x}{|x|} \right) xN(ds, dx) =: \tilde{U}_t^{(1)} - \tilde{U}_t^{(2)},
\]
so that
\[
J_2^3(t) = - \tilde{E} \left( \tilde{U}_t^{(1)} 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) + \tilde{E} \left( \tilde{U}_t^{(2)} 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) =: - J_2^{31}(t) + J_2^{32}(t).
\]
First, for \( J_2^{32}(t) \), note that
\[
\tilde{E} \left( Z_t^{(p)} 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) = \tilde{E} \left( Z_t^{(p)} 1_{\{Z_t \geq 0\}} \right) + \tilde{E} \left( Z_t^{(p)} \left( 1_{\{Z_t^{(p)} + Z_t^{(n)} \geq -\tilde{\gamma} t\}} - 1_{\{Z_t^{(p)} + Z_t^{(n)} \geq 0\}} \right) \right) \\
= t^\gamma \tilde{E} \left( Z_t^{(p)} 1_{\{Z_t \geq 0\}} \right) + t^\gamma \tilde{E} \left( Z_t^{(p)} \int_{-\tilde{\gamma} t - \tilde{p} \gamma}^{Z_t^{(p)}} f_{Z_t^{(n)}}(z) dz \right) \\
= t^\gamma \tilde{E} \left( Z_t^{(p)} 1_{\{Z_t \geq 0\}} \right) + \tilde{E} \left( Z_t^{(p)} f_{Z_t^{(n)}}(-Z_t^{(p)}) \right) + o(t), \quad t \to 0,
\]
since \( \sup_{z \in \mathbb{R}} f_{Z_t^{(n)}}(z) < \infty \). Similarly,
\[
\tilde{E} \left( Z_t^{(n)} 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) = t^\gamma \tilde{E} \left( Z_t^{(n)} 1_{\{Z_t \geq 0\}} \right) + \gamma t \tilde{E} \left( Z_t^{(n)} f_{Z_t^{(p)}}(-Z_t^{(n)}) \right) + o(t), \quad t \to 0.
\]
From (3.17)-(3.18) and the fact that \( \tilde{E} \left( Z_t^{(n)} f_{Z_t^{(p)}}(-Z_t^{(n)}) \right) = - \tilde{E} \left( Z_t^{(p)} f_{Z_t^{(n)}}(-Z_t^{(p)}) \right) \), we get
\[
J_2^{32}(t) = \alpha(1) \tilde{E} \left( Z_t^{(p)} 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) + \alpha(-1) \tilde{E} \left( Z_t^{(n)} 1_{\{Z_t \geq -\tilde{\gamma} t\}} \right) \\
= t^\gamma \left( \alpha(1) \tilde{E} \left( Z_t^{(p)} 1_{\{Z_t \geq 0\}} \right) + \alpha(-1) \tilde{E} \left( Z_t^{(n)} 1_{\{Z_t \geq 0\}} \right) \right) \\
+ \tilde{E} \left( Z_t^{(p)} f_{Z_t^{(n)}}(-Z_t^{(p)}) \right) + o(t), \quad t \to 0.
\]
For $J_{21}^{31}$, we will show that

$$J_{21}^{31}(t) = \mathbb{E} \left( \tilde{U}_t^{(1)} 1_{\{Z_t \geq -\bar{\gamma} t\}} \right) = \theta t + o(t), \quad t \to 0$$  \hfill (3.20)

where

$$\theta := C(1) \mathbb{P} (Z_1 \leq 0) \int_0^\infty (\alpha(1)x - \ln \bar{q}(x)) x^{-\gamma - 1} dx - C(-1) \mathbb{P} (Z_1 \geq 0) \int_{-\infty}^0 (\alpha(-1)x - \ln \bar{q}(x)) |x|^{-\gamma - 1} dx. \quad (3.21)$$

In order to do that, we define $f(x) := \varphi(x) + \alpha \left( \frac{x}{|x|} \right) x$ and, for $\varepsilon > 0$, further decompose $U_{\varepsilon}^{(1)}$ as

$$U_{\varepsilon}^{(1)} = \int_0^t \int f(x)N(ds,dx) = \int_0^t \int_{|x| \leq \varepsilon} f(x)N(ds,dx) + \int_0^t \int_{|x| > \varepsilon} f(x)N(ds,dx) =: U_{\varepsilon}^{(1,1)}(t) + U_{\varepsilon}^{(1,2)}(t), \quad (3.22)$$

and let

$$J_{21}^{31}(t) = \mathbb{E} \left( U_{\varepsilon}^{(1,1)}(t) 1_{\{Z_t \geq -\bar{\gamma} t\}} \right) + \mathbb{E} \left( U_{\varepsilon}^{(1,2)}(t) 1_{\{Z_t \geq -\bar{\gamma} t\}} \right) := \tilde{J}_{1,\varepsilon}(t) + \tilde{J}_{2,\varepsilon}(t). \quad (3.23)$$

For future reference, recall that $\varphi(x) = -\ln \bar{q}(x)$, and

$$\int |f(x)| \tilde{\nu}(dx) \leq \int \left| \alpha \left( \frac{x}{|x|} \right) x + 1 - \bar{q}(x) \right| \tilde{\nu}(dx) + \int |\bar{q}(x) - 1 - \ln \bar{q}(x)| \tilde{\nu}(dx) < \infty, \quad (3.24)$$

in light of (3.1), the boundedness of $\bar{q}$, and the fact that (2.1) implies (2.8) as proved in Lemma 2.1. Now note that $U_{\varepsilon}^{(1,2)}(t)$ is a compound Poisson process with drift; i.e., we can write

$$U_{\varepsilon}^{(1,2)}(t) = \beta^{(\varepsilon)} t + \sum_{i=1}^{N^{(\varepsilon)} \varepsilon} f(\eta^{(\varepsilon)} i),$$

where $\beta^{(\varepsilon)} := -\int_{|x| > \varepsilon} f(x) \tilde{\nu}(dx)$, $(N^{(\varepsilon)} \varepsilon)_{t \geq 0}$ is a counting process with intensity $\lambda^{(\varepsilon)} := \int_{|x| > \varepsilon} \tilde{\nu}(dx)$, and $(\eta^{(\varepsilon)} i)_{i \in \mathbb{N}}$ are i.i.d. random variables with probability measure $\tilde{\nu}(dx) 1_{\{|x| > \varepsilon\}} / \lambda^{(\varepsilon)}$. We can also write

$$Z_t = \int_0^t \int xN(ds,dx) = \int_0^t \int_{|x| \leq \varepsilon} xN(ds,dx) + \sum_{i=1}^{N^{(\varepsilon)} \varepsilon} \xi_i^{(\varepsilon)} - t \int_{|x| > \varepsilon} x \tilde{\nu}(dx) =: \tilde{Z}_t^{(\varepsilon)} + \sum_{i=1}^{N^{(\varepsilon)} \varepsilon} \xi_i^{(\varepsilon)} + c^{(\varepsilon)} t,$$

and, under $\mathbb{P}$, $t \mapsto \tilde{Z}_t^{(\varepsilon)} \xrightarrow{d} Z_t$ as $t \to 0$ (see [18], Proposition 1). Then, by conditioning on $N^{(\varepsilon)} \varepsilon$, we have

$$\tilde{J}_{2,\varepsilon}(t) = e^{-\lambda^{(\varepsilon)} t} \beta^{(\varepsilon)} t \mathbb{P} \left( Z_t \geq -\bar{\gamma} t \mid N^{(\varepsilon)} \varepsilon = 0 \right) + \lambda^{(\varepsilon)} t e^{-\lambda^{(\varepsilon)} t} \mathbb{E} \left( \left( \beta^{(\varepsilon)} t + f(\eta_1^{(\varepsilon)}) \right) 1_{\{Z_t \geq -\bar{\gamma} t\}} \right) 1_{\{N^{(\varepsilon)} \varepsilon = 1\}} + o(t)$$

$$= e^{-\lambda^{(\varepsilon)} t} \beta^{(\varepsilon)} t \mathbb{P} \left( t \mapsto \tilde{Z}_t^{(\varepsilon)} \geq -\bar{\gamma} + c^{(\varepsilon)} t \right)^{1-\frac{1}{\bar{\gamma}}} + \lambda^{(\varepsilon)} t e^{-\lambda^{(\varepsilon)} t} \mathbb{E} \left( f(\eta_1^{(\varepsilon)}) 1_{\{Z_t^{(\varepsilon)} \geq -\bar{\gamma} + c^{(\varepsilon)} t\}} \right) + o(t), \quad t \to 0,$$

and, thus,

$$\tilde{J}_{2,\varepsilon}(t) = \theta^{(\varepsilon)} t + o(t), \quad t \to 0,$$

(3.25)

where

$$\theta^{(\varepsilon)} := \beta^{(\varepsilon)} \mathbb{P} (Z_1 \geq 0) + \lambda^{(\varepsilon)} \mathbb{E} \left( f(\eta_1^{(\varepsilon)}) 1_{\{\xi_1^{(\varepsilon)} > 0\}} \right)$$

$$= -\mathbb{P} (Z_1 \geq 0) \int_{|x| > \varepsilon} f(x) \tilde{\nu}(dx) + \int_{|x| > \varepsilon} f(x) \tilde{\nu}(dx)$$

$$= C(1) \int_{|x| > \varepsilon} (\alpha(1)x - \ln \bar{q}(x)) x^{-\gamma - 1} dx - \mathbb{P} (Z_1 \geq 0) \int_{|x| > \varepsilon} C \left( \frac{x}{|x|} \right) \left( \alpha \left( \frac{x}{|x|} \right) x - \ln \bar{q}(x) \right) |x|^{-\gamma - 1} dx$$

$$= C(1) \mathbb{P} (Z_1 \leq 0) \int_{|x| > \varepsilon} (\alpha(1)x - \ln \bar{q}(x)) x^{-\gamma - 1} dx - C(-1) \mathbb{P} (Z_1 \geq 0) \int_{-\infty}^{-\varepsilon} (\alpha(-1)x - \ln \bar{q}(x)) |x|^{-\gamma - 1} dx. \quad (3.26)$$
Let us also remark that for \( \tilde{\vartheta} \) as in (3.21),
\[
\tilde{\vartheta} - \vartheta^{(c)} = C(1)\bar{P}(Z_1 \leq 0) \int_0^\xi (\alpha(1) x - \ln \tilde{q}(x)) x^{-Y-1} dx - C(-1)\bar{P}(Z_1 \geq 0) \int_{-\varepsilon}^0 (\alpha(1) x - \ln \tilde{q}(x)) |x|^{-Y-1} dx
\rightarrow 0, \quad \text{as } \varepsilon \to 0,
\]
in light of (3.21) and (3.24). For \( \bar{U}_{\varepsilon}^{(1,1)}(t) \) we note that by (3.24) and Theorem 10.15 in [13], we can write
\[
\bar{U}_{\varepsilon}^{(1,1)}(t) = \int_0^t \int_{|x| \leq \varepsilon} f(x) N(ds, dx) - t \int_{|x| \leq \varepsilon} f(x) \tilde{\vartheta}(dx),
\]
and, for each \( t > 0 \),
\[
|\tilde{J}_{1,\varepsilon}(t)| \leq \bar{E}|\bar{U}_{\varepsilon}^{(1,1)}(t)| \leq 2t \int_{|x| \leq \varepsilon} |f(x)| \tilde{\vartheta}(dx) =: K^{(c)} t \to 0, \quad \text{as } \varepsilon \to 0.
\]
Finally, (3.20) follows since, by (3.25) and (3.28),
\[
-K^{(c)} + \vartheta^{(c)} \leq \liminf_{t \to 0} \frac{\tilde{J}_{1,\varepsilon}(t)}{t} + \liminf_{t \to 0} \frac{\tilde{J}_{2,\varepsilon}(t)}{t} \leq \liminf_{t \to 0} \frac{J_{2,\varepsilon}^{(1)}(t)}{t} \leq \limsup_{t \to 0} \frac{J_{1,\varepsilon}(t)}{t} + \limsup_{t \to 0} \frac{J_{2,\varepsilon}(t)}{t} \leq K^{(c)} + \vartheta^{(c)},
\]
and the lower and upper bounds converge to \( \vartheta \) as \( \varepsilon \to 0 \) in view of (3.27) and (3.28). Combining (3.10)-(3.21) now gives the asymptotics of \( I_2(t) \),
\[
I_2(t) = t^+ \left( \alpha(1)\bar{E} \left( Z_1^{(p)} 1_{\{Z_1 \geq 0\}} \right) + \alpha(-1)\bar{E} \left( Z_1^{(n)} 1_{\{Z_1 \geq 0\}} \right) \right) + \tilde{\gamma}_t (\alpha(1) - \alpha(-1)) \bar{E} \left( Z_1^{(p)} f_{Z_1^{(p)}} \left( -Z_1^{(p)} \right) \right) + tC(1)\bar{P}(Z_1 \leq 0) \int_0^\infty (\tilde{q}(x) - 1 - \alpha(1)x) x^{-Y-1} dx \\
- tC(-1)\bar{P}(Z_1 \geq 0) \int_{-\infty}^0 (\tilde{q}(x) - 1 - \alpha(-1)x) |x|^{-Y-1} dx + o(t), \quad t \to 0,
\]
where the last two terms on the right-hand side above come from collecting the terms \( \vartheta_1 + \vartheta_2 - \vartheta - \eta \bar{P}(Z_1 \geq 1) \), using the expressions for \( \eta \) and \( \vartheta \) in (2.7) and (3.21), respectively, and noting that by Fubini’s theorem and simple manipulations we can write
\[
\vartheta_1 + \vartheta_2 = C(1) \int_0^\infty \int_{-\infty}^0 \left( e^{-x} - 1 \right) 1_{\{-\ln \tilde{q}(y) \leq x\}} dx \tilde{q}^{(p)} y^{-Y-1} dy - C(1) \int_0^\infty \int_{0}^\infty \left( e^{-x} - 1 \right) 1_{\{-\ln \tilde{q}(y) \geq x\}} dx \tilde{q}^{(n)} y^{-Y-1} dy \\
= C(1) \int_0^\infty \int_{-\ln \tilde{q}(y) > 0} \left( e^{-x} - 1 \right) dx \tilde{q}^{(p)} y^{-Y-1} dy - C(1) \int_0^\infty \int_{-\ln \tilde{q}(y) > 0} \left( e^{-x} - 1 \right) dx \tilde{q}^{(n)} y^{-Y-1} dy \\
= C(1) \int_0^\infty (\tilde{q}(y) - 1 - \ln \tilde{q}(y)) y^{-Y-1} dy.
\]
Finally, combining (3.7), (3.9), and (3.29), gives (3.2).

**Step 2:**

Now assume that \( X \) is a tempered stable-like process whose \( \tilde{q} \)-function satisfies (3.1), but not necessarily the additional conditions imposed in Step 1: (2.1-ii) and (2.1-iii). We would like to approximate it by a process whose \( \tilde{q} \)-function satisfies those conditions, and for which the result (3.2) is therefore known by Step 1. To do that, first note that since \( \tilde{q}(x) \to 1 \) as \( x \to 0 \), we can find \( \varepsilon_0 > 0 \) such that \( \inf_{|x| \leq \varepsilon_0} \tilde{q}(x) > 0 \). Next, for each \( \delta > 0 \), let \( (\Omega^{(0)}, F^{(0)}, \mathbb{P}^{(0)}) \) be an
extension of the original probability space \((\Omega, \mathcal{F}, \mathbb{P})\), carrying a Lévy process \(R^{(\delta)}\), independent of the original process \(X\), with Lévy triplet \((0, \beta^{(\delta)}, \nu^{(\delta)}_R)\) given by

\[
\nu^{(\delta)}_R(dx) := C \left(\frac{x}{|x|}\right) e^{-|x|/\delta^3} 1_{|x| \geq \varepsilon_0} |x|^{-1} dx, \quad \beta^{(\delta)} := \int_{|x| \leq 1} x \nu^{(\delta)}_R(dx). \tag{3.30}
\]

In particular, \(R\) is a compound Poisson process and can be written as

\[
R_t = \sum_{i=1}^{N_t^{(\delta)}} \xi_i^{(\delta)}, \tag{3.31}
\]

where \((N^{(\delta)}_t)_{t \geq 0}\) is a Poisson process with intensity \(\lambda^{(\delta)} := \int_{|x| \geq \varepsilon_0} \nu^{(\delta)}_R(dx)\), and \((\xi_i^{(\delta)})_{i \in \mathbb{N}}\) are i.i.d. random variables with probability measure \(\nu^{(\delta)}_R(dx)/\lambda^{(\delta)}\). Let us recall that, by the definition of a probability space extension (see [13]), the law of \(X\) under \(\mathbb{P}^{(\delta)}\) remains unchanged. Also, all expected values in the sequel will be taken with respect to the extended probability measure \(\mathbb{P}^{(\delta)}\), so for simplicity we denote the expectation under \(\mathbb{P}^{(\delta)}\) by \(\mathbb{E}\). Next, we approximate the law of the process \(X\) with that of the following process, again defined on the extended probability space \((\Omega^{(\delta)}, \mathcal{F}^{(\delta)}, \mathbb{P}^{(\delta)})\):

\[
X_t^{(\delta)} := X_t + R_t. \tag{3.32}
\]

Then the Lévy triplet \((b^{(\delta)}, \nu^{(\delta)}, \mathbb{P}^{(\delta)})\) of \(X^{(\delta)}\) is given by

\[
b^{(\delta)} := b + \beta^{(\delta)}, \quad \nu^{(\delta)}(dx) := C(x/|x|)|x|^{-1} \tilde{q}^{(\delta)}(x) dx := C(x/|x|)|x|^{-1} \left(\tilde{q}(x) + e^{-|x|/\delta^3} 1_{|x| \geq \varepsilon_0}\right) dx, \tag{3.33}
\]

so it is clear that \(\tilde{q}^{(\delta)}\) satisfies (3.1) and (2.1-iii). To show that \(\tilde{q}^{(\delta)}\) also satisfies (2.1-ii), note that, since \(\tilde{q}\) is bounded, for some \(B \in (0, \infty)\) and \(|x| \geq \varepsilon_0\),

\[
-\frac{1}{\delta} \leq \frac{\ln \tilde{q}^{(\delta)}(x)}{|x|} \leq \frac{B}{|x|},
\]

which clearly implies (2.1-ii). Hence, the probability measure \(\mathbb{P}\) can be defined as described in Section 3, using the jump measure of the process \(X^{(\delta)}\), and note that for \(\tilde{\gamma}^{(\delta)} := \mathbb{E}(X_1^{(\delta)})\), we have, using the expression (2.4),

\[
\tilde{\gamma}^{(\delta)} = b^{(\delta)} + \frac{C(1) - C(-1)}{Y - 1} + C(1) \int_0^1 x^{-Y} \left(1 - \tilde{q}^{(\delta)}(x)\right) dx - C(-1) \int_{-1}^0 |x|^{-Y} \left(1 - \tilde{q}^{(\delta)}(x)\right) dx
\]

\[
= b + \frac{C(1) - C(-1)}{Y - 1} + C(1) \int_0^1 x^{-Y} \left(1 - \tilde{q}(x)\right) dx - C(-1) \int_{-1}^0 |x|^{-Y} \left(1 - \tilde{q}(x)\right) dx = \tilde{\gamma},
\]

after plugging in the above expressions for \(b^{(\delta)}\) and \(\tilde{q}^{(\delta)}\). Now, since \(\tilde{q}^{(\delta)}\) satisfies the conditions in Step 1, we know that (3.2) holds for \(X^{(\delta)}\), i.e. that

\[
\mathbb{P}\left(X_t^{(\delta)} \geq 0\right) - \mathbb{P}(Z_1 \geq 0) = \sum_{k=1}^{n} dx t^{k(1-\frac{1}{Y})} + e t^{\frac{1}{Y} + f^{(\delta)} t + o(t)}, \quad t \rightarrow 0, \tag{3.34}
\]

where \(n := \max\{k \geq 3 : k \left(1 - \frac{1}{Y}\right) \leq 1\}\), and

\[
e := \alpha(1) \mathbb{E}\left(Z_1^{(p)} 1_{Z_1^{(p)} + Z_1^{(n)} \geq 0}\right) + \alpha(-1) \mathbb{E}\left(Z_1^{(n)} 1_{Z_1^{(p)} + Z_1^{(n)} \geq 0}\right), \tag{3.35}
\]

\[
d_k := \frac{(-1)^{k-1}}{k!} \tilde{\gamma}^k f_Z^{(k-1)}(0), \quad k \geq 1, \tag{3.36}
\]

are independent of \(\delta\), while

\[
f^{(\delta)} := \tilde{\gamma} \left(\alpha(1) - \alpha(-1)\right) \mathbb{E}\left(Z_1^{(p)} f_Z^{(n)} \left(-Z_1^{(p)}\right)\right) \tag{3.37}
\]

\[
+ \mathbb{P}(Z_1 \leq 0) C(1) \int_0^\infty \left(\tilde{q}^{(\delta)}(x) - 1 - \alpha(1) x\right) x^{-Y-1} dx
\]

\[
- \mathbb{P}(Z_1 > 0) C(-1) \int_{-\infty}^0 \left(\tilde{q}^{(\delta)}(x) - 1 - \alpha(-1) x\right) x^{-Y-1} dx.
\]
Now, from the triangle inequality, it follows that
\[
P\left(X_t^{(δ)} \geq 0\right) - \left|P\left(X_t \geq 0\right) - P\left(X_t^{(δ)} \geq 0\right)\right| \leq P\left(X_t \geq 0\right) \leq P\left(X_t^{(δ)} \geq 0\right) + \left|P\left(X_t \geq 0\right) - P\left(X_t^{(δ)} \geq 0\right)\right|,
\] (3.38)
and, by conditioning on the number of jumps of the process \(R\), we have
\[
R_t^{(δ)} := \left|P\left(X_t \geq 0\right) - P\left(X_t^{(δ)} \geq 0\right)\right| = \lambda^{(δ)} t e^{\lambda^{(δ)} t} \left|P\left(X_t \geq 0\right) - P\left(X_t + \xi_t^{(δ)} \geq 0\right)\right| + o(t), \quad t \to 0,
\] (3.39)
which, in particular, implies that
\[
\lim_{\delta \to 0} \lim_{t \to 0} t^{-1} R_t^{(δ)} = 0,
\] (3.40)
since \(\lim_{\delta \to 0} \lambda^{(δ)} = \lim_{\delta \to 0} \int_{|x| \geq \nu_0} e^{-|x|/\delta} \nu(dx) = 0\). By subtracting \(\tilde{e}(Z_1 \geq 0) + \sum_{k=1}^n d_k t^{k(1 - \delta)} + e t^{\delta} \) from the inequalities in (3.38), applying the expansion (3.34), dividing by \(t\), taking the limit as \(t \to 0\), and using (3.40), it is clear that
\[
\lim_{\delta \to 0} \frac{1}{t} \left(P\left(X_t \geq 0\right) - \tilde{e}(Z_1 \geq 0) - \sum_{k=1}^n d_k t^{k(1 - \delta)} - e t^{\delta}\right) = \lim_{\delta \to 0} f^{(δ)}.
\]
Therefore, to conclude, it suffices to show that \(\lim_{\delta \to 0} f^{(δ)} = f\), with \(f\) as in (3.5), which follows from (3.37) and the dominated convergence theorem.

4 Lévy jump model with stochastic volatility

In this section we consider the case when an independent continuous component is added to the pure-jump Lévy process \(X\). Concretely, let \(S_t := S_{0e^{X_t + V_t}}\), with \(X\) as in the previous section, while for the continuous component, \(V\), we consider an independent stochastic volatility process of the form
\[
dV_t = \mu(Y_t) dt + \sigma(Y_t) \left(\rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t\right), \quad V_0 = 0,
\] (4.1)
\[
dY_t = \alpha(Y_t) dt + \gamma(Y_t) dW^1_t, \quad Y_0 = y_0,
\] (4.2)
defined on the same probability space as \(X\). Here, \((W^1_t)_{t \geq 0}\) and \((W^2_t)_{t \geq 0}\) are standard Brownian motions, relative to the filtration \((\mathcal{F}_t)_{t \geq 0}\). \(-1 < \rho < 1\), \(\alpha(\cdot), \gamma(\cdot), \mu(\cdot), \text{ and } \sigma(\cdot)\), are such that \(V\) and \(Y\) are well defined. Moreover, it is assumed that \(\sigma_0 := \sigma(y_0) > 0\), and that there exists a bounded open interval \(I\), containing \(y_0\), on which \(\alpha(\cdot)\) is bounded, \(\gamma(\cdot)\) and \(\mu(\cdot)\) are Lipschitz continuous, and \(\sigma(\cdot)\) is a \(C^2\) function. In the sequel, \(\phi_{\tilde{e}}(\cdot) \) (resp. \(\phi(\cdot)\)) denotes the probability density function of a \(\mathcal{N}(0, \delta^2)\) (resp. \(\mathcal{N}(0, 1)\)) random variable, while
\[
\Psi(z) := \int_0^z \phi(x) dx, \quad z \in \mathbb{R}.
\] (4.3)
Let us also recall that \(L_Z\), defined in (2.11), denotes the infinitesimal generator of the strictly stable process \((Z_t)_{t \geq 0}\).

The next theorem gives an asymptotic expansion for the probability of a tempered stable-like process being positive, in the presence of a continuous component satisfying the previously described conditions.

Theorem 4.1. Let \(X\) be a tempered stable-like process as in Theorem 3.1 and \(V\) a diffusion process as described above. Then,
\[
P\left(X_t + V_t \geq 0\right) = \frac{1}{2} + \sum_{k=1}^n d_k t^{k \left(1 - \frac{Y}{2}\right)} + e t^{\frac{Y}{2}} + f t^{\frac{3Y - 1}{2}} + o(t^{\frac{3Y - 1}{2}}), \quad t \to 0,
\] (4.4)
where \(n := \max\{k \geq 3 : k \left(1 - \frac{Y}{2}\right) \leq \frac{3Y - 1}{2}\}\), and
\[
d_k := \frac{\sigma(y_0)^{-kY}}{k!} L^k_Z \Psi(0), \quad 1 \leq k \leq n,
\] (4.5)
\[
e := \left(\tilde{\gamma} + \mu(y_0) - \frac{\rho}{2} \sigma'(y_0) \gamma(y_0)\right) \phi_{\tilde{e}}(0),
\] (4.6)
\[
f := \left(\frac{\alpha(1)C(1) + \alpha(-1)C(-1)}{Y - 1} - \frac{C(1) + C(-1)}{\sigma^2(y_0)Y} \left(\tilde{\gamma} + \mu(y_0) - \frac{\rho}{2} \sigma'(y_0) \gamma(y_0) (1 + Y)\right)\right) \xi,
\] (4.7)
where $\xi := \int_0^\infty \phi_{\sigma_0}(x)x^{1-Y} \, dx$.

**Remark 4.2.** A few observations are in order:

(a) Using the notation of Remark 3.4, the terms can be ordered with regard to their rate of convergence as

$$d_1 t^{1-\frac{Y}{2}} \geq \cdots \geq d_m t^{m(1-\frac{Y}{2})} \geq \varepsilon t^\frac{Y}{2} \geq \cdots \geq d_n t^{n(1-\frac{Y}{2})} \geq f t^\frac{3-Y}{2}, \quad t \to 0,$$

where $m := \max\{k : k(1-Y/2) \leq 1/2\}$. A comparison of this to the expansion for ATM option prices given in Theorem 4.2 of [12], where the first and second order terms were of order $t^{\frac{Y}{2}}$ and $t^{\frac{3-Y}{2}}$, reveals that the convergence here is slower, as in the pure-jump case, unless $C(1) = C(-1)$ (see (4.8) below).

(b) As stated in the proof, there is another useful characterization of the $d_k$-coefficients in terms of a short-time expansion for a certain functional depending on (4.3). Concretely, the coefficients $d_1, \ldots, d_n$ are such that

$$\mathbb{E}(\Psi(Z_t)) = \mathbb{P}(Z_t + \sigma(y_0)W_t^1 \geq 0) = \frac{1}{2} + \sum_{k=1}^n d_k t^k + O(t^{n+1}), \quad t \to 0.$$

When $Z_t$ is symmetric (i.e., when $C(1) = C(-1)$), it follows that $\mathbb{E}(\Psi(Z_t)) = 0$, and all the $d_k$'s vanish. In that case, the expansion simplifies to

$$\mathbb{P}(X_t + V_t \geq 0) = \frac{1}{2} + \varepsilon t^\frac{Y}{2} + f t^\frac{3-Y}{2} + o(t^\frac{3-Y}{2}), \quad t \to 0. \tag{4.8}$$

(c) Interestingly enough, the correlation coefficient $\rho$ appears in the expansion. Moreover, so does $\sigma'(y_0)\gamma(y_0)$, i.e. the volatility of volatility. That is in sharp contrast to the expansions for option prices and implied volatility, given in Theorem 4.2 of [12], where the impact of replacing the Brownian component by a stochastic volatility process was merely to replace the volatility of the Brownian component, $\sigma$, by the spot volatility, $\sigma(y_0)$.

(d) Similarly, the leading order term here depends on the jump-component via $L_Z$, and the parameters $\tilde{\gamma}, \alpha(1)$, and $\alpha(-1)$, also appear, containing information on the tempering function $\tilde{q}$, i.e. the Lévy density away from the origin. That is again in contrast to what was observed in Theorem 4.2 of [12], where the leading order term only incorporated information on the spot volatility, $\sigma(y_0)$, and the approximation was altogether independent of the $\tilde{q}$-function.

(e) Tempered stable-like processes are a natural extension of stable Lévy processes. In the pure-jump case, the “deviation” of $X$ from a stable process does not appear in terms of order lower than $t^\frac{Y}{2}$ (see Remark 3.4-b). Here, recalling that $X_t$ has the stable representation $X_{t}^{stbl} := Z_t + \tilde{\gamma}t$ under $\mathbb{P}$, Theorem 4.1 implies that,

$$\mathbb{P}(X_t + V_t \geq 0) - \mathbb{P}(X_t^{stbl} + V_t \geq 0) = \frac{\alpha(1)C(1) + \alpha(-1)C(-1)}{Y-1} \int_0^\infty \phi_{\sigma_0}(x)x^{1-Y} \, dx \, t^{\frac{3-Y}{2}} + o(t^{\frac{3-Y}{2}}), \quad t \to 0.$$

In other words, for small $t$, one can explicitly approximate the positivity probability of $X_t + V_t$ by that of $X_t^{stbl} + V_t$, up to a term of order higher than $t^{(3-Y)/2}$.

(f) One can find a more explicit expression for the constant $f$ by noting that

$$\xi = \int_0^\infty \phi_{\sigma_0}(x)x^{1-Y} \, dx = \frac{(\sigma(y_0))^{1-Y}}{\sqrt{\pi}} \frac{\sqrt{\Gamma (1 - \frac{Y}{2})}}{(\sigma(y_0))^{1-Y}} \frac{(\sigma(y_0))^{1-Y} - 2 - \frac{\gamma}{Y}}{\sqrt{Y(Y-1)}} \frac{(3 - Y)}{2},$$

using the well-known moment formula for centered Gaussian random variables. Moreover, in Appendix A we show that the first two coefficients in (4.5) are given by

$$d_1 := \frac{C(1) - C(-1)}{(\sigma(y_0))^2 Y} \int_0^\infty (\phi(x) - \phi(0)) \, x^{-Y} \, dx = -(C(1) - C(-1)) \frac{(\sigma(y_0))^{1-Y} - 2 - \frac{\gamma}{Y}}{\sqrt{\Gamma (1 - \frac{Y}{2})}} \frac{(3 - Y)}{2} \frac{\sqrt{Y(Y-1)} \Gamma (1 - \frac{Y}{2})}{\sqrt{\pi}}, \tag{4.9}$$

$$d_2 := -\frac{1}{2} \frac{C^2(1) - C^2(-1)}{(\sigma(y_0))^2 Y^2} \int_0^\infty \int_0^\infty ((x + y)\phi(x + y) - x\phi(x) - y\phi(y))(xy)^{-Y} \, dx \, dy. \tag{4.10}$$
In the case when $e^{X_t+V_t}$ is a well defined $\mathbb{F}$-martingale, the previous result can be viewed as an asymptotic expansion for ATM digital call prices. Consequently, it can be combined with (1.2) and (1.12)-(1.13) to obtain an asymptotic expansion for the ATM implied volatility slope.

**Corollary 4.3.** Let $X$ and $V$ be as in Theorem 4.1, with $b$ as in (2.5), $\mu(\cdot) = -\frac{1}{2}\sigma(\cdot)$, and (4.1)-(4.2) such that $e^{V_t}$ is a true martingale. Then,

$$
- \frac{\partial \hat{\sigma}(k,t)}{\partial k} \mid_{k=0} = \sqrt{2\pi} \sum_{\nu=1}^{n} d_k t (1-\frac{y}{2})^{k-\frac{1}{2}} + \frac{c'}{\sigma_0} + \left( \sqrt{2\pi} f + \frac{1}{2} \sigma_1 \right) t^{1-\frac{y}{2}} + o(t^{1-\frac{y}{2}}), \quad t \to 0,
$$

(4.11)

$c' := \hat{\gamma} - \rho \sigma'(y_0) \gamma(y_0) \left( 1 - \frac{y^2}{2} \right)$, and, if $C(1) = C(-1)$, the expansion becomes

$$
- \frac{\partial \hat{\sigma}(k,t)}{\partial k} \mid_{k=0} = \frac{c'}{\sigma_0} + \left( \sqrt{2\pi} f + \frac{1}{2} \sigma_1 \right) t^{\frac{1}{2}} + o(t^{\frac{1}{2}}), \quad t \to 0.
$$

(4.12)

**Remark 4.4.** The previous result shows that the order of convergence and the sign of the ATM implied volatility slope can easily be recovered from the model parameters. In the asymmetric case, i.e. when $C(1) \neq C(-1)$, it blows up like $t^{\frac{1}{2}}$, and has the same sign as $C(1) - C(-1)$. However, when $C(1) = C(-1)$, the summation term in (4.11) vanishes, and the slope converges to a nonzero value, $-c'/\sigma_0$, as $t \to 0$, as in jump-diffusion models. In both cases it is observed that the slope behavior is less explosive than in the pure-jump case.

**Proof of Theorem 4.1.**

**Step 1:** We first show that (4.4) is true when $\mu(\cdot)$ and $\sigma(\cdot)$ are assumed to be bounded, by considering the stopped processes

$$
\tilde{\mu}_t := \mu(Y_{t\wedge \tau}), \quad \tilde{\sigma}_t := \sigma(Y_{t\wedge \tau}), \quad \tau := \inf \{ t : Y_t \notin I \},
$$

(4.13)

where $I$ is a bounded open interval containing $y_0$, such that $\sigma(\cdot)$ is $C^2$, $\mu(\cdot)$ and $\gamma(\cdot)$ are Lipschitz, and $\alpha(\cdot)$ is bounded on $I$. Note that due to the continuity of $\mu(\cdot)$ and $\sigma(\cdot)$ around $y_0$, we can find constants $0 < m < M < \infty$ such that $|\tilde{\mu}_t| < M$ and $m < \tilde{\sigma}_t < M$. Throughout the proof, we set $\sigma_0 = \sigma(y_0)$, $\mu_0 = \mu(y_0)$, $\alpha_0 = \alpha(y_0)$, $\gamma_0 = \gamma(y_0)$, and $\bar{\rho} := \sqrt{1 - \rho^2}$.

As in the pure-jump case, we also start by assuming that the $\tilde{q}$-function of $X$ satisfies (2.1-ii) and (2.1-iii). Then, the idea is to reduce the problem to the case where $\tilde{\mu}_t$ and $\tilde{\sigma}_t$ are deterministic, by conditioning the positivity probability on the realization of the process $(W^1_t)_{0 \leq t \leq 1}$. To do that we follow similar steps as in the proof of Theorem 4.2 in [12]. On a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ satisfying the usual conditions, we define independent processes $\tilde{X}$ and $\tilde{W}^2$, such that the law of $(\tilde{X}_t)_{0 \leq t \leq 1}$ under $\tilde{\mathbb{P}}$ is the same as the law of $(X_t)_{0 \leq t \leq 1}$ under $\mathbb{P}$, and $(\tilde{W}^2_t)_{0 \leq t \leq 1}$ is a standard Brownian motion. Also, for any deterministic functions $\tilde{\mu} := (\tilde{\mu}_s)_{s \in [0,1]}$, $\tilde{\sigma} := (\tilde{\sigma}_s)_{s \in [0,1]}$, and $\tilde{\bar{q}} := (\tilde{\bar{q}}_s)_{s \in [0,1]}$, belonging to $C([0,1])$, the set of continuous function on $(0,1)$, we define the process $(\tilde{V}^\tilde{\mu, \tilde{\sigma}, \tilde{\bar{q}}}_t)_{0 \leq t \leq 1}$ as follows:

$$
\tilde{V}^\tilde{\mu, \tilde{\sigma}, \tilde{\bar{q}}}_t := \int_0^t \tilde{\mu}_u du + \rho \tilde{\bar{q}}_u + \tilde{\rho} \int_0^t \tilde{\sigma}_u d\tilde{W}^2_u, \quad 0 \leq t \leq 1.
$$

(4.14)

With this notation at hand, we consider a functional $\Phi : [0,1] \times C([0,1]) \times C([0,1]) \times C([0,1]) \to [0,1]$, defined as

$$
\Phi(t, \tilde{\mu}, \tilde{\sigma}, \tilde{\bar{q}}) := \tilde{\mathbb{P}} \left( \tilde{X}_t + \tilde{V}^\tilde{\mu, \tilde{\sigma}, \tilde{\bar{q}}}_t \geq 0 \right).
$$

(4.15)

Then, for any $t \in [0,1]$,

$$
\mathbb{P} \left( X_t + V_t \geq 0 \middle| W^1_s, s \in [0,1] \right) = \Phi \left( t, (\tilde{\mu}_s)_{s \in [0,1]}, (\tilde{\sigma}_s)_{s \in [0,1]}, (\tilde{\bar{q}}_s)_{s \in [0,1]} \right),
$$

(4.16)

where

$$
\tilde{\bar{q}}_s := \int_0^s \tilde{\sigma}_u d\tilde{W}^2_u.
$$

(4.17)
For simplicity, we omit the superscripts in the process $\hat{V}^{\bar{\mu},\bar{\sigma},\hat{q}}$, unless explicitly needed. Throughout the proof, we assume that $\bar{\mu}$ and $\bar{\sigma}$ satisfy the same uniform boundedness conditions as $\hat{\mu}$ and $\hat{\sigma}$; namely, $m < \hat{\sigma} < M$ and $|\hat{\mu}| < M$ for any $t \in [0, 1]$.

As in the pure-jump case, we define the probability measure $\hat{\mathbb{P}}'$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ as described in Section 2, but replacing the jump measure $N$ of the process $X$ by the jump measure of $\hat{X}$. We also define the strictly stable process $\hat{Z}_t := \hat{X}_t - \hat{\gamma}t$, where $\hat{\gamma} := \hat{\mathbb{E}}'(\hat{X}_1)$, and $\hat{\mathbb{E}}'$ denotes the expectation with respect to the probability measure $\hat{\mathbb{P}}'$. Note that the law of $(\hat{V}_t)_{t \leq 1}$ under $\hat{\mathbb{P}}'$ remains unchanged and, under both $\hat{\mathbb{P}}$ and $\hat{\mathbb{P}}'$,

$$
t^{-\frac{1}{2}}\hat{V}_t \sim \mathcal{N}\left(t^{\frac{1}{2}}\hat{\mu}^*_t + t^{-\frac{1}{2}}\rho\hat{q}_t, \rho^2(\hat{\sigma}^*_t)^2\right)
$$

(4.18)

where, for $t \in (0, 1]$,

$$
\hat{\mu}^*_t := \frac{1}{t} \int_0^t \hat{\mu}_s ds \in [-M, M], \quad \hat{\sigma}^*_t := \sqrt{\frac{1}{t} \int_0^t \hat{\sigma}_s^2 ds} \in [m, M].
$$

(4.19)

Now, note that

$$
\mathbb{P}(X_t + V_t \geq 0) = \mathbb{P}\left(t^{-\frac{1}{2}}X_t + t^{-\frac{1}{2}}V_t \geq 0\right) \xrightarrow[t \to 0]{} \frac{1}{2},
$$

by Slutsky’s theorem, and the facts that $t^{-\frac{1}{2}}V_t \overset{\mathcal{D}}{=} \Lambda \sim \mathcal{N}(0, \sigma^2_0)$ (cf. [12], Eq. (4.47)), and $t^{-\frac{1}{2}}X_t \overset{\mathcal{D}}{=} Z_1$, where $Z_1$ is a strictly $Y$-stable random variable (cf. [18], Prop. 1). Next, in order to find higher order terms in the expansion, we investigate the limit of the process

$$
R_t := \mathbb{P}(X_t + V_t \geq 0) - \frac{1}{2},
$$

(4.20)

as $t \to 0$. In terms of the functional $\Phi$, $R_t$ can be expressed as

$$
R_t = \mathbb{E}\left(\mathbb{P}\left(X_t + V_t \geq 0|W^s_s, s \in [0, 1]\right) - \frac{1}{2}\right)
= \mathbb{E}\left(\Phi\left(t, (\bar{\mu}_s)_{s \in [0, 1]}, (\bar{\sigma}_s)_{s \in [0, 1]}, (\bar{q}_s)_{s \in [0, 1]}\right) - \frac{1}{2}\right)
=: \mathbb{E}\left(\bar{\Phi}\left(t, (\bar{\mu}_s)_{s \in [0, 1]}, (\bar{\sigma}_s)_{s \in [0, 1]}, (\bar{q}_s)_{s \in [0, 1]}\right)\right).
$$

(4.21)

Therefore,

$$
\bar{\Phi}(t, \bar{\mu}, \bar{\sigma}, \hat{q}) = \bar{\mathbb{E}}'\left(e^{-U_t}1_{\{t^{-\frac{1}{2}}\hat{V}_t \geq -t^{-\frac{1}{2}}\hat{Z}_t - \hat{\gamma}t\frac{1}{4}\}} - 1_{\{\hat{W}_t \geq 0\}}\right)
= \bar{\mathbb{E}}'\left(1_{\{t^{-\frac{1}{2}}\hat{V}_t \geq -t^{-\frac{1}{2}}\hat{Z}_t - \hat{\gamma}t\frac{1}{4}\}} - 1_{\{\hat{W}_t \geq 0\}}\right) + \bar{\mathbb{E}}'\left((e^{-U_t} - 1)1_{\{t^{-\frac{1}{2}}\hat{V}_t \geq -t^{-\frac{1}{2}}\hat{Z}_t - \hat{\gamma}t\frac{1}{4}\}}\right)
=: I_1(t, \bar{\mu}, \bar{\sigma}, \hat{q}) + I_2(t, \bar{\mu}, \bar{\sigma}, \hat{q}),
$$

(4.22)

and we analyze the two terms separately. For the first one, we use (4.18) to show that

$$
I_1(t, \bar{\mu}, \bar{\sigma}, \hat{q}) = \bar{\mathbb{E}}'\int_{-t^{-\frac{1}{2}}\hat{Z}_t - \hat{\gamma}t\frac{1}{4}}^{0} \phi_{\hat{\rho}\hat{\sigma}^*_t}(x) dx
= \bar{\mathbb{E}}'\left(\int_{-t^{-\frac{1}{2}}\hat{Z}_t - \hat{\gamma}t\frac{1}{4}}^{0} \phi_{\hat{\rho}\hat{\sigma}^*_t}(x) dx\right) + \bar{\mathbb{E}}'\left(\int_{-t^{-\frac{1}{2}}\hat{Z}_t - \hat{\gamma}t\frac{1}{4}}^{0} \phi_{\hat{\rho}\hat{\sigma}^*_t}(x) dx\right)
=: I^1_1(t, \bar{\mu}, \bar{\sigma}, \hat{q}) + I^2_1(t, \bar{\sigma}, \hat{q}) + I^3_1(t, \bar{\sigma}, \hat{q}) + I^4_1(t, \bar{\sigma}, \hat{q}),
$$

(4.23)
where we recall that $\phi_\sigma(\cdot)$ denotes the PDF of a $\mathcal{N}(0, \sigma^2)$ random variable. For the first term of (4.23), we have

$$
\mathbb{E} \left| I_1^t(t, \bar{\mu}, \bar{\sigma}, \bar{q}) \right| \leq t^{\frac{1}{2}} \phi_{\bar{\sigma} \rho m}(0) \mathbb{E} \left| \bar{\mu}_t - \mu_0 \right| \leq t^{\frac{1}{2}} \phi_{\bar{\sigma} \rho m}(0) \frac{1}{t} \int_0^t \mathbb{E} \left| \bar{\mu}_s - \mu_0 \right| ds = O(t), \quad t \to 0,
$$

(4.24)

where the last step follows from Lemma B.1-(i) below. Similarly, for the second part we use Lemma B.1-(ii), and the easily verifiable fact $\sup_{x \in \mathbb{R}} \left| \phi_{\bar{\sigma}_1 \rho \sigma_1}(x) - \phi_{\sigma_0}(x) \right| = \left| \phi_{\bar{\sigma}_1}(0) - \phi_{\sigma_0}(0) \right|$, to obtain

$$
\mathbb{E} \left| I_2^t(t, \bar{\sigma}, \bar{q}) \right| \leq t^{\frac{1}{2}} |\bar{\gamma} + \mu_0| \mathbb{E} \left| \phi_{\bar{\sigma}_1}(0) - \phi_{\sigma_0}(0) \right| \leq \frac{t^{\frac{1}{2}} |\bar{\gamma} + \mu_0|}{m^2 \bar{\rho} \sqrt{2\pi}} \mathbb{E} \left| \bar{\sigma}_t - \sigma_0 \right| = O(t), \quad t \to 0.
$$

(4.25)

We now consider the third part of (4.23), $I_3^t(t, \bar{q})$. To this end, assume $\bar{\gamma}^\prime := \bar{\gamma} + \mu_0 > 0$ (the analysis when $\bar{\gamma}^\prime < 0$ is identical), and write

$$
t^{-\frac{1}{2}} I_3^t(t, \bar{q}) - \bar{\gamma}^\prime \phi_{\sigma_0}(0) = t^{-\frac{1}{2}} \bar{E}^t \left( \int_{-t^{-\frac{1}{2}} \bar{Z}_2}^{-t^{-\frac{1}{2}} \bar{Z}_1} \phi_{\bar{\sigma} \rho \sigma}(x - t^{-\frac{1}{2}} \rho \sigma_0 \bar{w}^1) - \phi_{\bar{\sigma} \rho \sigma}(x - t^{-\frac{1}{2}} \rho \sigma_0 \bar{u}^1) \right) dx
\right) + t^{-\frac{1}{2}} \bar{E}^t \left( \int_{-t^{-\frac{1}{2}} \bar{Z}_1}^{t^{-\frac{1}{2}} \bar{Z}_1} \phi_{\bar{\sigma} \rho \sigma}(x - t^{-\frac{1}{2}} \rho \sigma_0 \bar{w}^1) - \phi_{\bar{\sigma} \rho \sigma}(x - t^{-\frac{1}{2}} \rho \sigma_0 \bar{u}^1) \right) dx
\right) + \bar{\gamma}^\prime \left( \phi_{\bar{\sigma} \rho \sigma}(x - t^{-\frac{1}{2}} \rho \sigma_0 \bar{w}^1) - \phi_{\sigma_0}(0) \right)
=: I_{3,1}^t(t, \bar{w}^1) + I_{3,2}^t(t, \bar{q}, \bar{w}^1) + I_{3,3}^{1/2}(t, \bar{w}^1),
$$

(4.26)

where $\bar{w}^1 \in \mathbb{R}$. First, for $I_{3,1}^t(t, \bar{w}^1)$, use Fubini’s theorem to write

$$
I_{3,1}^t(t, \bar{w}^1) = t^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 t^{-\frac{1}{2}} \bar{w}) - \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 t^{-\frac{1}{2}} \bar{w}^1) \right) J_t(x) dx,
$$

where $J_t(x) := \bar{E}^t \left( - t^{\frac{1}{2}} - \bar{\gamma} t^{1/2} x \right) \leq \bar{Z}_1 \leq \bar{t}^{-\frac{1}{2}} t^{1/2} x$. In Appendix A we show that there exists a constant $\tilde{\kappa}$ such that

$$
J_t(x) \leq \tilde{\kappa} t^{-\frac{3}{2}} |x|^{-Y-1},
$$

(4.27)

for all $x \neq 0$ and $t < 1$. Moreover, since $f_Z(x) \sim C \left( \frac{x}{|x|} \right) |x|^{-Y-1}$, as $|x| \to \infty$ (cf. [19], 14.37),

$$
J_t(x) = \bar{\gamma} t^{1/2} \int_{-\infty}^{x} f_Z \left( t^{-\frac{1}{2}} + u \right) du \sim \bar{\gamma} C \left( \frac{x}{|x|} \right) |x|^{-Y-1} t^{-\frac{3}{2}}, \quad t \to 0.
$$

(4.28)

On the other hand, it is easy to see that

$$
\mathbb{E} \left( \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) - \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) \right) = \phi_{\sigma_0}(x) - \phi_{\sigma_0}(0) = O(x^2), \quad x \to 0,
$$

so, in light of the above relations, the dominated convergence theorem can be applied to $\mathbb{E}(I_{3,1}^t(t, W^1_{1}))$, and

$$
\lim_{t \to 0} t^{-\frac{3}{2}} \mathbb{E} \left( I_{3,1}^t(t, W^1_{1}) \right) = \int_{-\infty}^{\infty} \mathbb{E} \left( \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) - \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) \right) \bar{\gamma} C \left( \frac{x}{|x|} \right) |x|^{-Y-1} dx
\right) \int_{-\infty}^{\infty} \mathbb{E} \left( \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) - \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) \right) x^{-Y-1} dx
\right) = \bar{\gamma} (C(1) + C(-1)) \int_{0}^{\infty} \mathbb{E} \left( \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) - \phi_{\bar{\sigma} \rho \sigma}(x - \rho \sigma_0 W^1_1) \right) x^{-Y-1} dx
\right),
$$

(4.29)

For the second part of (4.26), we can find a constant $\tilde{\kappa}$ such that

$$
\left| \phi_{\sigma_0}(x - t^{-\frac{1}{2}} \rho \bar{q}^1) - \phi_{\sigma_0}(x - t^{-\frac{1}{2}} \rho \sigma_0 \bar{w})^1 \right| \leq \tilde{\kappa} t^{-\frac{1}{2}} |\bar{q}^1 - t^{-\frac{1}{2}} \rho \sigma_0 \bar{w}|,
$$

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for all \( x \in \mathbb{R} \), and \( \mathbb{E}[t^{-\frac{1}{2}}\tilde{q}_t - \sigma_0t^{-\frac{1}{2}}W_t^1] = O(t^{\frac{3}{2}}) \) as \( t \to 0 \), by Lemma B.1-(v), so
\[
\mathbb{E}\left(t^{3/2}(t, \tilde{q}, W_t^1)\right) = O(t^{\frac{3}{2}}), \quad t \to 0. \tag{4.30}
\]
Finally, \( \mathbb{E}(I_1^{3,3}(t, W_t^1)) = 0 \) since
\[
\mathbb{E}\left(\phi_{\rho_0}(-\rho_0W_t^1)\right) = \frac{1}{\rho_0} \int_{\mathbb{R}} \phi(x) \phi\left(\frac{\rho x}{\rho_0}\right) dx = -\frac{1}{\sqrt{2\pi}\sigma_0} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_0^2}} dx = \phi_{\sigma_0}(0). \tag{4.31}
\]
Combining (4.26) and (4.29)-(4.31), gives
\[
\mathbb{E}\left(I_1^3(t, \tilde{q})\right) = \zeta' \phi_{\sigma_0}(0) t^{\frac{3}{2}} + \zeta' (C(1) + C(-1)) \int_0^{\infty} (\phi_{\sigma_0}(x) - \phi_{\sigma_0}(0)) x^{-1} dx t^{\frac{3}{2}} + o(t^{\frac{3}{2}}), \quad t \to 0. \tag{4.32}
\]
For the fourth term of (4.23), let us start by further decomposing it as follows:
\[
I_1^4(t, \tilde{q}, \tilde{q}) = \tilde{E}'\left(\int_{-t^{-\frac{1}{2}}-\frac{1}{2}t}^{t^{-\frac{1}{2}}} \phi_{\rho_0}(x) dx\right) + \tilde{E}'\left(\int_{t^{-\frac{1}{2}}+\frac{1}{2}t}^{-t^{-\frac{1}{2}}-\frac{1}{2}} \phi_{\rho_0}(x) dx\right)
\]
\[
= \tilde{E}'\left(\int_{-t^{-\frac{1}{2}}-\frac{1}{2}t}^{t^{-\frac{1}{2}}} \phi_{\rho_0}(x) dx\right) + \tilde{E}'\left(\int_{t^{-\frac{1}{2}}+\frac{1}{2}t}^{-t^{-\frac{1}{2}}-\frac{1}{2}} \phi_{\rho_0}(x) dx\right) =: J_1(t, \tilde{q}) + J_2(t). \tag{4.33}
\]
Next, using that \( \tilde{E}'(\tilde{Z}_t) = 0 \) and applying Fubini’s theorem,
\[
J_1(t, \tilde{q}) = \tilde{E}'\left(\int_{t^{-\frac{1}{2}}+\frac{1}{2}t}^{-t^{-\frac{1}{2}}-\frac{1}{2}} \phi_{\rho_0}(x) dx\right)
\]
\[
= \int_{B_0} (\phi_{\rho_0}(x) - \phi_{\rho_0}(0)) \left(\tilde{E}'\left(t^{\frac{3}{2}} - \frac{1}{2} t^{-\frac{1}{2}} x \leq \tilde{Z}_t \leq t^{\frac{3}{2}} + \frac{1}{2} t^{-\frac{1}{2}} x\right) - \tilde{E}'\left(t^{\frac{3}{2}} - \frac{1}{2} t^{-\frac{1}{2}} x \leq \tilde{Z}_t \leq t^{\frac{3}{2}} + \frac{1}{2} t^{-\frac{1}{2}} x\right)\right) dx
\]
\[
= \int_{B_0} (\phi_{\rho_0}(x) - \phi_{\rho_0}(0)) J_1(t, \tilde{x}) dx. \tag{4.34}
\]
Similar steps as in (4.27) can be used to show that, for any \( 0 < t < 1 \) and \( x \neq 0 \),
\[
|J_1(t, x, \tilde{x})| \leq \tilde{\kappa} t^{\frac{1}{2}} \left|\frac{\sigma_{\tilde{x}}}{\sigma_0} - 1\right| |x|^{-\frac{1}{2}}, \tag{4.35}
\]
where \( \tilde{\kappa} \) is a constant (the details can be found in Appendix A). Additionally, \( \mathbb{E}\left|\frac{\sigma_{\tilde{x}}}{\sigma_0} - 1\right| = O(t^{\frac{1}{2}}) \), as \( t \to 0 \), by Lemma B.1-(iii). Therefore, by the dominated convergence theorem,
\[
\lim_{t \to 0} t^{-\frac{3}{2}} \mathbb{E}\left(J_1(t, \tilde{q})\right) = \lim_{t \to 0} t^{-\frac{3}{2}} \int_{B_0} (\phi_{\rho_0}(0) - \phi_{\rho_0}(x)) \mathbb{E}\left(J_1(t, x, \tilde{x})\right) dx
\]
\[
= \int_{B_0} (\phi_{\rho_0}(0) - \phi_{\rho_0}(x)) t^{-\frac{3}{2}} \mathbb{E}\left(J_1(t, x, \tilde{x})\right) dx = 0, \tag{4.36}
\]
since, as verified in Appendix A,
\[
\mathbb{E}\left(J_1(t, x, \tilde{x})\right) = o(t^{\frac{3}{2}}), \quad t \to 0. \tag{4.37}
\]
For the second and third part of (4.33), first note that by Itô’s formula,
\[
\tilde{q}_t = \int_0^t \tilde{q}_s dW_s = \sigma_0 W_t^1 + \int_0^t \int_0^s \sigma'_u \gamma_u dW_s^1 dW_t^1 + \int_0^t \int_0^s \left(\sigma''_u \alpha_u + \frac{1}{2} \sigma''_u \gamma_u^2\right) dudW_s^1 =: \sigma_0 W_t^1 + \xi_t^1 + \xi_t^2, \tag{4.38}
\]
where
\[ \sigma'_u = \sigma'(Y_u)1_{(u < \tau)}, \quad \sigma''_u = \sigma''(Y_u)1_{(u < \tau)}, \quad \alpha_u := \alpha(Y_u \wedge \tau), \quad \gamma_u := \gamma(Y_u \wedge \tau). \]  
(4.38)

Also, define
\[ \xi_{t_1}^{1,0} := \int_0^t \int_0^s \sigma'_0 \gamma_0 dW_u^1 dW_s^1 = \frac{1}{2} \sigma'_0 \gamma_0 (W_t^1)^2, \]
and, for reals \( \bar{w}^1 \) and \( \bar{\xi} \), let
\[
J_2(t) + J_3(t, \bar{\sigma}, \bar{\xi}) = \mathbb{E}' \left( \int_{t-\frac{1}{2} \bar{\xi}}^{t-\frac{1}{2} \bar{\sigma}} \phi(\rho(\sigma_0 \bar{w} + \xi)) d\xi \right) + \mathbb{E}^0 \left( \int_{t-\frac{1}{2} \bar{\sigma}}^{t-\frac{1}{2} \bar{\sigma}} \phi(\rho(\sigma_0 \bar{w} + \xi)) d\xi \right)
\]
\[
+ \mathbb{E}' \left( \int_{t-\frac{1}{2} \bar{\sigma}}^{t-\frac{1}{2} \bar{\sigma}} \phi(\rho(\sigma_0 \bar{w} + \xi)) d\xi \right) + \mathbb{E}^0 \left( \int_{t-\frac{1}{2} \bar{\sigma}}^{t-\frac{1}{2} \bar{\sigma}} \phi(\rho(\sigma_0 \bar{w} + \xi)) d\xi \right)
\]
\[
= J_1(t, \bar{\sigma}, \bar{\xi}, \bar{w}^1, \bar{\xi}) + J_2(t, \bar{\sigma}, \bar{w}^1, \bar{\xi}) + J_3(t, \bar{\sigma}, \bar{w}^1) + J_4(t, \bar{\sigma}, \bar{w}^1).
\]
(4.40)

For the first term, we have
\[
\mathbb{E} \left( J_1 \left( t, \bar{\sigma}, \bar{\xi}, \bar{w}^1, \xi_{t_1}^{1,0} \right) \right) \leq \phi_{\rho \sigma_0}(0) \rho t \mathbb{E} \left( \xi_{t_1}^{1,0} \right) = O(t), \quad t \to 0,
\]
(4.41)
by Lemma B.1-(v). For the second term, Cauchy’s inequality, (4.39), and Lemma B.1-(iii), can be used to show that
\[
\mathbb{E} \left( \xi_{t_1}^{1,0} \right) = O(t^{\frac{3}{2}}), \quad t \to 0,
\]
and, therefore, using again that \( \sup_{x \in \mathbb{R}} |\phi(\rho(\sigma_0 x)) - \phi(\rho(\sigma_0 x))| = O(t) \),
\[
\mathbb{E} \left( J_2 \left( t, \bar{\sigma}, W_1^1, \xi_{t_1}^{1,0} \right) \right) = \mathbb{E} \left( \left. \mathbb{E}' \left( \int_{t-\frac{1}{2} \bar{\sigma}}^{t-\frac{1}{2} \bar{\sigma}} \phi(\rho(\sigma_0 \bar{w} + \xi)) d\xi \right) \right|_{\bar{w} = W_1^1} \right) + O(t), \quad t \to 0,
\]
where above we also used that \( (\xi_{t_1}^{1,0}, W_1^1) \mathbb{D} (\frac{1}{2} \sigma_0^2 \gamma_0 t ((W_1^1)^2 - 1), t^{1/2} W_1^1) \). Thus, by the dominated convergence theorem,
\[
\lim_{t \to 0} t^{-\frac{1}{2}} \mathbb{E} \left( J_2 \left( t, \bar{\sigma}, W_1^1, \xi_{t_1}^{1,0} \right) \right) = \frac{\rho}{2} \sigma_0^2 \gamma_0 \mathbb{E} \left( (W_1^1)^2 - 1 \right) \phi(\rho(\sigma_0 W_1^1)) = -\frac{\rho^3}{2} \phi(0) \sigma_0^2 \gamma_0,
\]
(4.42)
where the last step follows from elementary calculations as in (4.31). To find the second order term of \( J^2 \), define
\[
J_4(t, \bar{w}) := t^{-\frac{1}{2}} \mathbb{E}' \left( \int_{t-\frac{1}{2} \bar{\sigma}}^{t-\frac{1}{2} \bar{\sigma}} \phi(\rho(\sigma_0 \bar{w} + \xi)) d\xi \right) - \frac{\rho}{2} \sigma_0^2 \gamma_0 (\bar{w}^2 - 1) \phi(\rho(\sigma_0 \bar{w}))
\]
\[
= t^{-\frac{1}{2}} \mathbb{E}' \left( \int_{t-\frac{1}{2} \bar{\sigma}}^{t-\frac{1}{2} \bar{\sigma}} \phi(\rho(\sigma_0 \bar{w} + \xi)) d\xi \right) - \frac{\rho}{2} \sigma_0^2 \gamma_0 (\bar{w}^2 - 1) \phi(\rho(\sigma_0 \bar{w}))
\]
\[
= t^{-\frac{1}{2}} \int_{-\infty}^{\infty} (\phi(\rho(\sigma_0 x + \rho(\sigma_0 \bar{w}))) - \phi(\rho(\sigma_0 x + \rho(\sigma_0 \bar{w}))) J_4(x, \bar{w}) d\bar{w},
\]
where
\[
J_4(t, \bar{w}) := \mathbb{E}' \left( t^{-\frac{1}{2}} \bar{\sigma} \leq x \leq t^{-\frac{1}{2}} \bar{\sigma} + \frac{t}{2} \sigma_0^2 \gamma_0 (\bar{w}^2 - 1) \right) - \mathbb{E}' \left( t^{-\frac{1}{2}} \bar{\sigma} \leq x \leq t^{-\frac{1}{2}} \bar{\sigma} + \frac{t}{2} \sigma_0^2 \gamma_0 (\bar{w}^2 - 1) \right)
\]
\[
= \mathbb{E}' \left( t^{-\frac{1}{2}} \bar{\sigma} \leq x \leq t^{-\frac{1}{2}} \bar{\sigma} + \frac{t}{2} \sigma_0^2 \gamma_0 (\bar{w}^2 - 1) \right) - \mathbb{E}' \left( t^{-\frac{1}{2}} \bar{\sigma} \leq x \leq t^{-\frac{1}{2}} \bar{\sigma} + \frac{t}{2} \sigma_0^2 \gamma_0 (\bar{w}^2 - 1) \right)
\]
\[
= \int_{t^{-\frac{1}{2}} \bar{\sigma}}^{t^{-\frac{1}{2}} \bar{\sigma} + \frac{t}{2} \sigma_0^2 \gamma_0 (\bar{w}^2 - 1)} f(z) dz,
\]
20
for which we can establish a bound similar to the one in (4.27). Indeed, we can find a constant $\tilde{\kappa}$ such that

$$|J_t(x, \tilde{w})| \leq \tilde{\kappa} |\tilde{w}|^3 + (1 + |\tilde{w}|)^{-1/2} t^{1/2} |x|^{-Y-1},$$

(4.43)

for all $x \neq 0$ and $t \leq 1$ (see Appendix A). We will now apply the dominated convergence theorem twice to write

$$\lim_{t \to 0} t^{-\frac{3Y}{2}} \mathbb{E} \left[ J^2 \left( t, W_t^1 \right) \right] = \int_{-\infty}^{\infty} \lim_{t \to 0} t^{-\frac{3Y}{2}} \mathbb{E} \left[ \left( \phi_{\sigma_0, \rho}(x + \rho \sigma_0 W_t^1) - \phi_{\sigma_0, \rho} (\rho \sigma_0 W_t^1) \right) J_t(x, W_t^1) \right] dx$$

$$= \int_{-\infty}^{\infty} \mathbb{E} \left[ \left( \phi_{\sigma_0, \rho}(x + \rho \sigma_0 W_t^1) - \phi_{\sigma_0, \rho} (\rho \sigma_0 W_t^1) \right) \right] dt^{-\frac{3Y}{2}} J_t(x, W_t^1) dx$$

$$= \frac{\rho^3}{2} \sigma^3_0 \gamma_0 \int_{-\infty}^{\infty} \mathbb{E} \left[ \left( \phi_{\sigma_0, \rho}(x + \rho \sigma_0 W_t^1) - \phi_{\sigma_0, \rho} (\rho \sigma_0 W_t^1) \right) \left( |W_t^1|^2 - 1 \right) \right] C \left( \frac{x}{|x|} \right) |x|^{-Y-1} dx$$

(4.44)

where the last two equalities follow from the same steps as in (4.28), and standard calculations. The second application of the dominated convergence theorem above follows from (4.43), and the boundedness of $\phi_{\rho \sigma_0}$. The first application of it can also be justified using (4.43) for $|x| \geq 1$, but, for $|x| \leq 1$, we use Taylor’s theorem to switch the order of limit and integration. More precisely, we can write

$$\phi_{\rho \sigma_0}(x + \rho \sigma_0 \tilde{w}) = \phi_{\rho \sigma_0}(\rho \sigma_0 \tilde{w}) + x \phi'_{\rho \sigma_0}(x + \rho \sigma_0 \tilde{w}) \big|_{x=0} + \frac{1}{2} x^2 \phi''_{\rho \sigma_0}(x + \rho \sigma_0 \tilde{w}) \big|_{x=\xi_x} \tag{4.45}$$

where $0 \leq |\xi_x| \leq |x|$, so

$$
\mathbb{E} \left( \int_{-1}^{1} \left( \phi_{\rho \sigma_0}(x + \rho \sigma_0 W_t^1) - \phi_{\rho \sigma_0}(\rho \sigma_0 W_t^1) \right) J_t(x, W_t^1) \right) dx = \int_{-1}^{1} \mathbb{E} \left( \frac{1}{2} x^2 \phi''_{\rho \sigma_0}(x + \rho \sigma_0 \tilde{w}) \big|_{x=\xi_x} \right) J_t(x, W_t^1) dx,
$$

because $\phi'_{\rho \sigma_0}(x + \rho \sigma_0 \tilde{w}) \big|_{x=0} = -\frac{\rho^3}{\rho \sigma_0} \phi_{\rho \sigma_0}(\rho \sigma_0 \tilde{w})$, and

$$
\mathbb{E} \left( W_t^1 \phi_{\rho \sigma_0}(\rho \sigma_0 W_t^1) J_t(x, W_t^1) \right) = \int_{\mathbb{R}} \phi(w) w \phi_{\rho \sigma_0}(\rho \sigma_0 w) J_t(x, w) dw = 0,
$$

since the integrand is an odd function. Then, (4.43), and the fact that $\phi''(\cdot)$ is a bounded function, allows us to apply the dominated convergence theorem,

$$\lim_{t \to 0} t^{-\frac{3Y}{2}} \int_{-1}^{1} \mathbb{E} \left( \frac{1}{2} x^2 \phi''_{\rho \sigma_0}(x + \rho \sigma_0 \tilde{w}) \big|_{x=\xi_x} \right) J_t(x, W_t^1) dx = \int_{-1}^{1} \lim_{t \to 0} t^{-\frac{3Y}{2}} \mathbb{E} \left( \frac{1}{2} x^2 \phi''_{\rho \sigma_0}(x + \rho \sigma_0 \tilde{w}) \big|_{x=\xi_x} \right) J_t(x, W_t^1) dx$$

$$= \int_{-1}^{1} \lim_{t \to 0} t^{-\frac{3Y}{2}} \mathbb{E} \left( \left( \phi_{\rho \sigma_0}(x + \rho \sigma_0 W_t^1) - \phi_{\rho \sigma_0}(\rho \sigma_0 W_t^1) \right) J_t(x, W_t^1) \right) dx,$$

where we have again used (4.45). Finally, from (4.42) and (4.44) we get

$$\frac{2}{\rho^3 \sigma^3_0 \gamma_0} \mathbb{E} \left[ J^2 \left( t, \tilde{\sigma}, W_t^1, \xi_t^{1,0} \right) \right] = -\phi_{\sigma_0}(0) t^2 - \left( C(1) + C(-1) \right) \int_0^{\infty} \left( \phi_{\sigma_0}(x) - \phi_{\sigma_0}(0) \right) x^{-Y-1} dx t^{\frac{3Y}{2}}$$

$$+ \left( C(1) + C(-1) \right) \frac{1}{\sigma_0^3} \int_0^{\infty} \phi_{\sigma_0}(x) x^{-Y+1} dx t^{\frac{3Y}{2}} + o(t^{\frac{3Y}{2}}), \quad t \to 0. \tag{4.46}$$

Next, let $\tilde{P}$ denote the probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ defined in Section 2 and let $Z := (Z_t)_{t \geq 0}$ be the process defined in (2.9). Note that by the independence of $Z$ and $W_t^1$, and the fact that the law of $\tilde{Z}$ under $\tilde{P}$ is the same as
that of $Z$ under $\bar{P}$, the third term in (4.40) can be written as

$$
\mathbb{E} \left( J^3 (t, W^1_t) \right) = \mathbb{E} \left( \int_{-t^{-\frac{1}{2}} \bar{Z}_t}^{\infty} \phi_{\bar{P}_0} (x) dx \right) - \frac{1}{2}
$$

$$
= \mathbb{E} \left( \int_{-t^{-\frac{1}{2}} \bar{Z}_t}^{\infty} \int_{-\infty}^{\infty} \phi_{\bar{P}_0} (y - u) \phi_{\bar{P}_0} (u) du dy \right) - \frac{1}{2}
$$

$$
= \mathbb{E} \left( \int_0^t \phi_{\sigma_0} (y) dy \right)
$$

$$
= \mathbb{E} \left( Z(\sigma_0^t)^{-\frac{1}{2}} \phi (y) dy \right),
$$

where the second equality follows from a change of variables, $y = x + \rho \sigma_0 W^1_t$, while for the last equality we used the self-similarity relationship $s^{\frac{1}{2}} \bar{Z}_t \overset{\mathbb{P}}{=} Z_{st}$. Therefore, it is sufficient to find the asymptotic behavior, as $t \to 0$, of $\mathbb{E}(\Psi(\bar{Z}_t))$, with $\Psi$ as in (4.3). But, since $\Psi(z)$ has continuous and bounded derivatives of all orders, an iterated Dynkin-type formula (see [11], Eq. (1.6)) can be applied to obtain

$$
\mathbb{E} (\Psi(\bar{Z}_t)) = \Psi(0) + \sum_{k=1}^{n} \frac{\sigma_0^{-k} L^k \Psi(0)}{k!} t^k + \frac{n+1}{n!} \int_0^t (1 - \alpha)^n \mathbb{E} (L^{n+1} \Psi(\bar{Z}_{st})) \alpha dt,
$$

for any $n \in \mathbb{N}$, where $L_Z$ is the infinitesimal generator of the strictly stable process $Z$, defined in (2.11). Therefore,

$$
\mathbb{E} \left( J^3 (t, W^1_t) \right) = \sum_{k=1}^{n} \frac{\sigma_0^{-k} L^k \Psi(0)}{k!} t^k + O\left( t^{\frac{n+1}{2}} \right), \quad t \to 0.
$$

Finally, for the fourth part of (4.40), use Taylor’s theorem to write

$$
\phi_{\bar{P}_0^*}(x) = \frac{\partial \phi_{\bar{P}_0} (x)}{\partial \bar{P}_0} \left( \bar{P}_0^* - \bar{P}_0 \right) + h_x(\bar{P}_0^*) (\bar{P}_0^* - \bar{P}_0)^2,
$$

where $h_x(\bar{P}_0^*) \to 0$, as $\bar{P}_0^* \to \bar{P}_0$, and the boundedness of $\bar{P}_0^*$ away from 0 and $\infty$ allows us to find a constant $K$ such that $0 \leq |h_x(\bar{P}_0^*)| < K$, for all $t \leq 1$ and all $x \in \mathbb{R}$. In light of the latter condition and by Lemma B.1-(iii), it then follows that

$$
\mathbb{E} \left( J^4 (t, \bar{P}_0^*, W^1_t) \right) = -\mathbb{E} \left( (\bar{P}_0^*)^2 - \bar{P}_0 \right) \mathbb{E} \left( \left( \int_{-t^{-\frac{1}{2}} \bar{Z}_t}^{\infty} \phi_{\bar{P}_0} (x) dx \right) \left( \frac{1 - x^2}{\bar{P}_0^2 \sigma_0^2} \right) \right) + O(t)
$$

$$
= -\mathbb{E} \left( (\bar{P}_0^*)^2 - \bar{P}_0 \right) \mathbb{E} \left( \left( \int_{-t^{-\frac{1}{2}} \bar{Z}_t}^{\infty} \phi_{\bar{P}_0} (x) dx \right) \left( \frac{1 - x^2}{2 \sigma_0^2} \right) \right) + O(t), \quad t \to 0,
$$

where, for the second step above, we used the fact that

$$
\mathbb{E} \left( (\bar{P}_0^*)^2 - \bar{P}_0 \right) \left( \frac{1}{\bar{P}_0^* + \bar{P}_0} - \frac{1}{2 \sigma_0} \right) = \frac{1}{2 \sigma_0} \mathbb{E} (\bar{P}_0^* - \bar{P}_0)^2 = O(t), \quad t \to 0,
$$

again by Lemma B.1-(iii). Next note that, by Itô’s Lemma and the notation in (4.38),

$$
(\bar{P}_0^*)^2 - \bar{P}_0^2 = \frac{1}{t} \int_0^t (\bar{P}_0^* - \bar{P}_0)^2 ds = \frac{1}{t} \int_0^t \left( \int_0^s \left( 2 \bar{P}_0 \bar{P}_0^* \sigma_u + \frac{1}{2} (\bar{P}_0^*)^2 + \bar{P}_0 \bar{P}_0'' \sigma_u^2 \right) du + \int_0^s 2 \bar{P}_0 \bar{P}_0^* \gamma_u dW_u^1 \right) ds,
$$

and, hence,

$$
\mathbb{E} \left( (\bar{P}_0^*)^2 - \bar{P}_0^2 \right) \leq \frac{1}{t} \int_0^t \int_0^s \mathbb{E} \left( 2 \bar{P}_0 \bar{P}_0^* \sigma_u + \frac{1}{2} (\bar{P}_0^*)^2 + \bar{P}_0 \bar{P}_0'' \sigma_u^2 \right) \gamma_u^2 |duds = O(t), \quad t \to 0.
$$
To handle the last expression, let us first note that since the integrand in the last integral is bounded. Using that, and Lemma B.1-(vi), we get

\[
E \left( J^4(t, \tilde{\sigma}, W_1^1) \right) = -\frac{\sigma_0^2}{\sigma_0 \gamma_0} E \left( \frac{1}{t^3/2} \int_{t^{-3/2}}^t W_1^1 ds \int_{t^{-3/2}}^t \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) dx \right)
\]

To handle the last expression, let us first note that

\[
\int_{0}^{t} W_1^1 ds \bigg| W_1^1 \sim N \left( \frac{1}{2} W_1^1, \frac{t^3}{12} \right).
\]

Therefore, again using the probability measure \( \tilde{\mathbb{P}} \) and the process \( Z := (Z_t)_{t \geq 0} \), as in (4.47), we can write

\[
-\frac{t^{-3/2}}{\sigma_0^2} E \left( J^4(t, \tilde{\sigma}, W_1^1) \right) = \tilde{E} \left( \frac{1}{t^3/2} \int_{0}^{t} W_1^1 ds \int_{t^{-3/2}}^t \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) dx \bigg| W_1^1 \right)
\]

\[
= \tilde{E} \left( \frac{1}{2} t^{-3/2} W_1^1 \int_{t^{-3/2}}^t \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) dx \right)
\]

\[
= \frac{1}{2} E \left( W_1^1 \int_{Z(t^{-3/2})+\tilde{\rho}\sigma_0 W_1^1} \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) dx \right)
\]

\[
= \frac{1}{2} \tilde{E} \left( \tilde{\Psi} \left( Z \left( t^{-3/2} \right) \right) \right),
\]

where we have used the self-similarity relationships \( s \tilde{\rho} W_1^1 \overset{\mathcal{D}}{=} W_{s t}^1 \) and \( s \tilde{\rho} Z_i \overset{\mathcal{D}}{=} Z_{s t} \), and the notation

\[
\tilde{\Psi}(z) := \tilde{E} \left( W_1^1 \int_{z+\tilde{\rho}\sigma_0 W_1^1} \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) dx \right)
\]

Furthermore, we have

\[
\tilde{\Psi}(z) = \int_{z}^{\infty} \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) E \left( W_1^1 1_{\{\rho_0 \sigma_0 W_1^1 \geq z-x\}} \right) dx - \int_{-\infty}^{z} \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) E \left( W_1^1 1_{\{\rho_0 \sigma_0 W_1^1 \leq z-x\}} \right) dx
\]

\[
= \frac{\rho}{|\rho|} \int_{z}^{\infty} \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) \phi \left( \frac{x-z}{\rho \sigma_0} \right) dx + \frac{\rho}{|\rho|} \int_{-\infty}^{z} \phi_{\tilde{\rho}\sigma_0}(x) \left( 1 - \frac{x^2}{\tilde{\rho}^2 \sigma_0^2} \right) \phi \left( \frac{x-z}{\rho \sigma_0} \right) dx
\]

\[
= \frac{1}{\sigma_0} \phi_{\sigma_0}(z) (\sigma_0^2 - z^2) \tilde{\rho}^2 \rho.
\]

Hence, \( \tilde{\Psi}(z) \) has continuous and bounded derivatives of all orders, so we proceed as in (4.48) and obtain

\[
-2 \frac{\sigma_0}{\sigma_0 \gamma_0} E \left( J^4(t, \tilde{\mu}, \tilde{\sigma}, \tilde{\eta}, W_1^1) \right) = \tilde{\Psi}(0) t^{\frac{3}{2}} + L_Z \tilde{\Psi}(0) t^{\frac{3}{2}} + o(t^{\frac{3}{2}}), \quad t \to 0,
\]

(4.49)

where clearly \( \tilde{\Psi}(0) = \rho \tilde{\rho}^2 \phi(0) \), and, using (2.11), we have

\[
L_Z \tilde{\Psi}(0) = \int_{\mathbb{R}_0} \left( \tilde{\Psi}(u) - \tilde{\Psi}(0) - \tilde{\Psi}'(0) u \right) C \left( \frac{u}{|u|} \right) |u|^{-1} du
\]

\[
= \rho \tilde{\rho}^2 \int_{\mathbb{R}_0} \left( \frac{1}{\sigma_0} \phi_{\sigma_0}(u)(\sigma_0^2 - u^2) - \sigma_0 \phi_{\sigma_0}(0) \right) C \left( \frac{u}{|u|} \right) |u|^{-1} du
\]

\[
= (C(1) + C(-1)) \rho \tilde{\rho}^2 \int_{0}^{\infty} \left( \frac{1}{\sigma_0} \phi_{\sigma_0}(u)(\sigma_0^2 - u^2) - \sigma_0 \phi_{\sigma_0}(0) \right) u^{-1} du.
\]
Thus, combining (4.33), (4.35), (4.40)-(4.41), and (4.46)-(4.49), gives an asymptotic expansion for $\mathbb{E} \left( I_1^2(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) \right)$, which, together with (4.23)-(4.25) and (4.32), finally gives

$$
\mathbb{E} \left( I_1^2(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) \right) = \sum_{k=1}^{n} \frac{\sigma_0^{-kY} L_k^Y \Psi(0)}{k!} t^{k(1 - \frac{\gamma'}{2})} + \left( \gamma' - \frac{\rho}{2} \sigma_0^2 \gamma_0 \right) \phi_{\sigma_0}(0) t^{\frac{\gamma'}{2}}$

$$
- \frac{C(1) + C(-1)}{\sigma_0^Y} \int_{0}^{\infty} \phi_{\sigma_0}(x) x^{-Y+1} dx \left( \frac{2}{Y} \right) + o(t^{\frac{2}{Y}}),
$$

(4.50)

as $t \to 0$, where $n := \max \{ k \geq 3 : k(1 - Y/2) \leq (3 - Y)/2 \}$, and we have used integration by parts to write

$$
\int_{0}^{\infty} \phi_{\sigma_0}(x) x^{-Y+1} dx = - \frac{1}{\sigma_0^Y} \int_{0}^{\infty} \phi_{\sigma_0}(x) x^{-Y+1} dx.
$$

Now consider the second part of (4.42). We have

$$
I_2(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) = \mathbb{E}' \left( (e^{-\bar{U}_t} - 1) \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + (e^{-\bar{U}_t} - 1) \mathbb{E}' \left( (e^{-\bar{U}_t} - 1) \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + R(t, \bar{\sigma}, \bar{\mu}, \bar{\eta})
$$

$$
= - \mathbb{E}' \left( (e^{-\bar{U}_t} - 1) \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + \mathbb{E}' \left( (e^{-\bar{U}_t} - 1) \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + R(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}), \quad t \to 0,
$$

(4.51)

where clearly $\mathbb{E} \left( R(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) \right) = O(t)$, as $t \to 0$, while $\mathbb{E} \left( R(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) \right) = O(t)$ as well, because

$$
0 \leq \mathbb{E} \left( R(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) \right) \leq \mathbb{E}' \left( (e^{-\bar{U}_t} - 1) \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + \mathbb{E}' \left( (e^{-\bar{U}_t} - 1) \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + O(t)
$$

as $t \to 0$, where the last step follows from the treatment of the first two terms in (3.35). Next, recall the decompositions (2.9) and (3.15), where $\bar{U}_t^{(1)}$ was a finite variation process, so

$$
I_2(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) = \mathbb{E}' \left( (\bar{U}_t^{(2)})^2 \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + O(t)
$$

$$
= \alpha(1) \mathbb{E}' \left( (\bar{Z}_t^{(p)})^2 \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + \alpha(-1) \mathbb{E}' \left( (\bar{Z}_t^{(n)})^2 \mathbb{I}_{\{t > \frac{1}{2} Z_i \geq t^{-\frac{1}{2}} Z_i \}} \right) + O(t)
$$

(4.52)

and we look at the two terms separately. For the first one we have

$$
I_2(t, \bar{\sigma}, \bar{\mu}, \bar{\eta}) = \mathbb{E}' \left( (\bar{Z}_t^{(p)})^2 \int_{-t^{\frac{1}{2}}}^{t^{\frac{1}{2}}} (\bar{Z}_t^{(p)} + \bar{Z}_t^{(n)}) - (\bar{\gamma} + \bar{\mu}) t^{\frac{1}{2}} - t^{\frac{1}{2}} \bar{\rho}_t \phi_{\bar{\rho}_t}(x) dx \right)
$$

$$
= \mathbb{E}' \left( (\bar{Z}_t^{(p)})^2 \int_{-t^{\frac{1}{2}}}^{t^{\frac{1}{2}}} (\bar{Z}_t^{(p)} + \bar{Z}_t^{(n)}) - (\bar{\gamma} + \bar{\mu}) t^{\frac{1}{2}} - t^{\frac{1}{2}} \bar{\rho}_t \phi_{\bar{\rho}_t}(x) dx \right),
$$

(4.53)

and the third integral is zero since $\mathbb{E}(\bar{Z}_t^{(p)}) = 0$. For the first one, use the independence of $\bar{Z}_t^{(p)}$ and $\bar{Z}_t^{(n)}$ to write

$$
\mathbb{E}' \left( (\bar{Z}_t^{(p)})^2 \int_{-t^{\frac{1}{2}}}^{t^{\frac{1}{2}}} (\bar{Z}_t^{(p)} + \bar{Z}_t^{(n)}) - (\bar{\gamma} + \bar{\mu}) t^{\frac{1}{2}} - t^{\frac{1}{2}} \bar{\rho}_t \phi_{\bar{\rho}_t}(x) dx \right) \leq t^{\frac{1}{2}} \mathbb{E}' \left( (\bar{Z}_t^{(p)})^2 \phi_{\bar{\rho}_t}(0) \left( t^{\frac{1}{2}} + \frac{1}{2} \mathbb{E}' \left( (\bar{Z}_t^{(n)})^2 \right) \right) + (|\bar{\gamma}| + M)t^{\frac{1}{2}} \right)
$$

$$
= O(t^{\frac{3}{2}}) = o(t^{\frac{3}{2}}), \quad t \to 0.
$$

(4.54)
Finally, for the second integral, let \( \bar{\omega} \in \mathbb{R} \), and write

\[
\mathbb{E}' \left( Z_1^{(p)} \int_{t^{-\frac{1}{2}}}^{t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(x) dx \right) = \mathbb{E}' \left( \bar{Z}_1^{(p)} \int_{t^{-\frac{1}{2}}}^{t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(x) dx \right) + \mathbb{E}' \left( \bar{Z}_1^{(p)} \int_{t^{-\frac{1}{2}}}^{t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(x) dx \right)
\]

\[
+ \mathbb{E}' \left( \bar{Z}_1^{(p)} \int_{t^{-\frac{1}{2}}}^{t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(x) dx \right)
\]

\[
= J_1(t, \bar{\sigma}, \bar{\omega}, \bar{\omega}^1) + J_2(t, \bar{\sigma}, \bar{\omega}^1) + J_3(t, \bar{\sigma}, \bar{\omega}^1),
\]

and observe that

\[
\mathbb{E} \left| J_1(t, \bar{\sigma}, \bar{\omega}, W_t^1) + J_2(t, \bar{\sigma}, \bar{\omega}, W_t^1) \right| \leq 2\rho \phi_{\rho \sigma} (0) t^{\frac{1}{2} - \frac{1}{2} \rho \sigma} \mathbb{E}' \left| \bar{Z}_1^{(p)} \right| \mathbb{E} \left| \bar{\omega} - \sigma_0 W_t^1 \right| = O(t), \quad t \to 0,
\]

by Lemma B.1-(v), which implies that \( \mathbb{E} \left| \bar{\omega} - \sigma_0 W_t^1 \right| = O(t^{3/2}) \). Next, use (4.18) to write

\[
J_2(t, \bar{\sigma}, \bar{\omega}^1) = \mathbb{E}' \left( \bar{Z}_1^{(p)} \int_{t^{-\frac{1}{2}}}^{t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(x) dx \right)
\]

\[
= \mathbb{E}' \left( \bar{Z}_1^{(p)} \int_{t^{-\frac{1}{2}}}^{t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(x) dx \right)
\]

\[
= J_2^1(t, \bar{\sigma}, \bar{\omega}^1) - J_2^2(t, \bar{\sigma}, \bar{\omega}^1).
\]

Now, note that there exists \( \lambda > 0 \) such that for any \( 0 < t \leq 1 \) and \( x > 0 \) (see [10], Eq. (B.5)),

\[
t^{\frac{1}{2} + \frac{1}{2} - \frac{1}{2} \rho \sigma} \mathbb{E} \left( \bar{Z}_1^{(p)} \int_{0 \leq y \leq t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(y) d y \right) \leq \lambda x^{1 - \gamma},
\]

and also, from the standard estimate for stable densities (see, e.g. [19], (14.37)), it follows that for a fixed \( x > 0 \), there exists a \( 0 < t_0 < 1 \) and a constant \( \tilde{\kappa} \) such that

\[
t^{\frac{1}{2} + \frac{1}{2} - \frac{1}{2} \rho \sigma} w f_{\bar{Z}_1^{(p)}}(t^{\frac{1}{2} + \frac{1}{2}} w) \leq \tilde{\kappa} w^{\gamma}, \quad \forall w > x,
\]

where \( f_{\bar{Z}_1^{(p)}}(\cdot) \) is the density of \( \bar{Z}_1^{(p)} \). After writing

\[
J_2^1(t, \bar{\sigma}, \bar{\omega}^1) = t^{\frac{1}{2}} \int_{0}^{\infty} \mathbb{E} \left( \bar{Z}_1^{(p)} \int_{0 \leq y \leq t^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}} \rho \bar{\omega}} \phi_{\rho \bar{\sigma} \bar{t}}(y) d y \right) d y
\]

\[
= t^{\frac{1}{2}} \int_{0}^{\infty} \int_{0}^{\infty} z f_{\bar{Z}_1^{(p)}}(z) d z \phi_{\rho \bar{\sigma} \bar{t}}(y - t^{-\frac{1}{2}} \rho \sigma_0 \bar{\omega}) d y
\]

\[
= t^{\frac{1}{2}} \int_{0}^{\infty} \int_{0}^{\infty} w f_{\bar{Z}_1^{(p)}}(t^{\frac{1}{2} + \frac{1}{2}} w) d w \phi_{\rho \bar{\sigma} \bar{t}}(y - t^{-\frac{1}{2}} \rho \sigma_0 \bar{\omega}) d y,
\]

we can use (4.58)-(4.59), and

\[
\mathbb{E} \left( \phi_{\rho \bar{\sigma} \bar{t}}(y - t^{-\frac{1}{2}} \rho \sigma_0 W_t^1) \right) \leq \sqrt{\frac{M}{m}} \mathbb{E} \left( \phi_{\rho \sigma}(y - \rho \sigma_0 W_t^1) \right),
\]

to justify the use of the dominated convergence theorem,

\[
\lim_{t \to 0} t^{\frac{-1}{2}} \mathbb{E} \left( J_2^1(t, \bar{\sigma}, W_t^1) \right) = \int_{0}^{\infty} \int_{0}^{\infty} w^{-\gamma} \lim_{t \to 0} \frac{1}{(t^{\frac{1}{2} + \frac{1}{2}} w)^{\gamma - 1}} d w \mathbb{E} \left( \phi_{\rho \bar{\sigma} \bar{t}}(y - t^{-\frac{1}{2}} \rho \sigma_0 W_t^1) \right) d y
\]

\[
= C(1) \int_{0}^{\infty} \int_{0}^{\infty} w^{-\gamma} d w \mathbb{E} \left( \phi_{\rho \sigma}(y - \rho \sigma_0 W_t^1) \right) d y
\]

\[
= C(1) \frac{Y - 1}{Y} \int_{0}^{\infty} w^{-\gamma} \phi_{\sigma_0}(x) d y.
\]
because \( \mathbb{E} \left( \phi_{\rho \sigma_0} (y - \rho \sigma_0 W_t^1) \right) = \phi_{\sigma_0}(y) \). For the second part of (4.57), we have
\[
\mathbb{E} \left( J_2^2(t, \bar{\sigma}, W_t^1) \right) = o(t^{\frac{2 - \gamma}{2}}), \quad t \to 0, \tag{4.61}
\]
because the dominated convergence theorem can be applied as before, but now
\[
\lim_{t \to 0} t^{\frac{\gamma}{2} - \frac{1}{2}} w f_{Z_1^{(p)}}(t^{\frac{1}{2} - \frac{1}{2}} w) = 0,
\]
since \( w < 0 \) and the jump support of \( Z_1^{(p)} \) is concentrated on the positive axis. Combining (4.53)-(4.61) gives
\[
\mathbb{E} \left( I_1^2(t, \bar{\mu}, \bar{\sigma}, \bar{\rho}) \right) = \frac{C(1)}{Y - 1} \int_0^\infty x^{1-\gamma} \phi_{\sigma_0}(x) dx t^{\frac{2-\gamma}{2}} + o(t^{\frac{2-\gamma}{2}}), \quad t \to 0. \tag{4.62}
\]
Finally, for the term \( I_2^2(t) \) in (4.52), the same procedure can be used to obtain
\[
\mathbb{E} \left( I_2^2(t, \bar{\mu}, \bar{\sigma}, \bar{\rho}) \right) = \frac{C(-1)}{Y - 1} \int_0^\infty x^{1-\gamma} \phi_{\sigma_0}(x) dx t^{\frac{2-\gamma}{2}} + o(t^{\frac{2-\gamma}{2}}), \quad t \to 0, \tag{4.63}
\]
which, together with (4.62), yields
\[
\mathbb{E} \left( I_2(t, \bar{\mu}, \bar{\sigma}, \bar{q}) \right) = \frac{\alpha(1)C(1) + \alpha(-1)C(-1)}{Y - 1} \int_0^\infty \phi_{\sigma_0}(x)x^{1-\gamma} dx t^{\frac{2-\gamma}{2}} + o(t^{\frac{2-\gamma}{2}}), \quad t \to 0. \tag{4.64}
\]
Combining (4.20)-(4.22), (4.50), and (4.64), then gives (4.4).

**Step 2:**

The next step is to show that the expansion (4.4) extends to the case when the \( \bar{q} \)-function of \( X \) does not necessarily satisfy conditions (2.1-ii) and (2.1-iii). To do that, we proceed exactly as in Step 2 of the pure-jump case, as outlined below. Let \( \epsilon_0 > 0 \) be such that \( \inf_{x \geq \epsilon_0} \bar{q}(x) > 0 \), and on an extended probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), define a Lévy process \( R \), independent of the original process \( X \), with Lévy triplet given by \((0, \beta, \nu_R)\), where
\[
\nu_R(dx) := C(x/|x|) e^{-|x|} 1_{\{|x| \geq \epsilon_0\}} |x|^{-\gamma-1} dx, \quad \beta := \int_{|x| \leq 1} x \nu_R(dx), \tag{4.65}
\]
so \( R \) is a compound Poisson process and can be written as
\[
R_t = \sum_{i=1}^{N_t} \xi_i, \tag{4.66}
\]
where \((N_t)_{t \geq 0}\) is a Poisson process with intensity \( \lambda := \int_{|x| \geq \epsilon_0} \nu_R(dx) \), and \((\xi_i)_{i \in \mathbb{N}}\) are i.i.d. random variables with probability measure \( \nu_R(dx)/\lambda \). As before, we denote the expectation under \( \tilde{\mathbb{P}} \) by \( \mathbb{E} \), and approximate the law of the process \( X \) with that of the following process, again defined on the extended probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\),
\[
\tilde{X}_t := X_t + R_t, \tag{4.67}
\]
whose Lévy triplet \((0, \tilde{b}, \tilde{\nu})\) of \( \tilde{X} \) is given by
\[
\tilde{b} := b + \beta, \quad \tilde{\nu}(dx) := C(x/|x|)|x|^{-\gamma-1} \bar{q}(x) dx := C(x/|x|)|x|^{-\gamma-1} \left( \bar{q}(x) + e^{-|x|} 1_{|x| \geq \epsilon_0} \right) dx, \tag{4.68}
\]
where, as already shown in the pure-jump case, \( \bar{q} \) satisfies (3.1), (2.1-ii), and (2.1-iii). Hence, by Step 1 above, the asymptotic expansion (4.4) is true for \( \tilde{X} + V \). Finally, by conditioning on the number of jumps of the process \( R \), we have
\[
\mathcal{R}_t := \left| \mathbb{P}(X_t + V \geq 0) - \mathbb{P}(\tilde{X}_t + V \geq 0) \right| = O(t), \quad t \to 0, \tag{4.69}
\]
so, by the triangle inequality, every term in the expansion of \( P(X_t + V_t \geq 0) \) of order lower than \( t \), extends to the expansion of \( P(X_t + V_t \geq 0) \). Since \((3 - Y)/2 < 1\) for \(1 < Y < 2\), that implies that the asymptotic expansion (4.4) is also valid for \( X + V \).

**Step 3:**

Lastly, we will show that the expansion (4.4) extends to the general case when \( \sigma(\cdot) \) and \( \mu(\cdot) \) are not necessarily bounded functions. To that end, define a process \( \{\tilde{Y}_t\}_{t \leq 1} \) as in (4.1)-(4.2), but replacing \( \sigma(Y_t) \) and \( \mu(Y_t) \) with the stopped processes \( \tilde{\sigma}_t := \sigma(Y_{t \wedge \tau}) \) and \( \tilde{\mu}_t := \mu(Y_{t \wedge \tau}) \), introduced in (4.13). By Steps 1-2 above, the asymptotic expansion (4.4) holds for the process \( X + \tilde{V} \). For it to extend to the process \( X + \tilde{V} \), it is then sufficient to show that

\[
|P(X_t + V_t \geq 0) - P(X_t + \tilde{V} \geq 0)| = O(t), \quad t \to 0,
\]

because \((3 - Y)/2 < 1\) for \( Y \in (1, 2) \). The last identity easily follows from Lemma B.1-(iii). Indeed, since \( \tilde{V}_t = V_t \) for \( t < \tau \), we have

\[
|P(X_t + V_t \geq 0) - P(X_t + \tilde{V} \geq 0)| = |P(X_t + V_t \geq 0, \tau < t) - P(X_t + \tilde{V} \geq 0, \tau < t)| \leq 2P(\tau < t) = O(t),
\]

as \( t \to 0 \), by Lemma B.1-(iii).

\[ \square \]

### 5 Numerical examples

In this section we carry out a numerical analysis for the important class of the tempered stable Lévy processes, as defined in [5] and first introduced in [15], with and without a Brownian diffusion component. They are an extension of the CGMY model of [4], and characterized by a Lévy measure of the form

\[
\nu(dx) = C \left( \frac{x}{|x|} \right) |x|^{-Y-1} \left( e^{-Mx1_{\{x>0\}} + e^{-G|x|1_{\{x<0\}}} \right) dx,
\]

(5.1)

where \( C(1) \) and \( C(-1) \) are nonnegative such that \( C(1) + C(-1) > 0 \), \( G \) and \( M \) are strictly positive constants, and \( Y \in (1, 2) \). The martingale condition (2.5) also implies that \( M > 1 \). Note that in terms of the notation of Section 2, we have \( \alpha(1) = -M \) and \( \alpha(-1) = G \), and the constants \( \gamma \) and \( \eta \) defined in Eqs. (2.4) and (2.7) are given by

\[
\begin{align*}
\gamma &= -\Gamma(-Y) \left(C(1) \left((M-1)^Y - M^Y\right) + C(-1) \left((G+1)^Y - G^Y\right)\right), \\
\eta &= \Gamma(-Y) \left(C(1) M^Y + C(-1) G^Y\right).
\end{align*}
\]

In the pure-jump case, Theorem 3.1 presents an asymptotic expansion for digital call option prices,

\[
P(X_t \geq 0) - \tilde{P}(Z_1 \geq 0) = \sum_{k=1}^{n} d_k t^{k(1-\gamma)} + e \ t^\gamma + f \ t + o(t), \quad t \to 0,
\]

(5.2)

where \( n = \max\{k \geq 3 : k(1 - 1/Y) \leq 1\} \), and, in this case, the expressions for the coefficients become

\[
d_k = \frac{(-1)^{k-1}}{k!} \gamma^k f_Z^{(k-1)}(0), \quad 1 \leq k \leq n,
\]

(5.3)

\[
e = -M \tilde{E} \left(Z_1^{(p)} 1_{\{Z^{(p)} + Z^{(n)} \geq 0\}}\right) + G \tilde{E} \left(Z_1^{(n)} 1_{\{Z^{(p)} + Z^{(n)} \geq 0\}}\right),
\]

(5.4)

\[
f = -\gamma \left(M + G\right) \tilde{E} \left(Z_1^{(p)} f_{Z_1^{(n)}}(-Z_1^{(p)})\right) + \Gamma(-Y) \tilde{P}(Z_1 \leq 0) C(1) M^Y - \tilde{P}(Z_1 > 0) C(-1) G^Y.
\]

(5.5)

It is informative to note that the terms can be further simplified in the CGMY-case, i.e. when \( C := C(1) = C(-1) \). In that case, \( Z_1^{(n)} \overset{\Delta}{=} -Z_1^{(n)} \), which implies \( \tilde{E}(Z_1^{(p)} 1_{\{Z^{(p)} + Z^{(n)} \geq 0\}}) = \tilde{E}(Z_1^{(n)} 1_{\{Z^{(p)} + Z^{(n)} \geq 0\}}) \), and thus,

\[
e = \frac{G - M}{2\pi} \Gamma \left(1 - \frac{1}{Y} \right) \left(2C\Gamma(-Y) \left|\cos \left(\frac{\pi Y}{2}\right)\right|\right)^\gamma,
\]

(5.6)
where we have also used the expression for $\mathbb{E}(Z^+)$ given in Remark 3.4, and
\[
 f = -\tilde{\gamma}(M + G)\mathbb{E}\left(Z^{(p)}_1 f_{Z^{(p)}}(Z^{(p)}_1)\right) + \frac{CT(-Y)}{2}(M^Y - G^Y).
\]  
(5.7)

In the presence of a continuous Brownian component, Theorem 4.1 supplies an asymptotic expansion for ATM digital call prices:
\[
\mathbb{P}(X_t + V_t \geq 0) = \frac{1}{2} + \sum_{k=1}^{n} d_k t^{k(1-\nu/2)} + e t^{2\nu} + f t^{3\nu/2} + o(t^{3\nu/2}), \quad t \to 0,
\]  
(5.8)

where $V_t = \sigma W_t - \frac{1}{2}\sigma^2 t$, $n = \max\{k \geq 3 : k(1 - Y/2) \leq (3 - Y)/2\}$, and
\[
 d_k = \frac{\sigma^{-kY}}{k!} L_{\tilde{Z}}^k \Psi(0), \quad 1 \leq k \leq n,
\]  
(5.9)

\[
e = \frac{1}{\sqrt{2\pi}\sigma} \left(\tilde{\gamma} - \frac{1}{2}\sigma^2\right),
\]  
(5.10)

\[
f = \frac{\sigma^{-1-Y-2\nu/\sqrt{\pi}}}{{\nu}} \Gamma\left(1 - \frac{Y}{2}\right) \frac{(-MC(1) + GC(-1)) - C(1) + C(-1)}{\sigma^2Y} \left(\tilde{\gamma} - \frac{1}{2}\sigma^2\right).
\]  
(5.11)

The first two $d_k$-coefficients are given in Remark 4.2-(f), while in the CGMY-case, $C(1) = C(-1)$, and all the $d_k$’s vanish.

To assess the accuracy of the above approximations, we compare them to the true values of the ATM digital call prices, estimated by Monte Carlo simulation. Using the measure transformation introduced in Section 2, we can write
\[
\mathbb{P}(X_t + V_t \geq 0) = \mathbb{E}(\text{e}^{-U_t} 1_{\{X_t + V_t \geq 0\}}) = \mathbb{E}(\text{e}^{-\bar{U}_t - \eta t} 1_{\{Z_t + \tilde{\gamma} t + V_t \geq 0\}}),
\]

where
\[
V_t = \sigma W_t - \frac{\sigma^2}{2} t, \quad Z_t = Z_t^{(p)} + Z_t^{(n)}, \quad \bar{U}_t = MZ_t^{(p)} - GZ_t^{(n)},
\]

and $Z_t^{(p)}$ and $Z_t^{(n)}$ are strictly stable random variables with location parameter 0, skewness parameters 1 and $-1$, and scale parameters $(tC(1) \cos(\pi Y/2))\Gamma(-Y)^{1/Y}$ and $(tC(-1) \cos(\pi Y/2))\Gamma(-Y)^{1/Y}$, respectively.

As mentioned in the introduction, it is well known that in the presence of jumps, the convergence of many asymptotic expressions may be slow, and only satisfactory at extremely small time scales. However, the performance is highly parameter-dependent, and here we consider parameter values that are of relevance for financial applications. See, for example, [1, Tables 1 and 5], [14] and [20, p.82], where the tempered stable model is calibrated to observed option prices.

Figure 1 displays the asymptotic expansion (5.2) for ATM digital call log-prices, under a pure-jump tempered stable model, together with the true price estimated by Monte Carlo simulation as described above. The axes are on a log$_{10}$-scale, with time to maturity in years. The first and second order approximations are of order $t^{2\nu/3}$ and $t^{3\nu/2}$, respectively, and it is clear that the second order approximation significantly improves the first order approximation, and gives a good estimate for maturities up to a month. It performs particularly well in Panel (a), closely matching the true prices for maturities up to 0.1 years. Figure 2 then shows the implied volatility smile, together with the ATM implied volatility slope approximation given in Corollary 3.3. The maturity is 0.1 years, and the slope approximation captures the sign of the slope, and even gives a good estimate of its magnitude, which can be estimated numerically. In Panel (a), the slope approximation is 0.32 which coincides with the numerical estimate, but in Panel (b) the approximate value is $-0.46$ while the numerical estimate is $-0.52$.

Figure 3 carries out the same analysis for a tempered stable model with a nonzero Brownian component. Panel (a) compares the Monte Carlo estimate of the ATM digital call log-prices to the first- and second-order approximations from (5.8), which are of order $t^{2\nu/3}$ and $t^{3\nu/2}$, respectively. As in the pure-jump case, the second order approximation gives a good estimate for maturities up to at least a few trading days. Panels (b) and (c) then display the volatility smiles for maturities $t = 0.1$ and $t = 0.01$ years. They are not as pronounced as in the pure-jump case, but the ATM slope approximation from Corollary 4.3 nevertheless captures the sign of the slope. However, the magnitude estimate seems less precise than in the pure-jump case. For maturity $t = 0.1$, the approximation gives 0.093, compared to a numerical estimate of 0.043, but for maturity $t = 0.01$ it is 0.19, compared to the numerical estimate 0.23.
Figure 1: Comparison of ATM digital call option prices computed by Monte Carlo, and the first- and second-order approximations, under a pure-jump tempered stable model. Time is in years and both axes on a log₁₀-scale. Panel (a): \((C(1), C(-1), G, M, Y) = (0.0088, 0.0044, 0.41, 1.93, 1.5)\). Panel (b): \((C(1), C(-1), G, M, Y) = (0.015, 0.041, 2.318, 4.025, 1.35)\).

Figure 2: The red curve is the volatility smile as a function of log-strike, and the blue dashed line is the second order slope approximation. The maturity is \(t = 0.1\) years, and the model is the same as in Figure 1.
Figure 3: Panel (a) compares ATM digital call option prices computed by Monte Carlo, to the first- and second-order approximations. Time is in years and both axes on a log_{10}-scale. Panels (b) and (c) show the volatility smile (red) together with the slope approximation (blue) for maturities $t = 0.1$ and $t = 0.01$, respectively. The model is tempered stable with a Brownian component and parameters $(C(1), C(-1), G, M, Y, \sigma) = (0.0040, 0.0013, 0.41, 1.93, 1.5, 0.1)$.

A Proofs of technical results and lemmas

Proof of (1.19).

We have

$$
\left( e^{-\frac{1}{2} \sqrt{2\pi} \phi} \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)t}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) \right)^{-1} = \exp \left( \frac{1}{2} \left( \frac{\kappa}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^2 \right) \exp \left( \frac{1}{2} \left( \frac{1}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^2 \right),
$$

and, using $\frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{N-1} (-1)^n \frac{(2n-1)!}{(2x)^{2n}} + O(x^{-2N+1}e^{-x^2})$, as $x \to \infty$, we have for $\kappa > 0$

$$
\Phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)t}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) = \frac{1}{\sqrt{2\pi} \kappa} \frac{\hat{\sigma}(\kappa, t)\sqrt{t}}{\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)t} \exp \left( -\frac{1}{2} \left( \frac{\kappa}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^2 \right) \exp \left( -\frac{1}{2} \left( \frac{1}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^2 \right),
$$

as $t \to 0$, so

$$
e^\kappa \phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)t}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) \Phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)t}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) = \left( \frac{\hat{\sigma}^2(\kappa, t)t}{\kappa} \right)^{\frac{1}{2}} - \left( \frac{\hat{\sigma}^2(\kappa, t)t}{\kappa^2} \right)^{\frac{3}{2}} \left( 1 + \frac{\kappa}{2} \right) + O \left( \left( \hat{\sigma}^2(\kappa, t)t \right)^{\frac{5}{2}} \right). \quad (A.1)
$$

Next, using the Taylor expansion $x^\frac{1}{2} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + O((x-1)^3)$, $x \to 1$, and recalling the definition of $V_1(t, \kappa)$ from (1.17), we can write

$$
\left( \frac{\hat{\sigma}^2(\kappa, t)t}{\kappa^2} \right)^{\frac{1}{2}} = 1 + \frac{1}{2} V_1(t, \kappa) + o \left( \frac{1}{\log t} \right), \quad t \to 0,
$$

\[30\]
and, thus,

\[ \left( \frac{\hat{\sigma}^2(\kappa, t)}{\kappa^2} \right)^{\frac{1}{2}} = \frac{1}{(2 \log \frac{1}{\kappa})^{\frac{1}{2}}} \left( 1 + \frac{1}{2} V_1(t, \kappa) \right) + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right), \ t \to 0. \]

Doing the same for the second term of (A.1), using \( x^\frac{3}{2} = 1 + \frac{3}{2} (x - 1) + \frac{3}{8} (x - 1)^2 + O((x - 1)^3), \ x \to 1, \) we obtain

\[ e^\nu \phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^{\frac{1}{2}} \Phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) = \frac{1}{\sqrt{2 \log \frac{1}{\kappa}}} \left( 1 + \frac{1}{2} V_1(t, \kappa) \right) - \frac{1}{(2 \log \frac{1}{\kappa})^{\frac{1}{2}}} \left( 1 + \frac{\kappa}{2} \right) + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right), \ t \to 0. \] (A.2)

For \( \kappa < 0, \) note that (1.1) can be written

\[ \hat{\sigma}(\kappa, t) = \frac{-e^\nu \Phi \left( \frac{\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) + e^\nu \mathbb{P}(S_t \leq e^\nu)}{\sqrt{\nu}} \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right), \] (A.3)

and, as above, we get

\[ e^\nu \phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^{\frac{1}{2}} \Phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) = \frac{1}{\sqrt{2 \log \frac{1}{\kappa}}} \left( 1 + \frac{1}{2} V_1(t, \kappa) \right) - \frac{1}{(2 \log \frac{1}{\kappa})^{\frac{1}{2}}} \left( 1 + \frac{\kappa}{2} \right) + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right), \ t \to 0. \] (A.4)

Next, let \( K_\kappa := 4 \sqrt{\pi} a_0(\kappa)e^{-\kappa^2}/|\kappa|, \) and note that

\[ t \exp \left( \frac{1}{2} \left( \frac{\kappa}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^2 \right) = \exp \left( \frac{1}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right) \left( \frac{\kappa^2}{2 \hat{\sigma}(\kappa, t)\sqrt{t}} - \hat{\sigma}^2(\kappa, t)\sqrt{t} \right) = \exp \left( - \frac{\kappa^2}{2 \hat{\sigma}(\kappa, t)\sqrt{t}} V_1(t, \kappa) + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right) \right) \]

\[ = \exp \left( - \log \left( \frac{K_\kappa \left( \log \frac{1}{\kappa} \right)^{\frac{3}{2}}}{1 + V_1(t, \kappa) + o\left( \log \frac{1}{\kappa} \right)^{\frac{3}{2}}} \right) + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right) \right) \]

\[ = \frac{1}{K_\kappa \left( \log \frac{1}{\kappa} \right)^{\frac{3}{2}}} \exp \left( \log \left( K_\kappa \left( \log \frac{1}{\kappa} \right)^{\frac{3}{2}} \right) \right) V_1(t, \kappa) + o \left( \log \frac{1}{\kappa} \right)^{\frac{3}{2}} + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right), \] as \( t \to 0, \) so, when \( \kappa > 0, \)

\[ e^\nu \phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^{\frac{1}{2}} \mathbb{P}(X_\kappa + \sigma W_t \geq \kappa) = e^\nu \sqrt{2\pi} \exp \left( \frac{1}{2} \left( \frac{\kappa}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^2 \right) \exp \left( \frac{1}{2} \left( \frac{\hat{\sigma}(\kappa, t)\sqrt{t}}{2} \right)^2 \right) (t\nu|\kappa, \infty) + o(t)) \]

\[ = e^\nu \sqrt{2\pi} \frac{1}{K_\kappa} \nu|\kappa, \infty \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right)^{\frac{3}{2}} + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right), \ t \to 0. \] (A.5)

When \( \kappa < 0 \) we have

\[ e^\nu \phi \left( \frac{-\kappa + \frac{1}{2} \hat{\sigma}^2(\kappa, t)}{\hat{\sigma}(\kappa, t)\sqrt{t}} \right)^{\frac{1}{2}} \mathbb{P}(X_\kappa + \sigma W_t \leq \kappa) = e^\nu \sqrt{2\pi} \frac{1}{K_\kappa} \nu(-\infty, \kappa] \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right)^{\frac{3}{2}} + o \left( \frac{1}{(\log \frac{1}{\kappa})^{\frac{1}{2}}} \right), \ t \to 0. \] (A.6)
Combining (1.1) and (A.2)-(A.6) now results in (1.19). \[\square\]

**Proof of Lemma 2.1.**

For (2.6), recall that \(\varphi(x) = -\ln \bar{q}(x)\) and, thus, by (2.1-iii) and the boundedness of \(\bar{q}\), there exist finite constants \(K_1\) and \(K_2\) such that, for \(|x| \leq \frac{1}{2}\),

\[
\left( e^{\frac{\varphi(x)}{2}} - 1 \right)^2 \leq K_1 \varphi(x)^2 \leq K_2 \left( \bar{q}(x) - 1 \right)^2 \leq 2K_2 \left( \bar{q}(x) - 1 - \alpha \left( \frac{x}{|x|} \right) x \right)^2 + 2K_2 \left( \alpha \left( \frac{x}{|x|} \right) x \right)^2.
\] (A.7)

The second term on the right-hand side of (A.7) is clearly integrable on \([-1/2, 1/2]\) with respect to \(\nu\). For the first term, note that, by the boundedness of \(\bar{q}\), there exists a finite constant \(K_3\) such that

\[
\left( \bar{q}(x) - 1 - \alpha \left( \frac{x}{|x|} \right) x \right)^2 \leq K_3 \left| \bar{q}(x) - 1 - \alpha \left( \frac{x}{|x|} \right) x \right|,
\]

for any \(|x| \leq \frac{1}{2}\) and, thus, (2.1-i) suffices for the integral of the first term to be finite for \(|x| \leq \frac{1}{2}\). For \(|x| > \frac{1}{2}\), write

\[
\int_{|x| > \frac{1}{2}} \left( e^{\frac{\varphi(x)}{2}} - 1 \right)^2 \nu(dx) = \int_{|x| > \frac{1}{2}} \left( 1 - e^{-\varphi(x)/2} \right)^2 \bar{\nu}(dx) \leq 2 \left( \int_{|x| > \frac{1}{2}} \bar{\nu}(dx) + \int_{|x| > \frac{1}{2}} e^{-\varphi(x)} \bar{\nu}(dx) \right),
\]

which is finite since \(e^{-\varphi(x)} = \bar{q}(x)\) and \(\bar{q}(x)\) is bounded. For (2.8), note that there exists a constant \(K\) such that

\[
\left| e^{-\varphi(x)} - 1 - \varphi(x) \right| \leq K \varphi(x)^2,
\]

for any \(|x| \leq \frac{1}{2}\) and, thus, it can be handled as in (A.7). For \(|x| \geq \frac{1}{2}\), (2.1-ii)-(2.1-iii) and the fact that \(q(x)\) is bounded ensure that the integral is finite. \[\square\]

**Proof of Lemma 2.3.**

The proof is similar to the proof of Lemma 3.3 in [10] and is based on a suitable “small/large” jump decomposition of \(Z\) and \(\bar{U}\). Concretely, fix \(\varepsilon > 0\), and define

\[
\bar{Z}^{(e)}_{t} := \int_{0}^{t} \int_{|x| > \varepsilon} \varphi(x) N(ds, dx), \quad \bar{U}^{(e)}_{t} := \bar{U}_{t} - \bar{Z}^{(e)}_{t}, \quad (A.8)
\]

and

\[
Z^{(e)}_{t} := \int_{0}^{t} \int_{|x| > \varepsilon} xN(ds, dx), \quad Z^{(e)}_{t} := Z_{t} - Z^{(e)}_{t}. \quad (A.9)
\]

Let \(\{N^{(e)}_{t}\}_{t \geq 0}\) and \(\{\xi^{(e)}_{t}\}_{t \geq 1}\) denote the counting process and jump sizes of \(\{\bar{Z}^{(e)}_{t}\}_{t \geq 0}\), respectively, and let \(\lambda_{e} := \mathbb{E}(N^{(e)}_{1})\) be the jump intensity. Conditioning on \(N^{(e)}_{t}\) gives:

\[
t^{-1} \bar{\mathbb{P}} \left( Z_{t} + \bar{\gamma} t \geq 0, \bar{U}_{t} \geq v \right) = t^{-1} e^{-\lambda_{e}t} \bar{\mathbb{P}} \left( Z^{(e)}_{t} + \bar{\gamma} t \geq 0, \bar{U}^{(e)}_{t} \geq v \right) + e^{-\lambda_{e}t} \bar{\mathbb{P}} \left( Z^{(e)}_{t} + \bar{\gamma} t + \xi^{(e)}_{1} \geq 0, \bar{U}^{(e)}_{t} + \varphi(\xi^{(e)}_{1}) \geq v \right) + O(t).
\]

The first term above can be made \(O(t)\) by taking \(0 < \varepsilon < \varepsilon_{0}\), for some small enough \(\varepsilon_{0} > 0\) (see, e.g., [19, Section 26]). For the second term, note that, since \(\varphi(x) \to 0\) as \(x \searrow 0\), there exists \(\varepsilon_{0} > 0\) such that, for all \(0 < \varepsilon < \varepsilon_{0}\),

\[
a_{0}(v) := \lambda_{e} \bar{\mathbb{P}} \left( \xi^{(e)}_{1} \geq 0, \varphi(\xi^{(e)}_{1}) \geq v \right) = \int_{\mathbb{R}_{0}} 1_{\{x \geq 0, \varphi(x) \geq v\}} \bar{\nu}(dx) = C(1) \int_{0}^{\infty} 1_{\{v(x) \geq v\}} x^{-\gamma - 1} dx.
\]

Also, since

\[
F_{v,e}(z, u) := \bar{\mathbb{P}} \left( z + \xi^{(e)}_{1} \geq 0, u + \varphi(\xi^{(e)}_{1}) \geq v \right)
\]
is continuous at \((z, u) = (0, 0)\), for any fixed \(0 < \varepsilon < \varepsilon_0\), the function 
\[
A_{\varepsilon}(v) := t^{-1} \tilde{P} \left( Z_t + \tilde{\gamma} t \geq 0, \tilde{U}_t \geq v \right) - a_0(v)
\]
is such that 
\[
\lim_{t \to 0} A_{\varepsilon}(v) = \lambda_0 \lim_{t \to 0} \tilde{E} \left( Z_t^{(\varepsilon)} + \tilde{\gamma} t + \xi_t^{(\varepsilon)} \geq 0, \tilde{U}_t^{(\varepsilon)} + \varphi(\xi_t^{(\varepsilon)}) \geq v \right) - \tilde{P} \left( \xi_t^{(\varepsilon)} \geq 0, \varphi(\xi_t^{(\varepsilon)}) \geq v \right)
\]
\[
= \lambda_0 \lim_{t \to 0} \tilde{E} \left( Z_t^{(\varepsilon)} + \tilde{\gamma} t, \tilde{U}_t^{(\varepsilon)} \right) - \tilde{E}(0, 0)
\]
\[
= \lambda_0 \tilde{E} \left( \lim_{t \to 0} F_{v, x} \left( Z_t^{(\varepsilon)} + \tilde{\gamma} t, \tilde{U}_t^{(\varepsilon)} \right) - F_{v, x}(0, 0) \right)
\]
\[
= 0,
\]
where we had used the dominated convergence theorem to obtain the last equality. The other relation is proved in the same way.

**Proof of (4.9)-(4.10).**

First note that \(\Psi'(x) = \phi(x)\) and \(\Psi''(x) = -x\phi(x)\). Then recall that \(\tilde{\nu}(dx) = \C \left( \frac{x}{|x|} \right) |x|^{-Y-1} dx\), and let \(\tilde{\nu}^*(dx) := -\frac{1}{Y} \text{sgn}(x) \C \left( \frac{x}{|x|} \right) |x|^{-Y} dx\). For the first term, integration by parts then gives 
\[
L_Z \Psi(0) = \int_{\mathbb{R}^2} (\Psi(u) - \Psi(0) - \Psi'(0) u) \tilde{\nu}(du)
\]
\[
= -\int_{\mathbb{R}^2} (\Psi'(u) - \Psi'(0)) \tilde{\nu}^*(du)
\]
\[
= \int_{\mathbb{R}^2} (\phi(u) - \phi(0)) \tilde{\nu}^*(du)
\]
\[
= \frac{C(1) - C(-1)}{Y} \int_0^{\infty} (\phi(u) - \phi(0)) u^{-Y} du.
\]

To obtain the second term, we again integrate by parts 
\[
L_Z^2 \Psi(0) = \int_{\mathbb{R}^2} (L_Z \Psi(u) - L_Z \Psi(0) - u(L_Z \Psi)'(0)) \tilde{\nu}(du) = -\int_{\mathbb{R}^2} ((L_Z \Psi)'(u) - (L_Z \Psi)'(0)) \tilde{\nu}^*(du),
\]
since \(L_Z \Psi(u) - L_Z \Psi(0) - u(L_Z \Psi)'(0)\) is of order \(O(u^2)\), as \(|u| \to 0\), and of order \(O(u)\), as \(|u| \to \infty\). We have 
\[
(L_Z \Psi)'(u) = \int_{\mathbb{R}^2} \Psi'(v + u) - \Psi'(u) - \Psi'(u)v \tilde{\nu}(dv) = \int_{\mathbb{R}^2} (\phi(v + u) - \phi(u) + uv\phi(u)) \tilde{\nu}(dv),
\]
so 
\[
L_Z^2 \Psi(0) = -\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \phi(v + u) - \phi(u) + uv\phi(u) - \phi(v) + \phi(0) \right) \tilde{\nu}(dv) \tilde{\nu}^*(du),
\]
and, since the integrand is of order \(u^2\) as \(|v| \to 0\), and bounded as \(|v| \to \infty\), integrating by parts w.r.t. \(v\) gives 
\[
L^2 \Psi(0) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( -(v + u)\phi(v + u) + u\phi(u) + v\phi(v) \right) \tilde{\nu}^*(du) \tilde{\nu}^*(dv)
\]
\[
= -\frac{C^2(1) - C^2(-1)}{Y^2} \int_0^{\infty} \int_0^{\infty} \left( (v + u)\phi(v + u) - u\phi(u) - v\phi(v) \right) (uv)^{-Y} dv du,
\]
since the terms with \(\text{sgn}(uv) < 0\) cancel each other out.

**Proof of (4.27).**
First note that we can find $R > 0$ such that
\[ f_Z(z) \leq R|z|^{-Y-1}, \quad (A.10) \]
for all $z \in \mathbb{R}$ (see, e.g. [19], (14.37)). Denote $\|f_Z\|_{\infty} := \sup_{z \in \mathbb{R}} f_Z(z)$ and let $H := 2\gamma > 0$. Then, by (A.10), we have for $x \neq 0$ and $t \leq 1$:
\[
\begin{align*}
J_t(x) &\leq R \int_{-t^\frac{1}{Y} + \frac{x}{\sigma_0}}^{t^\frac{1}{Y} + \frac{x}{\sigma_0}} |x|^{-Y-1} dz \mathbf{1}_{\{x > H\}} + \int_{-t^\frac{1}{Y} + \frac{x}{\sigma_0}}^{0} f_Z(z)dz \mathbf{1}_{\{x < H\}} \\
&\leq R \gamma t^{\frac{1}{Y} + \frac{Y}{Y-1}} \left( |t^{\frac{1}{Y} + \frac{x}{\sigma_0}} + \gamma t^{1-Y} - x|^{-Y-1} + |t^{\frac{1}{Y} + \frac{x}{\sigma_0}} - x|^{-Y-1} \right) \mathbf{1}_{\{x > H\}} + \gamma t^{1-Y} \|f_Z\|_{\infty} \mathbf{1}_{\{x < H\}} \\
&\leq R \gamma t^{\frac{1}{Y} + \frac{Y}{Y-1}} \mathbf{1}_{\{x > H\}} + \gamma t^{1-Y} \left( (R + \|f_Z\|_{\infty} H^{Y+1}) |t^{\frac{1}{Y} + \frac{x}{\sigma_0}} - x|^{-Y-1} \right) \\
&\leq R t^{\frac{3}{Y} - \frac{Y}{Y-1}} |x|^{-Y-1},
\end{align*}
\]
where $\tilde{\kappa}$ is a constant, and we used the facts that $t \leq 1$ and $H = 2\gamma$. \hfill $\Box$

**Proof of (4.34).**

Using (A.10), we have for $x \neq 0$,
\[
|\tilde{J}_1(t, x, \bar{\sigma})| \leq R \left| \int_{t^\frac{1}{Y} + \frac{x}{\sigma_0}}^{t^\frac{1}{Y} + \frac{x}{\sigma_0}} |x|^{-Y-1} dz \right| \\
\leq R t^{\frac{1}{Y} + \frac{Y}{Y-1}} |x| \left( \bar{\sigma}_t^Y \sigma_0^{-1} - 1 \right) \left( |t^{\frac{1}{Y} + \frac{x}{\sigma_0}} \bar{\sigma}_t^Y \sigma_0^{-1}|^{-Y-1} + |t^{\frac{1}{Y} + \frac{x}{\sigma_0}} - x|^{-Y-1} \right) \\
\leq \tilde{\kappa} t^{\frac{1}{Y} - \frac{Y}{Y-1}} \frac{\bar{\sigma}_t^Y \sigma_0^{-1} - 1}{x} |x|^{-Y},
\]
where $\tilde{\kappa} := R \left( (m/M)^{-Y-1} + 1 \right)$ is a constant. \hfill $\Box$

**Proof of (4.36).**

For simplicity, we assume that $x > 0$ (the case $x < 0$ is almost identical). Let us start by noting that, due to the expansion of the strictly stable density $f_Z$ (see [19], 14.34), one can find a constant $\tilde{\kappa}$ such that for $0 < t < 1$:
\[
|f_Z(zt^{\frac{1}{Y} + \frac{x}{\sigma_0}}) - C(1) (zt^{\frac{1}{Y} + \frac{x}{\sigma_0}})^{-Y-1}| \leq \tilde{\kappa} (zt^{\frac{1}{Y} + \frac{x}{\sigma_0}})^{-2Y-1}, \quad \forall z \in x \left( \frac{m}{M}, \frac{M}{m} \right), \quad (A.11)
\]
Next,
\[
\begin{align*}
\mathbb{E} (\tilde{J}_1(t, x, \bar{\sigma})) &= t^{\frac{1}{Y} + \frac{x}{\sigma_0}} \mathbb{E} \left( \int_x^{x \frac{\bar{\sigma}_t^Y \sigma_0^{-1}}{\bar{\sigma}_t^Y \sigma_0^{-1}}} f_Z(zt^{\frac{1}{Y} + \frac{x}{\sigma_0}})dz - \int_x^{x \frac{\bar{\sigma}_t^Y \sigma_0^{-1}}{\bar{\sigma}_t^Y \sigma_0^{-1}}} f_Z(zt^{\frac{1}{Y} + \frac{x}{\sigma_0}})dz \right) \\
&= t^{\frac{1}{Y} + \frac{x}{\sigma_0}} \mathbb{E} \left( \int_x^{x \frac{\bar{\sigma}_t^Y \sigma_0^{-1}}{\bar{\sigma}_t^Y \sigma_0^{-1}}} \left( C(1)(zt^{\frac{1}{Y} + \frac{x}{\sigma_0}})^{-Y-1}dz + (f_Z(zt^{\frac{1}{Y} + \frac{x}{\sigma_0}}) - C(1)(zt^{\frac{1}{Y} + \frac{x}{\sigma_0}})^{-Y-1}) \right)dz \right) \\
&= C(1) \frac{1}{Y} t^{\frac{1}{Y} - \frac{Y}{Y-1}} x^{-Y} \mathbb{E} \left( \left( 1 - \left( \bar{\sigma}_t^Y \sigma_0^{-1} \right)^{-Y} \right)^+ + \left( \left( \bar{\sigma}_t^Y \sigma_0^{-1} \right)^{-Y} - 1 \right)^+ \right) + o \left( t^{\frac{x}{Y}} \right), \quad t \to 0,
\end{align*}
\]
where we have used (A.11) and the fact that $\mathbb{E}|1 - (\tilde{\sigma}_t^*/\sigma_0)|^{-2Y} = O(t^{1/2})$, which can be shown in the same way as Lemma B.1-(vii), from which it also follows that

$$
\mathbb{E} \left( \tilde{J}_1(t, x, \tilde{\sigma}) \right) = C(1) \frac{1}{Y} t^{1-\frac{1}{Y}} x^{-Y} \mathbb{E} \left( 1 - \left( \frac{\tilde{\sigma}_t^*}{\sigma_0} \right)^{-Y} \right) + o(t^{\frac{1}{2}}) = o(t^{\frac{1}{2}}), \quad t \to 0.
$$

\hfill \Box

**Proof of (4.43).**

Assume $x > 0$ (the case when $x < 0$ is similar), and let $a := \frac{2}{5}\sigma_0^* (\tilde{w}^2 - 1)$. When $a < 0$, use (A.10) to write

$$
|J_t(x, \tilde{w})| = R \int_{\frac{t}{x} - \frac{1}{x}}^{\frac{t}{x} + 1} \left| z^{-Y-1} \right| dz \leq R |a| t^{-1-\frac{1}{Y}} (t^{\frac{1}{2}} + x)^{-Y-1} \leq R |a| t^{-\frac{3}{2}Y} x^{-Y-1}.
$$

When $a > 0$, let $H := 2a$, and again use (A.10) to write

$$
|J_t(x, \tilde{w})| \leq R \int_{\frac{t}{x} - \frac{1}{x}}^{\frac{t}{x} + 1} \left| z^{-Y-1} \right| dz \leq R a t^{-1-\frac{1}{Y}} (t^{\frac{1}{2}} + x)^{-Y-1} \left( \frac{t^{\frac{1}{2}} + x - at^{-\frac{1}{Y}}}{t^{\frac{1}{2}} + x} \right)^{-Y-1} + a^{-1-\frac{1}{Y}} \| f_z \|_{H^Y} \left( \frac{t^{\frac{1}{2}} + x}{t^{\frac{1}{2}} + x} \right)^{-Y-1} \leq (R 2^{Y+1} + \| f_z \|_{H^Y}) \| a \|_{H^Y} (t^{\frac{1}{2}} + x)^{-Y-1},
$$

where we have used $t \leq 1$ and $H = 2a$. By doing the same for $x < 0$ and using the definition of $a$, one obtains

$$
|J_t(x, \tilde{w})| \leq \kappa |\tilde{w}| - 1 |(1 + |\tilde{w}|^2 - 1)^{-1} | t^{\frac{3}{2}Y} |x|^{-Y-1},
$$

where $\kappa$ is a constant that does not depend on $\tilde{w}$. \hfill \Box

**B Some useful estimates for $\bar{\mu}$ and $\tilde{\sigma}$**

**Lemma B.1.** Let $V$ be as in (4.1)-(4.2), with $\mu(Y_t)$ and $\sigma(Y_t)$ replaced by $\bar{\mu}_t$ and $\tilde{\sigma}_t$, defined in (4.13). Also let $\bar{\sigma}_t$, $\tilde{\sigma}_t''$, $\bar{\alpha}_t$, and $\bar{\gamma}_t$, be as in (4.38), and $\tilde{\sigma}_t^* := \sqrt{\frac{1}{2} \int_0^t \tilde{\sigma}_s^2 ds}$. Then the following relations hold for any $p \geq 1$:

(i) $\mathbb{E} |\bar{\mu}_t - \mu_0|^p = O(t^{\frac{p}{2}})$, \quad $t \to 0$.

(ii) $\mathbb{E} |\tilde{\sigma}_t - \sigma_0|^p = O(t^{\frac{p}{2}})$, \quad $t \to 0$.

(iii) $\mathbb{E} |\bar{\sigma}_t - \sigma_0|^p = O(t^{\frac{p}{2}})$, \quad $t \to 0$.

(iv) For $\tau$ as in (4.13), we have $\mathbb{P} (\tau < t) = O(t^p)$, \quad $t \to 0$.

(v) For $\xi_t^1, \xi_t^2$ and $\xi_t^{1,0}$, as in (4.37)-(4.39), we have $\mathbb{E} |\xi_t^1| = O(t)$ and $\mathbb{E} |\xi_t^2| + \mathbb{E} |\xi_t^1 - \xi_t^{1,0}| = O(t^{\frac{3}{2}})$, \quad $t \to 0$.

(vi) $\mathbb{E} |\int_0^t \int_0^1 (\bar{\sigma}_u \tilde{\sigma}_u^* \gamma_u - \sigma_0 \gamma_u^2) \tilde{w}_u^1 \tilde{u}_u^1 ds| = O(t^2)$, \quad $t \to 0$.

(vii) $\lim_{t \to 0} t^{-\frac{1}{2}} \mathbb{E} (\sigma_0^Y - (\tilde{\sigma}_t^* - Y) = 0.$
Proof. Let $L$ be a common Lipschitz constant for $\tilde{\mu}_t$, $\tilde{\sigma}_t$, and $\tilde{\gamma}_t$.

(i) By the Lipschitz continuity of $\tilde{\mu}_t$, and the Burkholder-Davis-Gundy (BDG) inequality, we can find a constant $C_p$ such that
\[
\mathbb{E} |\tilde{\mu}_t - \mu_0|^p \leq L^p \mathbb{E} |Y_{t \wedge \tau} - y_0|^p \leq L^p C_p \left( \mathbb{E} \left( \int_0^t \tilde{\alpha}_s ds \right)^p + \mathbb{E} \left( \int_0^t \tilde{\gamma}_s^2 ds \right)^{\frac{p}{2}} \right) = O(t^{\frac{p}{2}}), \quad t \to 0,
\]
since $\tilde{\alpha}_s$ and $\tilde{\gamma}_s$ are bounded.

(ii) is proved in a similar way, and for (iii) we use the boundedness of $\tilde{\sigma}_t$, Jensen’s inequality, and (ii) to write
\[
\mathbb{E} |\tilde{\sigma}_t^* - \sigma_0|^p \leq \frac{1}{(2m)^p} \mathbb{E} \left( \frac{1}{t} \int_0^t (\tilde{\sigma}_s^2 - \sigma_0^2) ds \right)^p \leq \left( \frac{M}{m} \right)^p \frac{1}{t} \int_0^t \mathbb{E} (\tilde{\sigma}_s - \sigma_0)^p ds = O(t^{\frac{p}{2}}), \quad t \to 0.
\]
For the proof of (iv) we refer to the proof of Lemma 4.1 in [12].

(v) By Cauchy’s inequality and Itô’s isometry we have
\[
\mathbb{E} |\xi_t^2| \leq \sqrt{\int_0^t \mathbb{E} \left( \int_0^s \left( \tilde{\sigma}_u^* \tilde{\alpha}_u + \frac{1}{2} \tilde{\sigma}_u^* \tilde{\gamma}_u^2 \right) du \right)^2 ds = O(t^{\frac{p}{2}}), \quad t \to 0.
\]
Similarly,
\[
\mathbb{E} |\xi_t^1 - \xi_t^{1,0}| \leq \sqrt{\int_0^t \int_0^s \mathbb{E} (\tilde{\sigma}_u^* \tilde{\gamma}_u - \sigma_0^* \gamma_0)^2 duds = O(t^{\frac{p}{2}}), \quad t \to 0,
\]
because by the boundedness of $\tilde{\sigma}_u^*$ and $\tilde{\gamma}_u$, we can find a constant $K$ such that
\[
\mathbb{E} (\tilde{\sigma}_u^* \tilde{\gamma}_u - \sigma_0^* \gamma_0)^2 \leq K \mathbb{E} (\tilde{\gamma}_u - \gamma_0)^2 + K \mathbb{E} (\tilde{\sigma}_u^* - \sigma_0^*)^2 \leq 2KLK (Y_{u \wedge \tau} - y_0)^2 = O(u), \quad t \to 0,
\]
where in the last step we again used the BDG inequality. Similarly, Cauchy’s inequality and Itô’s isometry yield \(\mathbb{E} |\xi_t^1| = O(t)\).

(vi) By Itô’s isometry we have
\[
\mathbb{E} \left( \frac{1}{t} \int_0^t \int_0^s (\tilde{\sigma}_u^* \tilde{\gamma}_u - \sigma_0^* \gamma_0) dW_u^1 ds \right)^2 \leq \frac{1}{t} \int_0^t \mathbb{E} \left( \int_0^s (\tilde{\sigma}_u^* \tilde{\gamma}_u - \sigma_0^* \gamma_0) dW_u^1 \right)^2 ds
\leq \frac{1}{t} \int_0^t \int_0^s \mathbb{E} (\tilde{\sigma}_u^* \tilde{\gamma}_u - \sigma_0^* \gamma_0)^2 duds = O(t^{\frac{p}{2}}), \quad t \to 0,
\]
where the last step can be justified like the last step in the proof of (iv), using the boundedness and Lipschitz continuity of $\tilde{\sigma}_u$, $\tilde{\sigma}_u^*$, and $\tilde{\gamma}_u$.

(vii) Using that $0 < m < \hat{\sigma}_t < M < \infty$, we can write $(\hat{\sigma}_t)^{-1} = \sigma_0^{-1} - Y \sigma_0^{-1} (\hat{\sigma}_t - \sigma_0) + h(\hat{\sigma}_t) (\hat{\sigma}_t - \sigma_0)^2$, where $0 \leq |h(\hat{\sigma}_t)| < K$, for all $t < 1$ and some constant $K$. Therefore,
\[
\mathbb{E} (\sigma_0^{-1} - (\hat{\sigma}_t)^{-1}) = Y \sigma_0^{-2} \mathbb{E} (\hat{\sigma}_t - \sigma_0) + \mathbb{E} \left( h(\hat{\sigma}_t) (\hat{\sigma}_t - \sigma_0)^2 \right)
\]
References