A note on high-order short-time expansions for close-to-the-money option prices under the CGMY model

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Abstract
The short-time asymptotic behavior of option prices for a variety of models with jumps has received much attention in recent years. In the present work, a novel third-order approximation for close-to-the-money European option prices under the CGMY Lévy model is derived, and extended to a model with an additional independent Brownian component. The asymptotic regime considered, in which the strike is made to converge to the spot stock price as the maturity approaches 0, is more relevant in applications since the most liquid options have strikes that close to the spot price. Our results shed new light on the connection between both the volatility of the continuous component and the jump parameters and the behavior of option prices near expiration when the strike is close to the spot price. In particular, a new type of transition phenomenon is uncovered in which the third order term exhibits two distinct asymptotic regimes depending on whether $1 < Y < \frac{3}{2}$ or $\frac{3}{2} < Y < 2$.

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1 Introduction

Stemming in part from its importance for model calibration and testing, small-time asymptotics of option prices have received a lot of attention in recent years (see, e.g., [3], [5], [6], [10], [11], [12], [13], [15], and references therein). In the present paper, we study the small-time behavior for close-to-the-money European call option prices

$$E\left[\left(S_t - S_0 e^{\kappa t}\right)^+\right] = S_0 E\left[\left(e^{X_t} - e^{\kappa t}\right)^+\right], \quad t \geq 0,$$

(1.1)

where $t \mapsto \kappa_t$ is a deterministic function such that $\kappa_t \to 0$ as $t \to 0$, under the exponential Lévy model

$$S_t := S_0 e^{X_t}, \quad \text{with} \quad X_t := L_t + \sigma W_t, \quad t \geq 0,$$

(1.2)

where $L = (L_t)_{t \geq 0}$ is a CGMY Lévy process (cf. [4]), while $W = (W_t)_{t \geq 0}$ is an independent standard Brownian motion. Throughout, $x^+ := x 1_{\{x > 0\}}$ and $x^- := x 1_{\{x < 0\}}$ denote the positive and negative parts of a real $x$. It is well known that the first-order asymptotic behavior of (1.1) changes radically depending on whether the parameter $Y$ of the process $L$ is smaller or larger than 1 (see, e.g., [15]). We focus here on the latter case, which arguably is more relevant for financial applications in light of some recent empirical evidence based on high-frequency data supporting this assumption (cf. [1]). For $1 < Y < 2$, the first order asymptotic behavior of the at-the-money (ATM) option price (i.e., $\kappa_t = 0$ in (1.1)) for all $t \geq 0$ in short-time takes the form:

$$\lim_{t \to 0} t^{-\frac{1}{2}} \frac{1}{S_0} E\left[ (S_t - S_0)^+ \right] = E(\bar{Z}^+),$$

(1.3)

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where \( Z \) is a symmetric \( \alpha \)-stable random variable under \( \mathbb{P} \). When \( \sigma \neq 0 \), \( Z \sim \mathcal{N}(0, \sigma^2) \) (\( \alpha = 2 \)) and, thus, \( \mathbb{E}(Z^+) = \sigma / \sqrt{2\pi} \) (see [13] and [15]). When \( \sigma = 0 \), \( \alpha = Y \) and the characteristic function of \( Z \) is explicitly given (see [5] and [15]) by

\[
\mathbb{E}(e^{iuZ}) = \exp \left\{ -2C\Gamma(-Y) \left| \cos \left( \frac{\pi Y}{2} \right) \right| |u|^Y \right\}.
\]

In that case, (see (25.6) in [14]),

\[
\mathbb{E}(Z^+) = \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{Y} \right) \left( 2C\Gamma(-Y) \left| \cos \left( \frac{\pi Y}{2} \right) \right| \right)^\frac{1}{Y}.
\]

Moreover (see [7]), in the pure-jump CGMY case (\( \sigma = 0 \)), the second-order asymptotic behavior of the ATM call option price (i.e., \( \kappa = 0 \) in (1.1), for all \( t \geq 0 \)) in short-time is of the form

\[
\frac{1}{S_0} \mathbb{E} [(S_t - S_0)^+] = d_1 t^\frac{1}{Y} + d_2 t + o(t), \quad t \to 0,
\]

while in the case of a non-zero independent Brownian component (\( \sigma \neq 0 \)), it is

\[
\frac{1}{S_0} \mathbb{E} [(S_t - S_0)^+] = d_1 t^\frac{1}{Y} + d_2 t^\frac{3-Y}{2} + o \left( t^\frac{3-Y}{2} \right), \quad t \to 0,
\]

for different constants \( d_1 \) and \( d_2 \), which are explicitly given in the sequel. For extensions of these results to a more general class of Lévy processes, we refer the reader to [7].

In this note, we derive the third-order asymptotic behavior of the close-to-the-money option prices (1.1) in the CGMY model both with and without a Brownian component, when the log-moneyness \( \kappa_t \) converges to 0 (so that the strike \( K \) converges to \( S_0 \) at a suitable rate, as the maturity \( t \) goes to 0). In particular, we obtain the third-order asymptotic behavior of the ATM option prices (i.e., \( \kappa_t = 0 \) in (1.1) for all \( t \geq 0 \)) under both models. Moreover, these results are in line with the intuitive idea that if \( K \) is close enough to \( S_0 \) (relative to the time-to-expiration \( t \)), then the third-order expansion for ATM option prices is still accurate. See also [11] and [8] for other pieces of work where close-to-the-money asymptotics of first and second order are proposed.

There is an important motivation to consider the third-order expansion. As shown in the numerical examples provided in [7, Section 6], though being a significant improvement over the first-order expansion, in some cases, the second-order expansion is not that accurate unless \( t \) is extremely small, particularly under the presence of a Brownian component. As shown in the sequel (see Figure 1 below), the third order expansions, derived here, can significantly improve the approximation accuracy. Moreover, in the same way as the asymptotic behavior of the leading term substantially changes when \( Y \) transitions at 1, we uncover a similar phenomenon for the third order term, but this time when \( Y \) transitions at 3/2. This identifies the value of \( Y = 3/2 \) as another transition point for the asymptotic behavior of ATM option prices.

As in [7], an important ingredient in our approach is a change of probability measure under which the process \( (L_t)_{t \geq 0} \) becomes a stable Lévy process, which, in turn, enable us to exploit some precise high-order asymptotic results for the transition density and the tail distribution of such processes. However, it is important to emphasize that the generalization from the second to the third order is quite intricate and requires new results and techniques beyond those used in [7]. For instance, an important step in obtaining the asymptotic expansion in the presence of a nonzero Brownian component \( W \) is to determine the short-time asymptotic behavior of the following type of quantities:

\[
R_t^{(k)} := \int_0^\infty \mathbb{E} \left[ (\sigma W_1)^k 1_{\{0 \leq \sigma W_1 \leq tz\}} \right] (p_Z(z) - Cz^{-Y-1}) \, dz, \quad k = 0, 1,
\]

where \( p_Z \) is the density of a symmetric stable random variable \( Z \) with Lévy density \( C|z|^{-Y-1} \) so that (see (14.34) in [14])

\[
p_Z(z) = Cz^{-Y-1} + Dz^{-2Y-1} + o \left( z^{-2Y-1} \right), \quad z \to \infty,
\]
for an appropriate constant \( D \) (see (2.16) below). A natural approach to attack this problem would go along the following lines:

\[
R^{(k)}_t = t^{-1} \int_0^\infty \mathbb{E} \left[ (\sigma W_1)^k \mathbf{1}_{\{0 \leq \sigma W_1 \leq u\}} \right] \left[ p_Z(t^{-1} u) - C(t^{-1} u)^{-Y - 1} \right] \, du
\]

\[
\sim Dt^{2Y} \int_0^\infty \mathbb{E} \left[ (\sigma W_1)^k \mathbf{1}_{\{0 \leq \sigma W_1 \leq u\}} \right] u^{-2Y - 1} \, du \quad (t \to 0)
\]

\[
= -D^{2k-2}Y / 4Y t^{2Y} \mathbb{E} \left( |W_1|^{k-2Y} \right),
\]

where in the last equality, Fubini’s theorem and the symmetry of \( W_1 \) would be used. However, the fact that the expectation in the last expression above is infinite, since \( k - 2Y < -1 \) when \( Y > 1 \), makes the above argument useless. As it turns out, with Fourier analysis techniques for tempered distributions, we show that

\[
R^{(0)}_t \sim t^{2Y-1} E^{(0)}, \quad R^{(1)}_t \sim t^{2Y} E^{(1)}, \quad t \to 0,
\]

for some explicitly defined constants \( E^{(k)}, k = 0, 1 \).

Clearly, a method which is able to yield asymptotics of arbitrary order for a relatively general model is desirable. This is indeed achievable for out-the-money option prices (cf. [5] and [6]), but asymptotics for ATM option prices are notoriously technically hard to obtain and attempts to extend the results in this manuscript to even the fourth order or a more general tempered-stable-like process have yet to be successful.

The remaining of the paper is organized as follows. Section 2 contains preliminary results on the CGMY model, some probability measure transformations, and asymptotic results for stable Lévy processes which will be needed throughout the paper. Section 3 establishes the third-order asymptotics of close-to-the-money call option prices, some probability measure transformations, and asymptotic results for stable Lévy processes which will be needed throughout the paper. Section 3 also establishes the third-order asymptotics of close-to-the-money call option prices, some probability measure transformations, and asymptotic results for stable Lévy processes which will be needed throughout the paper. The proofs of our main results are deferred to the Appendix.

## 2 Setup and preliminary results

Throughout, \( W = (W_t)_{t \geq 0} \) and \( L = (L_t)_{t \geq 0} \) respectively stand for a standard Brownian motion and a CGMY Lévy process independent of each other (cf. [4]) defined on a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). As usual, we denote the parameters of \( L \) by \( C, G, M > 0 \), and \( Y \in (0, 2) \) so that the Lévy measure of \( L \) is given by

\[
\nu(dx) = \left( \frac{Ce^{-Mx}}{|x|^{1+Y}} \mathbf{1}_{\{x > 0\}} + \frac{Ce^{Gx}}{|x|^{1+Y}} \mathbf{1}_{\{x < 0\}} \right) dx.
\]

Hereafter, we assume \( Y \in (1, 2) \), \( M > 1 \), zero interest rate, and that \( \mathbb{P} \) is a martingale measure for the exponential Lévy model (1.2) with log-return process \( X_t := L_t + \sigma W_t, t \geq 0 \). In particular, the characteristic function of \( X_1 \) is given by

\[
\mathbb{E} \left( e^{iuX_1} \right) = \exp \left\{ icu - \frac{\sigma^2 u^2}{2} + CT(-Y) \left( (M - iu)^Y + (G + iu)^Y - M^Y - G^Y \right) \right\}, \quad (2.1)
\]

with \( c := -CT(-Y) \left( (M - 1)^Y + (G + 1)^Y - M^Y - G^Y \right) - \sigma^2 / 2 \). The following notation is used in what follows:

\[
M^* = M - 1, \quad G^* = G + 1, \quad e^* = c + \sigma^2, \quad \varphi(x) := M^* x \mathbf{1}_{\{x > 0\}} - G^* x \mathbf{1}_{\{x < 0\}}, \quad \nu^*(dx) = e^* \nu(dx). \quad (2.2)
\]

We will make use of two density transformations of the Lévy process (see [14, Definition 33.4]). Hereafter, \( \mathbb{P}^* \) and \( \mathbb{P}^{\dagger} \) are probability measures on \((\Omega, \mathcal{F})\) such that for any \( t \geq 0 \):

\[
\frac{d\mathbb{P}^*|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{X_t}, \quad \frac{d\mathbb{P}^{|_{\mathcal{F}_t}}}{d\mathbb{P}^*|_{\mathcal{F}_t}} = e^{U_t}, \quad (2.3)
\]
where

\[ U_t := \lim_{\epsilon \to 0} \left[ \sum_{s \leq t: |\Delta X_s| > \epsilon} \varphi(\Delta X_s) - t \int_{|x| > \epsilon} (e^{\varphi(x)} - 1) \nu^*(dx) \right]. \]

Throughout, \( \mathbb{E}^* \) and \( \mathbb{E}^\ast \) denote the expectations under \( \mathbb{P}^* \) and \( \mathbb{P}^\ast \), respectively.

From the density transformation and the Lévy-Itô decomposition of a Lévy process (see [14, Theorems 19.2 and Theorem 33.1]), \((X_t)_{t \geq 0}\) can be written as

\[ X_t = L_t^* + \sigma W_t^*, \quad t \geq 0, \tag{2.4} \]

where, under \( \mathbb{P}^* \), \((W_t^*)_{t \geq 0}\) is again a Wiener process while \((L_t^*)_{t \geq 0}\) is still a CGMY process, independent of \( W^* \), but with parameters \( C, Y, M = M^* \), and \( G = G^* \). The Lévy triplet of \((X_t)_{t \geq 0}\) under \( \mathbb{P}^* \) is given by \((b^*, (\sigma^*)^2, \nu^*)\) with \( \sigma^* := \sigma \) and

\[ b^* := c^* - \int_{|x| > 1} x \nu^*(dx) - C \gamma Y (Y^{-1} - (G^*)^{Y^{-1}} - (G^*)^{Y^{-1}}). \tag{2.5} \]

Similarly, under the measure \( \mathbb{P}^\ast \), the process \((L_t^*)_{t \geq 0}\) remains a stable Lévy process while \((W_t^*)_{t \geq 0}\) remains a Wiener process independent of \( L^* \). Concretely, setting

\[ \tilde{\nu}(dx) := C|x|^{-Y-1}dx, \quad \tilde{b} = b^* + \int_{|x| \leq 1} x(\tilde{\nu} - \nu^*)(dx), \]

under \( \mathbb{P}^\ast \), \((X_t)_{t \geq 0}\) is a Lévy process with Lévy triplet \((\tilde{b}, \sigma^2, \tilde{\nu})\). In particular,

\[ \tilde{\gamma} := \mathbb{E}(X_1) = -CT(-Y) ((M - 1)^Y + (G + 1)^Y - M^Y - G^Y) + \frac{\sigma^2}{2}, \tag{2.6} \]

and the centered process

\[ Z_t := L_t^* - t\tilde{\gamma}, \quad t \geq 0, \tag{2.7} \]

is symmetric and strictly \( Y \)-stable under \( \mathbb{P}^\ast \), and is therefore self-similar; i.e., \((t^{-1/Y}Z_{ut})_{u \geq 0} \overset{\mathbb{P}}{=} (Z_u)_{u \geq 0}\), for any \( t > 0 \).

The process \((U_t)_{t \geq 0}\) can be expressed in terms of the jump-measure \( N(dt, dx) := \#\{(s, \Delta X_s) \in dt \times dx\} \) of \((X_t)_{t \geq 0}\) and its compensator \( \bar{N}(dt, dx) := N(dt, dx) - \tilde{\nu}(dx)dt \) (under \( \mathbb{P}^\ast \)), namely,

\[ U_t := \bar{U}_t + \eta t := M^* \bar{U}_t^{(p)} - G^* \bar{U}_t^{(n)} + \eta t, \quad t \geq 0, \tag{2.8} \]

where

\[ \bar{U}_t^{(p)} := \int_0^t \int_{(0, \infty)} x \bar{N}(ds, dx), \quad \bar{U}_t^{(n)} := \int_0^t \int_{(-\infty, 0)} x \bar{N}(ds, dx), \tag{2.9} \]

\[ \eta := C \int_0^\infty \left( e^{-M^* x} - 1 + M^* x \right) x^{-Y-1} dx + C \int_{-\infty}^0 \left( e^{G^* x} - 1 - G^* x \right) x^{-Y-1} dx \]

\[ = CT(-Y) \left[ (M - 1)^Y + (G + 1)^Y \right]. \tag{2.10} \]

Finally, let us also note the following decomposition of the process \( X \) in terms of the previously defined processes:

\[ X_t = Z_t + t\tilde{\gamma} + \sigma W_t^* = \bar{U}_t^{(p)} + \bar{U}_t^{(n)} + t\tilde{\gamma} + \sigma W_t^*, \quad t \geq 0. \tag{2.11} \]

To conclude this section, we recall some well-known results of stable Lévy processes needed in the sequel. First, under \( \mathbb{P}^\ast \), \((\bar{U}_t^{(p)})_{t \geq 0}\) and \((-\bar{U}_t^{(n)})_{t \geq 0}\) are independent and identically distributed one-sided \( Y \)-stable processes with
scale, skewness, and location parameters given by $C|\cos(\pi Y/2)|\Gamma(-Y)$, 1, and 0, respectively. The common transition density of $U_1^{(p)}$ and $-U_1^{(n)}$ is hereafter denoted by $p_U(t, x)$, $t \geq 0$. The following second-order approximation of $p_U(1, x)$ is well-known\(^1\) (see e.g., (14.34) in [14]):

$$p_U(1, x) = Cx^{-Y-1} - \frac{C^2}{2\pi} \sin(2\pi Y)\Gamma(2Y + 1)\Gamma^2(-Y)x^{-2Y-1} + o(x^{-2Y-1}), \quad x \to \infty. \quad (2.12)$$

In particular,

$$\widetilde{P} \left( \tilde{U}_{1}^{(p)} \geq x \right) = \widetilde{P} \left( -\tilde{U}_{1}^{(n)} \geq x \right) = C \frac{Y}{x} - \frac{C^2}{2\pi} \sin(2\pi Y)\Gamma(2Y)\Gamma^2(-Y)x^{-2Y} + o(x^{-2Y}), \quad x \to \infty. \quad (2.13)$$

The following result sharpens (2.12) and (2.13). Its proof is presented in the Appendix.

**Lemma 2.1.** There exist constants $0 < K_1, K_2 < \infty$ such that, for any $x > 0$,

(i) $\widetilde{P} \left( \tilde{U}_{1}^{(p)} \geq x \right) \leq K_1 x^{-Y}$, (ii) $\left| \widetilde{P} \left( \tilde{U}_{1}^{(p)} \geq x \right) - C \frac{Y}{x}x^{-Y} \right| = \left| \widetilde{P} \left( -\tilde{U}_{1}^{(n)} \geq x \right) - C \frac{Y}{x}x^{-Y} \right| \leq K_2 x^{-2Y}. \quad (2.14)$

Similarly, the tail distribution and the probability density of $Z_1$, hereafter denoted by $p_Z(1, z)$, admit the following asymptotic behaviors\(^2\) (see (14.34) in [14]),

$$\widetilde{P}(Z_1 \geq z) = C z^{-Y} - \frac{C^2}{\pi Y} \cos^2\left(\frac{\pi Y}{2}\right) \Gamma(2Y + 1)\Gamma^2(-Y)z^{-2Y} + o(z^{-2Y}), \quad z \to \infty, \quad (2.15)$$

$$p_Z(1, z) = Cz^{-Y-1} - \frac{2C^2}{\pi} \sin(\pi Y)\cos^2\left(\frac{\pi Y}{2}\right) \Gamma(2Y + 1)\Gamma^2(-Y)z^{-2Y-1} + o(z^{-2Y-1}), \quad z \to \infty. \quad (2.16)$$

As in the proof of Lemma 2.1, there exists a constant $0 < K_3 < \infty$ such that, for any $z > 0$,

$$\widetilde{P}(Z_1 \geq z) \leq K_3 z^{-Y}. \quad (2.17)$$

Finally, the following identity will also be of use:

$$\mathbb{E} \left( e^{-\tilde{U}_1} \right) = \mathbb{E} \left( e^{-t^{1/Y}\tilde{U}_1} \right) = \mathbb{E}^* \left( e^{-t^{1/Y}M^*\tilde{U}_1^{(p)}} \right) \mathbb{E}^* \left( e^{t^{1/Y}G^*\tilde{U}_1^{(n)}} \right) = e^{\kappa t}, \quad t \geq 0. \quad (2.18)$$

### 3 The main results

In this section, we present the high-order asymptotic behavior for near at-the-money call option prices for both the pure-jump ($\sigma = 0$) and the mixed case ($\sigma \neq 0$) models. The expansion for the latter is more explicit and of greater use for financial application in light of some recent empirical evidence, based on high-frequency data, which tends to support a mixed model over either a pure-jump or a purely-continuous one (cf. [2]). However, since the proof for the mixed model is more intricate, we first present the pure-jump case for easiness of exposition. The proofs of all the results are deferred to the Appendix.

**Theorem 3.1.** Let $t \to \kappa_t$ be a deterministic function such that

$$\kappa_t = \alpha t + e_{2} t^{2-1/Y} + o(t^{2-1/Y}), \quad as \quad t \to 0,$$

for some constants $e_1, e_2 \in \mathbb{R}$. Then, under the exponential CGMY model (1.2) without Brownian component,

$$\Pi(t) := \frac{1}{S_0} \mathbb{E} \left[ \left( S_t - S_0 e^{\kappa_t} \right)^2 \right] = d_{11} t^{\frac{\gamma}{2}} + d_{22} t + d_{31} t^{2-\frac{\gamma}{2}} + d_{32} t^{\frac{\gamma}{2}} + o \left( t^{2-\frac{\gamma}{2}} \right) + o \left( t^{\frac{\gamma}{2}} \right), \quad t \to 0, \quad (3.1)$$

\(^1\)In terms of the parameterization $(\alpha, \beta, \tau, c)$ introduced in Definition 14.16 in [14], $(\alpha, \beta, \tau, c)$ of $U_1^{(p)}$ and $-U_1^{(n)}$ is $(Y, 1, 0, C|\cos(\pi Y/2)|\Gamma(-Y))$.

\(^2\)In terms of the parametrization in [14, Definition 14.16], $(\alpha, \beta, \tau, c)$ of $Z_1$ therein is $(Y, 0, 0, 2C|\cos(\pi Y/2)|\Gamma(-Y))$.\]
where
\[ d_1 := \tilde{E}(Z_t^+) = \frac{1}{\pi} \left( 1 - \frac{1}{Y} \right) \left( 2CT(-Y) \mid \cos \left( \frac{\pi Y}{2} \right) \right)^\frac{1}{Y}, \]
\[ d_2 := \frac{CT(-Y)}{2} [ (M - 1)^Y - M^Y - (G + 1)^Y + G^Y ] - \frac{e_1}{2}, \]
\[ d_{31} := \frac{(\gamma - e_1)^2}{2} p_Z(1,0) - \frac{e_2}{2} = \frac{(\gamma - e_1)^2}{2\pi} \Gamma \left( 1 + \frac{1}{Y} \right) \left( -2CT(-Y) \cos \left( \frac{\pi Y}{2} \right) \right)^{1/2} - \frac{e_2}{2}, \]
\[ d_{32} := -\frac{1}{2} \tilde{E} \left[ \left( Z_t^+ + \tilde{U}_1 \right)^2 1_{[Z_t^+ + \tilde{U}_1 \leq 0]} \right] - \int_0^\infty w \left[ \tilde{p} \left( Z_t^+ + \tilde{U}_1 \geq w \right) - \frac{CM^Y}{Yw^Y} - \frac{C(G + 1)^Y}{Yw^Y} \right] dw. \]

In particular, if \( 1 < Y < 3/2 \), the third-order term is \( d_{31}t^{2-Y} \), while if \( 3/2 < Y < 2 \), the third-order term is \( d_{32}t^{2-Y} \).

The following result provides the asymptotic behavior of the corresponding Black-Scholes implied volatility, which hereafter is denoted by \( \hat{\sigma} \).

**Proposition 3.2.** Let \( d_3 = d_{31}1_{\{Y \leq 3/2\}} + d_{32}1_{\{Y > 3/2\}} \). Then, under the exponential CGMY model (1.2) without Brownian component, as \( t \to 0 \),
\[ \frac{1}{\sqrt{2\pi}} \hat{\sigma}(t) = \begin{cases} 
   d_1 t^{\frac{3}{Y} - \frac{1}{2}} + d_2 t^{\frac{3}{Y} - \frac{1}{2}} + o \left( t^{\frac{3}{Y} - \frac{1}{2}} \right), & \text{if } 1 < Y \leq \frac{3}{2}, \\
   d_1 t^{\frac{3}{Y} - \frac{1}{2}} + d_2 t^{\frac{3}{Y} - \frac{1}{2}} + d_3 t^{\frac{3}{Y} - \frac{1}{2}} + o \left( t^{\frac{3}{Y} - \frac{1}{2}} \right), & \text{if } \frac{3}{2} < Y < 2.
\end{cases} \]

We now analyze the case of a CGMY model with non-zero Brownian component. In that instance, it was shown in [7, Section 5] that, the second order correction term for the ATM European call option price is given by
\[ \frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0)^+ \right] = d_1 t^{\frac{3}{Y} - \frac{1}{2}} + d_2 t^{\frac{3}{Y} - \frac{1}{2}} + o \left( t^{\frac{3}{Y} - \frac{1}{2}} \right), \quad t \to 0, \]
with
\[ d_1 := \mathbb{E}^* \left[ (\sigma W_t^+)^+ \right] = \frac{\sigma}{\sqrt{2\pi}}, \quad d_2 := \frac{C\sigma^{1-Y}}{Y(Y - 1)} \mathbb{E} \left[ |W_t^+|^{1-Y} \right] = \frac{C^2}{\sqrt{\pi Y(Y - 1)}} \Gamma \left( 1 - \frac{Y}{2} \right). \]

As seen in the previous expressions, the first-order term only synthesizes the information about the continuous volatility parameter \( \sigma \), while the second-order term incorporates also the information on the degree of jump activity \( Y \) and the overall jump-intensity parameter \( C \). However, these two-terms do not reflect the relative intensities of the negative or positive jumps (as given by the parameters \( G \) and \( M \), respectively). This fact suggests the need of a high-order approximation as described in the following theorem.

**Theorem 3.3.** Let \( t \to \kappa_t \) be a deterministic function such that
\[ \kappa_t = e_1 t + e_2 t^{\frac{3}{Y} - Y} + o \left( t^{1+(\frac{3}{Y} - Y)} \right), \quad t \to 0, \]
for some constants \( e_1, e_2 \in \mathbb{R} \). Let also
\[ d_{31} := -CT(-Y) \left[ (G + 1)^Y - G^Y \right] + \frac{\gamma - e_1}{2}, \]
\[ d_{32} := -\frac{1}{\pi} \sigma^{1-2Y} C^2 \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2 (-Y) 2^{Y-\frac{1}{2}} \Gamma \left( Y - \frac{1}{2} \right) - \frac{e_2}{2}. \]

Then, under the exponential CGMY model (1.2) with non-zero Brownian component,
\[ \Pi(t) := \frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0 e^{\kappa_t})^+ \right] = d_1 t^{\frac{3}{Y} - \frac{1}{2}} + d_2 t^{\frac{3}{Y} - \frac{1}{2}} + d_{31} t + d_{32} t^{\frac{3}{Y} - Y} + o(t) + o \left( t^{\frac{3}{Y} - Y} \right), \quad t \to 0, \]
where \( d_1 \) and \( d_2 \) are given in (3.8). In particular, if \( 1 < Y < 3/2 \), the third-order term is \( d_{31} t \), while if \( 3/2 < Y < 2 \), the third-order term is \( d_{32} t^{\frac{3}{Y} - Y} \).
Figure 1: Comparisons of ATM call option prices with the first-, second-, and third-order approximations for $Y = 1.2$ (left panel) and $Y = 1.6$ (right panel). In both cases, $C = 0.5$, $G = 2$, $M = 3.6$, and $\sigma = 0.5$.

**Remark 3.4.** The third order expansion can significantly improve the approximation accuracy as illustrated in Figure 1. Interestingly, the third order term only shed light on the parameter $G$, which controls the relative intensity of negative jumps, but only if $Y < 1/2$.

Our final proposition gives the small-time asymptotic behavior for the Black-Scholes implied volatility, denoted again by $\hat{\sigma}$, corresponding to the option prices of Theorem 3.3. Unlike the pure-jump case, we can only derive the second order asymptotic behavior using Theorem 3.3. In fact, the first order term of the call option price under the generalized CGMY model is the same as the one under the Black-Scholes model. The third-order term of $\hat{\sigma}$ requires higher order asymptotics of the ATM call option price.

**Proposition 3.5.** Let $d_3 = d_{31} 1_{\{Y \leq 3/2\}} + d_{32} 1_{\{Y \geq 3/2\}}$. Then, under the exponential CGMY model (1.2) with non-zero Brownian component, as $t \to 0$,

$$\frac{1}{\sqrt{2\pi}} \hat{\sigma}(t) = \begin{cases} \sigma + d_2 t^{1-Y} + o(t^\frac{1}{2}), & \text{if } 1 < Y \leq \frac{3}{2}, \\ \sigma + d_2 t^{1-Y} + d_3 t^2 - Y + o(t^\frac{3}{2}-Y), & \text{if } \frac{3}{2} < Y < 2. \end{cases}$$

### A Proofs

For simplicity, throughout all the proofs, we fix $S_0 = 1$.

**Proof of Lemma 2.1.** From the leading term in the expansion (2.13), there exists $N > 0$ such that, for any $x > 0$,

$$\mathbb{P}\left(\bar{U}_1^{(p)} \geq x\right) = \mathbb{P}\left(\bar{U}_1^{(p)} \geq x\right) (1_{\{x \geq N\}} + 1_{\{x < N\}}) \leq \frac{2C}{Y} x^{-Y} 1_{\{x \geq N\}} + \frac{NY}{x^Y} 1_{\{x < N\}} \leq (2CY^{-1} + NY) x^{-Y},$$

and the first relationship in (2.14) follows by setting $K_1 = 2CY^{-1} + NY$. Similarly, from (2.13), there exists $N > 0$ such that, for any $x > 0$,

$$\left| \mathbb{P}\left(\bar{U}_1^{(p)} \geq x\right) - \frac{C}{Y} x^{-Y} \right| = \left| \mathbb{P}\left(\bar{U}_1^{(p)} \geq x\right) - \frac{C}{Y} x^{-Y} \right| (1_{\{x \geq N\}} + 1_{\{x < N\}}) \leq C^2 |\sin(2\pi Y)| \Gamma(2Y) \Gamma^2(-Y) x^{-2Y} 1_{\{x \geq N\}} + \left| \mathbb{P}\left(\bar{U}_1^{(p)} \geq x\right) + \frac{C}{Y} x^{-Y} \right| 1_{\{x < N\}} \leq \left( \frac{C^2}{\pi} |\sin(2\pi Y)| \Gamma(2Y) \Gamma^2(-Y) + N^{2Y} + CNY^{-1} \right) x^{-2Y}.$$
The second identity in (2.14) follows by setting $K_2 = C^2 |\sin(2\pi Y)| \Gamma(2Y) / \pi + N^{2Y} + CN^{Y}Y^{-1}$. □

**Proof of Theorem 3.1** Set $\tilde{\gamma}_t := t^{-1/Y}(\tilde{\gamma} - t^{-1}\kappa_t)$ and $\vartheta := -CT(-Y) \left( M^Y + (G^*)^Y \right)$ and note that, in view of (2.6) and (2.10), $d_2 = \vartheta + \eta + \tilde{\gamma}_i/2 - \varepsilon_1/2$. For future reference, it is also convenient to write $\vartheta$ as

$$\vartheta = \frac{C}{\sqrt{Y}} \left[ M^Y + (G^*)^Y \right] \int_0^{\infty} \frac{e^{-t^{1/Y}v} - 1}{t^{1/Y}v} v^{-Y} dv; \quad (A.1)$$

which follows from the identity (see (14.18) in [14]):

$$-Y \Gamma(-Y) = \Gamma(1-Y) = \int_0^{\infty} (e^{-y} - 1) y^{-Y} dy = \int_0^{\infty} \frac{e^{-t^{1/Y}v} - 1}{t^{1/Y}v} v^{-Y} dv. \quad (A.2)$$

Next, note the following decomposition for the option price derived from (2.3), (2.8), (2.11), (2.18), and the fact that $(1 - e^{-x})^+ = 1 - e^{-x^+}$,

$$\Pi(t) = \mathbb{E} \left[ e^{X_t} (1 - e^{-X_t + \kappa_t})^+ \right] = \mathbb{E}^* \left[ \left( 1 - e^{-\tilde{X}_t} \right)^+ \right] = e^{-\eta_t} \mathbb{E} \left[ e^{-\bar{U}_t} \left( 1 - e^{-\tilde{X}_t} \right)^+ \right] = 1 - e^{-\eta_t} \mathbb{E} \left( e^{-\bar{U}_t} - \tilde{X}_t^+ \right),$$

where above $\tilde{X}_t := X_t - \kappa_t$. Set

$$\Delta_1(t) := t^{1/Y} \mathbb{E} \left[ 1 - e^{-\tilde{U}_t + \tilde{X}_t^+} - \left( \tilde{U}_t + \tilde{X}_t^+ \right) \right], \quad \Delta_2(t) := t^{1/Y} \left( \tilde{X}_t^+ - \mathbb{E}(Z_t^+) \right).$$

Then, recalling that $\mathbb{E}(\tilde{U}_t) = 0$ and $\mathbb{E}(Z_t^+) = t^{1/Y} \mathbb{E}(Z_t^+)$, we have the decomposition:

$$A(t) := t^{1/Y} \Pi(t) - \mathbb{E}(Z_t^+) - d_2$$

$$= \left( t^{1/Y} \Delta_1(t) - \vartheta \right) + \left( t^{1/Y} \Delta_2(t) - \frac{\tilde{\gamma}_t - \varepsilon_1}{2} \right) - \frac{e^{-\eta t} - 1}{t} + \eta \mathbb{E} \left( e^{-\vartheta_t} - \tilde{X}_t^+ \right) - \eta \bar{Y} \mathbb{E} \left( \tilde{X}_t^+ \right)$$

$$\quad := A_1(t) + A_2(t) + A_3(t) + A_4(t) - A_5(t). \quad (A.3)$$

We shall prove that $A_1(t) = O(t^{1/Y})$ and $A_2 = O(t^{1/Y})$ and so $t^{1/Y} \Delta_1(t) = O(t)$. These results, in turn, imply that $A_i(t) = O(t) = o(t^{1/Y}) = o(t^{1/Y})$, $i = 3, 4$, and $A_5(t) = o(t^{1/Y}) = o(t^{1/Y}) = o(t^{1/Y})$ since $k_t t^{-1/Y} \to 0$. So, it remains to analyze the asymptotic behaviors of $A_1(t)$ and $A_2(t)$. These two cases are analyzed in two steps:

**Step 1.** Using the identity

$$\mathbb{E} \left[ 1 - e^{-V} - V \right] = \int_0^{\infty} (e^{-y} - 1) \mathbb{P}(V \geq y) dy - \int_0^{\infty} (e^{-y} - 1) \mathbb{P}(V \leq -y) dy,$$

which implies

$$A_1(t) = \left[ \int_0^{\infty} \frac{e^{-t^{1/Y}v} - 1}{t^{1/Y}v} v^{-Y} dv - \vartheta \right] - \int_0^{\infty} \frac{e^{t^{1/Y}v} - 1}{t^{1/Y}v} \mathbb{P} \left( X_t^+ + U_t \geq v \right) dv$$

$$\quad := B_1(t) - B_2(t). \quad (A.4)$$

For $B_2(t)$, using the decompositions (2.8)-(2.11) as well as the self-similarity of $(Z_t, \tilde{U}_t)$ for $t > 0$, and

$$\lim_{t \to 0} t^{1/Y} B_2(t) = \lim_{t \to 0} \int_0^{\infty} \frac{e^{t^{1/Y}v} - 1}{t^{1/Y}v} \mathbb{P} \left( (Z_t + \tilde{\gamma}_t)^+ + \tilde{U}_1 \leq -v \right) dv = \int_0^{\infty} v \mathbb{P} \left( Z_1^+ + \tilde{U}_1 \leq -v \right) dv, \quad (A.5)$$

where the second equality follows from the dominated convergence theorem, which applies in view of the following direct consequences of (2.18):

$$\frac{e^{t^{1/Y}v} - 1}{t^{1/Y}v} \mathbb{P} \left( t^{1/Y} \tilde{X}_t^+ + U_1 \leq -v \right) \leq \frac{e^{t^{1/Y}v} - 1}{t^{1/Y}v} \mathbb{P} \left( \tilde{U}_1 \leq -v \right) \leq ve^{t^{1/Y}v} \mathbb{E} \left( e^{-\vartheta_t} \right) \leq e^{\eta v e^{r - 1}} \leq e^{\eta v - v/2}$$. 

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We now analyze the asymptotic behavior of $B_1(t)$, which is shown to be $O(t^{\gamma - 1})$. To this end, again use the decompositions (2.8)-(2.11) as well as the self-similarity of $\{(Z_t, \bar{U}_t)\}_{t \geq 0}$ to express $t^{1-\gamma} B_1(t)$ as:

$$t^{1-\gamma} B_1(t) = \int_0^\infty \frac{e^{-t^{\gamma} v} - 1}{t^{\gamma}} \left[ \bar{P} \left( Z_1 + \bar{\gamma}_t > 0, Z_1 + \bar{\gamma}_t + \bar{U}_1 \geq v \right) - \frac{C M Y}{Y v^Y} \right] dv$$

$$+ \int_0^\infty \frac{e^{-t^{\gamma} v} - 1}{t^{\gamma}} \left[ \bar{P} \left( Z_1 + \bar{\gamma}_t \leq 0, \bar{U}_1 \geq v \right) - \frac{C (G^*)^Y}{Y v^Y} \right] dv$$

$$:= B_{11}(t) + B_{12}(t),$$

(A.6)

where we have used (A.1). As suggested from the previous equation, the limit of each of the terms therein can be obtained by passing $\lim_{t \to 0}$ into the various integrals to get

$$\lim_{t \to 0} t^{1-\gamma} B_1(t) = - \int_0^\infty v \left[ \bar{P} \left( Z_1^+ + \bar{U}_1 \geq v \right) - \frac{C M Y}{Y v^Y} - \frac{C (G^*)^Y}{Y v^Y} \right] dv.$$

We now proceed to show that the latter operation is indeed valid. We begin with $B_{11}(t)$, applying first the decomposition

$$\bar{P} \left( Z_1 + \bar{\gamma}_t > 0, Z_1 + \bar{\gamma}_t + \bar{U}_1 \geq v \right) = \bar{P} \left( \bar{U}_1^{(p)} + \bar{U}_1^{(n)} + \bar{\gamma}_t > 0, M \bar{U}_1^{(p)} - G \bar{U}_1^{(n)} + \bar{\gamma}_t \geq v \right)$$

$$= \bar{P} \left( \bar{U}_1^{(p)} \geq \frac{v + G \bar{U}_1^{(n)} - \bar{\gamma}_t}{M}, -\bar{U}_1^{(n)} < \frac{v + M^* \bar{\gamma}_t}{M + G} \right)$$

$$+ \bar{P} \left( \bar{U}_1^{(p)} + \bar{\gamma}_t \geq -\bar{U}_1^{(n)} \geq \frac{v + M^* \bar{\gamma}_t}{M + G} \right),$$

where we have used that $Z_1 = \bar{U}_1^{(p)} + \bar{U}_1^{(n)}$ and $\bar{U}_1 = M^* \bar{U}_1^{(p)} - G^* \bar{U}_1^{(n)}$. Hence,

$$B_{11}(t) := \int_0^\infty \frac{e^{-t^{\gamma} v} - 1}{t^{\gamma}} \left[ \bar{P} \left( \bar{U}_1^{(p)} \geq \frac{v + G \bar{U}_1^{(n)} - \bar{\gamma}_t}{M}, -\bar{U}_1^{(n)} < \frac{v + M^* \bar{\gamma}_t}{M + G} \right) - \frac{C M Y}{Y v^Y} \right] dv$$

(A.7)

$$+ \int_0^\infty \frac{e^{-t^{\gamma} v} - 1}{t^{\gamma}} \left[ \bar{P} \left( \bar{U}_1^{(p)} \geq -\bar{U}_1^{(n)} \geq \frac{v + M^* \bar{\gamma}_t}{M + G} \right) \right] dv.$$ 

(A.8)

By (2.14-i), for any $v > 0$ and $t$ small enough (so that $G^*|\bar{\gamma}_t| < 1$ and $M^*|\bar{\gamma}_t| < 1$), the expression inside the integral in (A.8), which we denote $b^{(2)}_{11}(t; v)$, is such that

$$\left| b^{(2)}_{11}(t; v) \right| \leq v \bar{P} \left( \bar{U}_1^{(p)} \geq \frac{v + M^* \bar{\gamma}_t}{M + G} + \bar{\gamma}_t \right) \bar{P} \left( \bar{U}_1^{(n)} \geq \frac{v + M^* \bar{\gamma}_t}{M + G} \right) \leq v 1_{\{v \leq 1\}} + v 1_{\{v > 1\}} \min \left\{ 1, K_1^2 (M + G)^2 Y v^{-2Y} \right\},$$

where $K_1 \in (0, \infty)$ is given as in (2.14-i). Hence, by the dominated convergence theorem,

$$\lim_{t \to 0} \int_0^\infty b^{(2)}_{11}(t; v) dv = - \int_0^\infty v \bar{P} \left( \bar{U}_1^{(p)} \geq \frac{v + M^* \bar{\gamma}_t}{M + G} \right) dv.$$ 

(A.9)

We now bound the expression inside the integral in (A.7), which we denote $b^{(1)}_{11}(t; v)$. It suffices to consider $v > 1$, since $|b^{(1)}_{11}(t; v)| \leq v(1 + CY^{-1} M Y v^{-Y})$, which is integrable on $\{v \leq 1\}$. We also let $t$ be small enough so that $|\bar{\gamma}_t| < 1$, $G^*|\bar{\gamma}_t| < 1$, and $M^*|\bar{\gamma}_t| < 1$. Then, for any $v > 1$,

$$\left| b^{(1)}_{11}(t; v) \right| \leq v \int_{-\infty}^{v + M^* \bar{\gamma}_t} p_U(1, y) \left| \bar{P} \left( \bar{U}_1^{(p)} \geq \frac{y - G \bar{U}_1^{(n)} - \bar{\gamma}_t}{M} \right) - \frac{C M Y}{Y v^Y} \right| dy$$

$$\leq v \int_{-\infty}^{v + M^* \bar{\gamma}_t} p_U(1, y) \left| \bar{P} \left( \bar{U}_1^{(p)} \geq \frac{y - G \bar{U}_1^{(n)} - \bar{\gamma}_t}{M} \right) \right| dy$$

$$+ v \int_{-\infty}^{v + M^* \bar{\gamma}_t} p_U(1, y) \frac{C M Y}{Y v^Y - 1} \left| (v - G \bar{U}_1^{(n)} - \bar{\gamma}_t)^{-Y} - v^{-Y} \right| dy + \frac{C M Y}{Y v^Y - 1} \bar{P} \left( \bar{U}_1^{(p)} \geq \frac{v + M^* \bar{\gamma}_t}{M + G} \right)$$

$$:= D^{(1)}_t(v) + D^{(2)}_t(v) + D^{(3)}_t(v).$$

(A.10)
Next, since
\[
\frac{v - Gy - \tilde{\gamma}_t}{M} \geq \frac{v - G\frac{v + M^*\tilde{\gamma}_t}{M + G} - \tilde{\gamma}_t}{M} = \frac{v - G^*\tilde{\gamma}_t}{M + G} > 0, \quad \text{for any } y \leq \frac{v + M^*\tilde{\gamma}_t}{M + G}, \quad (A.11)
\]
the first integral in (A.10) can be bounded, using (2.14-ii), via:
\[
D_t^{(1)}(v) \leq K_2 v \int_{-\infty}^{v + M^*\tilde{\gamma}_t} f_y(1, y) M^2 Y \left(\frac{v - Gy - \tilde{\gamma}_t}{M + G}\right)^2 dy \leq K_2 (M + G)^2 Y v^{1 - 2Y}, \quad \text{for any } v > 1, \quad (A.12)
\]
where $K_2 \in (0, \infty)$ is given as in (2.14-ii). Moreover, using the convexity and monotonicity of the function $f(x) = x^{-Y}$ on $(0, \infty)$ and (A.11), the second integral in (A.10) can be upper estimated as
\[
D_t^{(2)}(v) \leq C M^Y v \int_{-\infty}^{v + M^*\tilde{\gamma}_t} f_y(1, y) v^{-Y - 1} |Gy + \tilde{\gamma}_t| dy \leq C M^Y v^{-Y} \left(\frac{G^*}{Y v^Y} \left|U_1^{(p)}\right| + 1\right). \quad (A.13)
\]
Finally, by (2.14-i), the last term in (A.10) can be upper bounded via
\[
D_t^{(3)}(v) \leq K_1 C M^Y Y^{-1} v^{1 - 2Y}, \quad \text{for any } v > 1. \quad (A.14)
\]
Combining (A.10) and (A.12)-(A.14), and by the dominated convergence theorem,
\[
\lim_{t \to 0} \int_0^\infty b_{11}^{(1)}(t; v) dv = -\int_0^\infty v \left[\mathbb{P} \left(\tilde{U}_1^{(p)} \geq \frac{v + G\tilde{U}_1^{(n)}}{M}, \tilde{U}_1^{(n)} \leq \frac{v}{M + G} - \frac{C M^Y}{Y v^Y}\right)\right] dv. \quad (A.15)
\]
Putting together (A.9) and (A.15), we obtain
\[
\lim_{t \to 0} B_{11}(t) = -\int_0^\infty v \mathbb{P} \left(\tilde{U}_1^{(p)} \geq \frac{v + G\tilde{U}_1^{(n)}}{M}, \tilde{U}_1^{(n)} \leq \frac{v}{M + G} - \frac{C M^Y}{Y v^Y}\right) dv - \int_0^\infty v \mathbb{P} \left(\tilde{U}_1^{(p)} \geq \frac{v + G\tilde{U}_1^{(n)}}{M}, \tilde{U}_1^{(n)} \leq \frac{v}{M + G} - \frac{C M^Y}{Y v^Y}\right) dv.
\]
Applying the same arguments to the decomposition
\[
\mathbb{P} \left(Z_1 + \tilde{\gamma}_t \leq 0, \tilde{U}_1 \geq v\right) = \mathbb{P} \left(-\tilde{U}_1^{(p)} - \tilde{\gamma}_t \geq \tilde{U}_1^{(p)} \geq \frac{v + G^*\tilde{\gamma}_t}{M + G}\right) \mathbb{P} \left(-\tilde{U}_1^{(n)} \geq \frac{v - M^*\tilde{U}_1^{(p)}}{G^*}, \tilde{U}_1^{(p)} \leq \frac{v - G^*\tilde{\gamma}_t}{M + G}\right),
\]
it can be shown that
\[
\lim_{t \to 0} B_{12}(t) = -\int_0^\infty v \mathbb{P} \left(-\tilde{U}_1^{(p)} \geq \tilde{U}_1^{(p)} \geq \frac{v}{M + G}\right) dv - \int_0^\infty v \mathbb{P} \left(-\tilde{U}_1^{(n)} \geq \frac{v - M^*\tilde{U}_1^{(p)}}{G^*}, \tilde{U}_1^{(p)} \leq \frac{v}{M + G} - \frac{C (G^*)^Y}{Y v^Y}\right) dv
\]
\[
= -\int_0^\infty v \left[\mathbb{P} \left(Z_1 < 0, \tilde{U}_1 \geq v - \frac{C (G^*)^Y}{Y v^Y}\right)\right] dv. \quad (A.17)
\]
Combining (A.6), (A.16), and (A.17), we obtain
\[
\lim_{t \to 0} t^{1 - \frac{P}{2}} B_1(t) = -\int_0^\infty v \left[\mathbb{P} \left(Z_1^* + \tilde{U}_1 \geq v - \frac{C M^Y}{Y v^Y} - \frac{C (G^*)^Y}{Y v^Y}\right)\right] dv. \quad (A.18)
\]
Combining (A.4), (A.5) and (A.18) together with the identity $\int_0^\infty v \mathbb{P} (V \leq -v) dv = \mathbb{E} ((V^-)^2) / 2$, we get
\[
\lim_{t \to 0} t^{1 - \frac{P}{2}} A_1(t) = -\int_0^\infty v \left[\mathbb{P} \left(Z_1^* + \tilde{U}_1 \geq v - \frac{C M^Y}{Y v^Y} - \frac{C (G^*)^Y}{Y v^Y}\right)\right] dv - \frac{1}{2} \mathbb{E} \left[\left(\left(Z_1^* + \tilde{U}_1\right)^-\right)^2\right] := d_{32}. \quad (A.19)
\]
Step 2. Now, we analyze the behavior of \( A_2 = t^{\frac{1}{4}} \Delta_2(t) - (\hat{\gamma} - e_1)/2 \). By the self-similarity of \((Z_t)_{t \geq 0}\),

\[
\Delta_2(t) = \mathbb{E} \left[ (Z_1 + \hat{\gamma}_t)^+ - Z_1^+ \right] = \int_0^\infty \left( \mathbb{P}(Z_1 \geq u - \hat{\gamma}_t) - \mathbb{P}(Z_1 \geq u) \right) du = \int_0^\infty \int_{u-\hat{\gamma}_t}^0 p_Z(w) dw du,
\]

where for simplicity we have written \( p_Z(u) \) for the density \( p_Z(1, u) \) of \( Z_1 \). By the symmetry of \( Z_1 \), \((\hat{\gamma} - e_1)/2 = (\hat{\gamma} - e_1) \int_0^\infty p_Z(u) du \) and, thus, recalling that \( \hat{\gamma}_t := t^{-1/\gamma} (\hat{\gamma} - t^{-1} \kappa_t) \),

\[
A_2(t) = t^{\frac{1}{4}} \int_0^\infty \left( \frac{1}{\gamma_t} \int_{u-\hat{\gamma}_t}^u p_Z(w) dw - p_Z(u) \right) du + \frac{1}{2} (e_1 - t^{-1} \kappa_t) := A_{21}(t) + A_{22}(t).
\]

The identity

\[
p_Z(w) = p_Z(u) + (w - u) \int_0^1 p_Z'(u + \beta(u - w)) d\beta,
\]

followed by the change of variables \( v = \hat{\gamma}_t^{-1}(w - u) \), gives

\[
A_{21}(t) = t^{\frac{1}{4}} \int_0^\infty \left[ \int_{-1}^0 \left( \int_0^1 p_Z'(u + \beta v) d\beta \right) dv \right] du.
\]

By Fubini’s theorem,

\[
A_{21}(t) = t^{\frac{1}{4}} \int_0^\infty \left( \int_{-1}^0 \left( \int_0^1 p_Z'(u + \beta v) d\beta \right) dv \right) du = -t^{\frac{1}{4}} \int_{-1}^0 \left( \int_0^1 p_Z' \left( \beta v t^{\frac{1}{4}} \right) d\beta \right) dv.
\]

It is now clear that

\[
\lim_{t \to 0} t^{\frac{1}{4}} A_2(t) = \frac{(\hat{\gamma} - e_1)^2 p_Z(0) - e_2}{2} := d_{3,1}, \tag{A.20}
\]

since \( t^{1/\gamma - 1}(e_1 - t^{-1} \kappa_t) \to -e_2 \) by assumption. Next, using the power series representation of \( p_Z \) around \( z = 0 \) as given in (14.30) in [14], it follows that \( d_{3,1} \) as above reduces to the expression in (3.4). Finally, combining (A.19) and (A.20) with (A.3) (together with the remarks thereafter), we obtain (3.1).

**Proof of Proposition 3.2.** The small-time asymptotic behavior of the ATM call-option price \( C_{BS}(t, \sigma) \) at maturity \( t \) under the Black-Scholes model with volatility \( \sigma > 0 \) and zero interest rate (and fixing for simplicity \( S_0 = 1 \), is such that:

\[
C_{BS}(t, \sigma) = \frac{\sigma}{\sqrt{2\pi}} t^{\frac{1}{4}} - \frac{\sigma^3}{24\sqrt{2\pi}} t^2 + O(t^{\frac{5}{2}}), \quad t \to 0, \tag{A.21}
\]

(see, e.g., [9, Corollary 3.4]). To derive the small-time asymptotics for the implied volatility, we need a result analogous to (A.21) when \( \sigma \) is replaced by \( \hat{\sigma}(t) \). To obtain it, combining first the following representation

\[
C_{BS}(t, \sigma) = F(\sigma \sqrt{t}) \quad \text{with} \quad F(w) := \int_0^w \Phi \left( \frac{v}{2} \right) dv = \frac{1}{\sqrt{2\pi}} \int_0^w \exp \left( -v^2/8 \right) dv,
\]

originating in [13, Lemma 3.1], together with the Taylor expansion of \( F \) at \( w = 0 \) (see [13, Lemma 5.1]), we get

\[
F(w) = \frac{1}{\sqrt{2\pi}} w - \frac{1}{24\sqrt{2\pi}} w^3 + O(w^5), \quad w \to 0.
\]

Then, since \( \hat{\sigma}(t) \to 0 \) as \( t \to 0 \) (see, e.g., [15, Proposition 5]), we conclude that

\[
C_{BS}(t, \hat{\sigma}(t)) = \frac{\hat{\sigma}(t)}{\sqrt{2\pi}} t^{\frac{1}{4}} - \frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}} t^2 + O \left( \left( \hat{\sigma}(t) t^{\frac{1}{4}} \right)^5 \right), \quad t \to 0. \tag{A.22}
\]

Returning to the proof of Proposition 3.2, by comparing the first order terms in (3.1) and (A.22), it follows that \( \mathbb{E}(Z_1^+) t^{\frac{1}{4}} \sim (2\pi)^{-1/2} \hat{\sigma}(t) \sqrt{t} \) as \( t \to 0 \), and thus,

\[
\hat{\sigma}(t) \sim \sqrt{2\pi} \mathbb{E}(Z_1^+) t^{\frac{1}{4} - \frac{1}{2}} := \sigma t^{\frac{1}{4} - \frac{1}{2}}, \quad t \to 0. \tag{A.23}
\]
Next, set $\tilde{\sigma}(t) := \tilde{\sigma}(t) - \sigma_1 t^{1/2 - \frac{1}{2}}$. Comparing the first two terms in (3.1) with the first term in (A.22) (noting that the second term in (A.22) is $O(t^{1/2})$), leads to

$$\tilde{\sigma}(t) \sim \sqrt{\frac{\pi}{2}} C \Gamma(-Y) \left( (M-1)^Y - M^Y - (G+1)^Y + G^Y \right)^{\sqrt{t}} := \sigma_2 \sqrt{t}, \quad t \to 0. \quad (A.24)$$

Finally, to obtain the third-order expansion, set $\tilde{\sigma}(t) := \tilde{\sigma}(t) - \sigma_1 t^{1/2 - \frac{1}{2}} - \sigma_2 \sqrt{t}$. By comparing the first three terms in (3.1) with the first term in (A.22), it follows that

$$\frac{\tilde{\sigma}(t)}{\sqrt{2\pi}} t^{\frac{1}{2}} \sim d_3 t^{2-\frac{1}{2}} 1_{(1 < Y \leq 2)} + d_3 t^{\frac{1}{2}} 1_{\{2 < Y < 2\}}. \quad (A.25)$$

which leads to (3.6).

**Proof of Theorem 3.3.** Fix

$$\tilde{\gamma}_t := t^{\frac{1}{2}} (\tilde{\gamma} - t^{-\frac{1}{2}} \kappa_t) = (\tilde{\gamma} - e_1) t^{\frac{1}{2}} - e_2 t^{2-Y} + o \left(t^{\frac{1}{2}v(2-Y)}\right),$$

and let

$$\Delta_0(t) := \frac{1}{\sqrt{t}} E \left[(S_t - e^{\kappa_t})^+ - d_1\right], \quad (A.26)$$

with the constant $d_1$ given in (3.8). First, note the following easy representation

$$\frac{1}{\sqrt{t}} E \left[(S_t - e^{\kappa_t})^+\right] = \frac{1}{\sqrt{t}} E \left[e^{X_t} (1 - e^{-X_t + \kappa_t})^+\right] = \frac{1}{\sqrt{t}} E^* \left(1 - e^{-\tilde{X}_t^*}\right) = \int_0^\infty e^{-\sqrt{Y} v} P^* \left(t^{-1/2} \tilde{X}_t \geq v\right) dv,$$

where above $\tilde{X}_t := X_t - \kappa_t$ and, in the last equality, we used the identity

$$E^*(1 - e^{-\tilde{X}_t^*}) = \int_0^\infty e^{-x} P^*(\tilde{X}_t \geq x) dx,$$

together with the change of variables $v = t^{-1/2} x$. Next, recalling that $X_t = L_t^* + \sigma W_t^* = \tilde{\gamma}_t + Z_t + \sigma W_t^*$ and using the self-similarity of $W^*$ and the change of variables $y = v - \tilde{\gamma}_t$, it follows that

$$\Delta_0(t) = \int_{-\tilde{\gamma}_t}^\infty e^{-\sqrt{Y} y - \sqrt{Y} t} P^* \left(\sigma W_t^* \geq y - t^{-\frac{1}{2}} Z_t\right) dy - \int_0^\infty P^* (\sigma W_t^* \geq y) dy.$$

Moreover, by changing the probability measure $P^*$ to $\bar{P}$, recalling that $U_t = \bar{U}_t + \eta t$, and using the self-similarity of $\{(Z_t, U_t)\}_{t \geq 0}$, we get

$$\Delta_0(t) = \int_{-\tilde{\gamma}_t}^\infty e^{-\sqrt{Y} y - \sqrt{Y} t} \bar{P} \left(\sigma W_t^* \geq y - t^{-\frac{1}{2}} Z_t\right) dy - \int_0^\infty \bar{P} (\sigma W_t^* \geq y) dy.$$  \hspace{1cm} (A.27)

Since $\tilde{\gamma}_t \to 0$, as $t \to 0$, it is clear that

$$\lim_{t \to 0} t^{\frac{1}{2}} A_3(t) = \lim_{t \to 0} \int_0^\infty \frac{e^{-\sqrt{Y} y - \sqrt{Y} t} - 1}{\sqrt{t}} \bar{P} (\sigma W_t^* \geq y) dy \to - \int_0^\infty \bar{P} (\sigma W_t^* \geq y) dy = - \frac{\sigma^2}{4}. \quad (A.28)$$

For $A_2(t)$, we consider two cases. First, assume $Y < 1/2$. Then, by changing variables $u = t^{-1/2} y$, the integral appearing in the $A_2(t)$ term can be written as

$$t^{\frac{1}{2}} \int_{-(\tilde{\gamma}_t - 1) + R_t}^{+\infty} \bar{P} \left(\sigma W_t^* \geq t^{\frac{1}{2}} u - t^{-\frac{1}{2}} \tilde{X}_t^*\right) du,$$
where $R_t := e_2 t^{3/2-Y} + o(t^{3/2-Y})$. The above integral can further be decomposed as $t^{1/2} \int_{-\gamma}^0 \rho_{\gamma-e_1}(t) + t^{1/2} \int_{-\gamma-e_1}^\infty \rho_{-t^2/2}$. By the dominated convergence theorem, the first integral is clearly equal to $t^{1/2} / (\gamma - e_1) / 2 + o(t^{1/2})$. For the second integral, by changing variables $u = (u + (\gamma - e_1)) t^{-3/2-Y}$, it becomes

$$e^{-t(\gamma - e_1) t^{-2} Y} \int_{\gamma+e_1}^0 e^{-t^{3/2} Y} w^{-1} \{ \sigma \leq (\gamma - e_1) - t^{3/2} Y \} \left( \sigma \leq (\gamma - e_1) - t^{3/2} Y \right) dw,$$

Proceeding as before and decomposing the last integral as $t^{-2} Y \int_{\gamma+e_1}^0 t^{2} Y \int_{\gamma+e_1}^0$, so that, by the dominated convergence theorem, the first term is $-t^{-2} Y e_2 / 2 + o(t^{-2} Y)$, while the second is $o(t^{-2} Y)$ since the integrand is uniformly bounded by 1. If $Y \geq 3/2$, we may instead change the variables $u = t^{-2} Y$ and follow a similar argument. In either case,

$$A_2(t) = \frac{\gamma - e_1}{2} t^{-2} Y + o \left( \frac{t^{3/2-Y}}{2} \right), \quad t \to 0.$$  
(A.29)

It remains to analyze the asymptotic behavior of $A_1(t)$. To this end, let us first decompose it as:

$$A_1(t) = e^{-(\eta^2 + \sqrt{\gamma})} \left( e^{-t^{\gamma}} \int_{0}^{\infty} \left\{ \sigma W_t, \sigma W_t + t^{1/2} \right\} \right) e^{-\sqrt{\gamma} Y} dy$$

Next, by Fubini's theorem and the independence of $W_t$ and $(Z_t, U_t)$,

$$B_1(t) = e^{-(\eta^2 + \sqrt{\gamma})} \left( e^{-\sigma W_t} \int_{0}^{\infty} \left\{ \sigma W_t, \sigma W_t + t^{1/2} \right\} \right) e^{-t^\gamma Y} du$$

where the expression for the last term above is derived by conditioning on $\sigma W_t$, and since by the symmetry of $Z_t$, it also follows that:

$$\mathbb{E} \left( Z_1 \mathbb{1}_{\{ Z_1 \geq -t^{1/2} \} } \right) = \mathbb{E} \left( Z_1 \left( 1 - \mathbb{1}_{\{ Z_1 \leq -t^{1/2} \} } \right) \right) = \mathbb{E} \left( -Z_1 \mathbb{1}_{\{ Z_1 \leq -t^{1/2} \} } \right) = \mathbb{E} \left( Z_1 \mathbb{1}_{\{ Z_1 \geq t^{1/2} \} } \right).$$
In order to obtain the asymptotic behavior of $B_{11}(t)$, let

$$
B_{11}^{(1)}(t) := \int_{-\infty}^{0} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \tilde{P} \left( Z_1 \geq 0, \bar{U}_1 \leq u \leq \bar{U}_1 + Z_1 \right) du,
$$

$$
B_{11}^{(2)}(t) := \int_{0}^{\infty} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \tilde{P} \left( Z_1 \geq 0, \bar{U}_1 \leq u \leq \bar{U}_1 + Z_1 \right) du.
$$

For $B_{11}^{(1)}(t)$, we use arguments as in (A.5). Concretely, for $u < 0$, by (2.18),

$$
\frac{e^{-t^{\frac{1}{2}} u} - 1}{t^{\frac{1}{2}}} \tilde{P} \left( Z_1 \geq 0, \bar{U}_1 \leq u \leq \bar{U}_1 + Z_1 \right) \leq (-u) e^{-t^{\frac{1}{2}} u} \tilde{P} \left( \bar{U}_1 \leq u \right) \leq (-u) e^{(1-t^{\frac{1}{2}}) u} \tilde{P} \left( e^{-\bar{U}_1} \right) \leq e^{\gamma} (-u) e^{\frac{\gamma}{2}},
$$

and thus, by the dominated convergence theorem,

$$
B_{11}^{(1)}(t) = t^{\frac{1}{2}} \int_{-\infty}^{0} (-u) \tilde{P} \left( Z_1 \geq 0, u \leq \bar{U}_1 + Z_1 \right) du + o(t^{\frac{1}{2}}), \quad t \to 0. \tag{A.32}
$$

For $B_{11}^{(2)}(t)$, we use arguments similar to those used to obtain (A.16). Concretely, let

$$
B_{11}^{(21)}(t) := \int_{0}^{\infty} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \tilde{P} \left( Z_1 \geq 0, u \leq \bar{U}_1 + Z_1 \right) du
$$

$$
= t^{\frac{1}{2}} \int_{0}^{\infty} \frac{e^{-t^{\frac{1}{2}} u} - 1}{t^{\frac{1}{2}}} \left[ \tilde{P} \left( Z_1 \geq 0, u \leq \bar{U}_1 + Z_1 \right) - \frac{CM}{Y M Y} \right] du - t^{1-\frac{1}{2}} C \Gamma (-Y) M Y,
$$

where in the second equality the identity (A.2) was used. The integral on the right-hand side of the previous identity is precisely the first integral defined in (A.6), after setting $\gamma = 0$. Then, one uses arguments as those leading to (A.16) to conclude that

$$
B_{11}^{(21)}(t) = -t^{1-\frac{1}{2}} C \Gamma (-Y) M Y + O(t^{\frac{1}{2}}), \quad t \to 0.
$$

Via a similar approach, it is shown that

$$
B_{11}^{(22)}(t) := \int_{0}^{\infty} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \tilde{P} \left( Z_1 \geq 0, u \leq \bar{U}_1 \right) du = -t^{1-\frac{1}{2}} C \Gamma (-Y) (M^*) Y + O(t^{\frac{1}{2}}), \quad t \to 0.
$$

Therefore,

$$
B_{11}^{(2)}(t) = -t^{1-\frac{1}{2}} C \Gamma (-Y) (M Y - (M^*) Y) + O(t^{\frac{1}{2}}), \quad t \to 0,
$$

which, together with (A.32), implies that the term $B_{11}(t)$ introduced in (A.31) is such that

$$
B_{11}(t) = -\frac{1}{2} t^{\frac{1}{2}} C \Gamma (-Y) (M Y - (M^*) Y) + O(t^{\frac{1}{2} - \frac{1}{2}}), \quad t \to 0. \tag{A.33}
$$

To deal with $B_{12}(t)$, first make the change of variables $x = t^{\frac{1}{2}} u$ in the integral of this term so that

$$
B_{12}(t) = t^{-\frac{1}{2}} e^{-(t + \sqrt{\gamma} u)} \int_{0}^{\infty} \left[ \int_{\mathbb{R}} \left( e^{-x} - 1 \right) \tilde{P} \left( -t^{\frac{1}{2}} x \leq Z_1 \leq 0, \bar{U}_1 + Z_1 \leq t^{\frac{1}{2}} x \leq \bar{U}_1 \right) dx \right] e^{-\sqrt{\gamma} w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi \sigma^2}} dw.
$$

We next prove that $B_{12}(t) = o(t^{1/2})$ as $t \to 0$. To this end, let

$$
B_{12}^{(1)}(t) = \int_{0}^{\infty} \left[ \int_{0}^{\infty} \left( 1 - e^{-x} \right) \tilde{P} \left( -t^{\frac{1}{2}} x \leq Z_1 \leq 0, \bar{U}_1 + Z_1 \leq t^{\frac{1}{2}} x \leq \bar{U}_1 \right) dx \right] e^{-\sqrt{\gamma} w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi \sigma^2}} dw,
$$

$$
B_{12}^{(2)}(t) = \int_{-\infty}^{0} \left[ \int_{-\infty}^{0} \left( e^{-x} - 1 \right) \tilde{P} \left( -t^{\frac{1}{2}} x \leq Z_1 \leq 0, \bar{U}_1 + Z_1 \leq t^{\frac{1}{2}} x \leq \bar{U}_1 \right) dx \right] e^{-\sqrt{\gamma} w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi \sigma^2}} dw.
$$
For $B_{12}^{(1)}(t)$, since by (2.14-i), for any $x > 0$ and $t > 0$,

$$
\frac{1}{t} \mathbb{P}\left(-t^{\frac{1}{2}} \frac{\dot{w}}{\sqrt{2r}} \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq t^{-\frac{1}{2}} x \leq \tilde{U}_1 \right) \leq \frac{1}{t} \mathbb{P}\left(\tilde{U}_1 \geq t^{-\frac{1}{2}} x \right) \leq \frac{1}{t} \mathbb{P}\left(\tilde{U}_1^{(p)} \geq \frac{t^{-\frac{1}{2}} x}{2M^*} \right) + \frac{1}{t} \mathbb{P}\left(-\tilde{U}_1^{(n)} \geq \frac{t^{-\frac{1}{2}} x}{2G^*} \right)
$$

$$
\leq 2^2 K_1 \left( (M^*)^Y + (G^*)^Y \right) x^{-Y}.
$$

(A.34)

it follows from the dominated convergence theorem that,

$$
\lim_{t \to 0} \frac{1}{t} B_{12}^{(1)}(t) = \int_0^\infty \left\{ \int_0^\infty \left(1 - e^{-x} \right) \left[ \lim_{t \to 0} \frac{1}{t} \mathbb{P}\left(-t^{\frac{1}{2}} \frac{\dot{w}}{\sqrt{2r}} \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq t^{-\frac{1}{2}} x \leq \tilde{U}_1 \right) \right] dx \right\} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw.
$$

(A.35)

Moreover, for any $t > 0$, $w > 0$ and $x > 0$, $P_t(x, w) := \mathbb{P}\left(-t^{\frac{1}{2}} \frac{\dot{w}}{\sqrt{2r}} \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq t^{-\frac{1}{2}} x \leq \tilde{U}_1 \right)$ is such that

$$
P_t(x, w) = \mathbb{P}\left(-t^{\frac{1}{2}} \frac{\dot{w}}{\sqrt{2r}} \leq \tilde{U}_1^{(p)} + \tilde{U}_1^{(n)} \leq 0, (M^* + 1) \tilde{U}_1^{(p)} + (1 - G^*) \tilde{U}_1^{(n)} \leq t^{-\frac{1}{2}} x \leq M^* \tilde{U}_1^{(p)} - G^* \tilde{U}_1^{(n)} \right)
$$

$$
\leq \mathbb{P}\left(\frac{t^{-\frac{1}{2}} x}{M^* + G^*} \leq -\tilde{U}_1^{(n)} \leq \frac{t^{-\frac{1}{2}} x + (M^* + 1) \tilde{U}_1^{(n)} / (M^* + G^*)}{M^*} \leq \tilde{U}_1^{(p)} \right)
$$

$$
\leq \mathbb{P}\left(\frac{t^{-\frac{1}{2}} x}{M^* + G^*} \leq -\tilde{U}_1^{(n)} \leq \frac{t^{-\frac{1}{2}} x + G^* (M^* + 1) \tilde{U}_1^{(n)} / (M^* (M^* + G^*))}{M^*} \leq \tilde{U}_1^{(p)} \right).
$$

Hence, in view of (2.14-i),

$$
\lim_{t \to 0} \frac{1}{t} P_t(x, w) \leq K_1 (M^* + G^*)^{-Y} x^{-Y} \lim_{t \to 0} \mathbb{P}\left(\tilde{U}_1^{(p)} \geq \frac{t^{-\frac{1}{2}} [M^* x - G^* (M^* + 1) \sqrt{7w}]}{M^* (M^* + G^*)} \right) = 0.
$$

(A.36)

Combining (A.35) and (A.36) leads to $B_{12}^{(1)}(t) = o(t)$. For $B_{12}^{(2)}(t)$, note that, for any $t > 0$, $w > 0$ and $x < 0$,

$$
\frac{1}{t} P_t(x, w) \leq \frac{1}{t} \mathbb{P}\left(\tilde{U}_1 + Z_1 \leq t^{-\frac{1}{2}} x \right) \leq \frac{1}{t} \mathbb{P}\left(\tilde{U}_1^{(p)} \leq \frac{t^{-\frac{1}{2}} x}{2(M^* + 1)} \right) + \frac{1}{t} \mathbb{P}\left(-\tilde{U}_1^{(n)} \leq -\frac{t^{-\frac{1}{2}} x}{2(G^* - 1)} \right).
$$

Using that $M, G > 0$, it follows

$$
\frac{1}{t} P_t(x, w) \leq \frac{2}{t} \mathbb{P}\left(\tilde{U}_1^{(p)} \leq \frac{t^{-\frac{1}{2}} x}{2(M^* + G^*)} \right) \leq \frac{2}{t} \mathbb{E}\left(e^{-\tilde{U}_1^{(p)}}\right) \exp\left\{\frac{t^{-\frac{1}{2}} x}{2(M^* + G^*)}\right\} \to 0, \quad t \to 0.
$$

Therefore, by the dominated convergence theorem,

$$
\lim_{t \to 0} \frac{1}{t} B_{12}^{(2)}(t) = \int_0^\infty \left\{ \int_{-\infty}^0 \left(1 - e^{-x} \right) \left( \lim_{t \to 0} \frac{1}{t} P_t(x, w) \right) dx \right\} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw = 0,
$$

which in turn implies that

$$
B_{12}(t) = t^{-\frac{1}{2}} e^{-(\eta t + \sqrt{7} t)} \left( B_{12}^{(2)}(t) - B_{12}^{(1)}(t) \right) = o(t^{\frac{1}{2}}), \quad t \to 0.
$$

(A.37)

Finally, we deal with $B_{13}(t)$ and analyze the asymptotic behavior of the following expression:

$$
\tilde{B}_{13}(t) := t^{\frac{1}{2}} - 1 B_{13}(t) - \frac{C \sigma^{-Y}}{2(Y-1)} \mathbb{E}\left(\left|W_1^*\right|^{1-Y}\right).
$$

(A.38)
First, $\tilde{B}_{13}(t)$ is further decomposed as:
\[
\tilde{B}_{13}(t) = t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \left[ e^{-\sqrt{\tau} W_t^*} 1_{\{W_t^* \geq 0\}} \int_{t^{\frac{1}{3}} + \frac{1}{4} \sigma W_t^*}^\infty z p_Z(1, z) \, dz \right] - \tilde{E} \left( 1_{\{W_t^* \geq 0\}} \int_{t^{\frac{1}{3}} + \frac{1}{4} \sigma W_t^*}^\infty C z^{-Y} \, dz \right)
\]
\[
= t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \left( e^{-\sqrt{\tau} W_t^*} - 1 \right) \tilde{E} \left( 1_{\{W_t^* \geq 0\}} \int_{t^{\frac{1}{3}} + \frac{1}{4} \sigma W_t^*}^\infty z p_Z(1, z) \, dz \right)
\]
\[
+ t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \tilde{E} \left( e^{-\sqrt{\tau} W_t^*} - 1 \right) \int_{t^{\frac{1}{3}} + \frac{1}{4} \sigma W_t^*}^\infty z p_Z(1, z) \, dz
\]
\[
+ t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \tilde{E} \left( 1_{\{W_t^* \geq 0\}} \int_{t^{\frac{1}{3}} + \frac{1}{4} \sigma W_t^*}^\infty z \left( p_Z(1, z) - C z^{-Y - 1} \right) \, dz \right)
\]
\[
:= B_{13}^{(1)}(t) + B_{13}^{(2)}(t) + B_{13}^{(3)}(t).
\]
(A.39)

As shown next,
\[
B_{13}^{(1)}(t) = O \left( t^{1\land(\frac{2}{3} - Y)} \right) \Rightarrow O(\sqrt{t}), \quad B_{13}^{(2)}(t) = O(\sqrt{t}), \quad t \to 0.
\]
(A.40)

Indeed, for $B_{13}^{(1)}(t)$, we first rewrite the expectation as
\[
\tilde{E} \left( e^{-\sqrt{\tau} W_t^*} 1_{\{W_t^* \geq 0\}} \int_{t^{\frac{1}{3}} + \frac{1}{4} \sigma W_t^*}^\infty z p_Z(1, z) \, dz \right) = \int_{0}^{\infty} \left( \int_{t^{\frac{1}{3}} + \frac{1}{4} \sigma w}^\infty z p_Z(1, z) \, dz \right) e^{-\sqrt{\tau} w} \frac{e^{w^2}}{\sqrt{2\pi \sigma^2}} \, dw.
\]
(A.41)

Next, by (2.16), there exists $H_1 > 0$ such that, for any $z \geq H_1$,
\[
p_Z(1, z) \leq 2C z^{-Y - 1}.
\]
(A.42)

Hence, for any $w > 0$,
\[
t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \int_{t^{\frac{1}{3}} + \frac{1}{4} w}^\infty z p_Z(1, z) \, dz \leq t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \left[ \int_{t^{\frac{1}{3}} + \frac{1}{4} w}^\infty 2Cu^{-Y} du + \int_{t^{\frac{1}{3}} + \frac{1}{4} w < H_1}^{H_1} H_1^{-1} w^{1-Y} \right]
\]
\[
\leq \frac{2C w^{1-Y}}{Y-1} + H_1^{-1} w^{1-Y},
\]
where to derive the second term in the last inequality we used that $H_1 \geq t^{\frac{1}{3} - \frac{1}{4} w} < H_1$, when $t^{\frac{1}{3} - \frac{1}{4} w} \leq 1$. Together with (A.41) and since $Y \in (1, 2)$, the first relation in (A.40) is obtained, since $\sqrt{\tau} t = (\gamma - e_1) t = e_2 t^{5/2-Y} + o(t^{(5/2-Y)_0})$. The second relation therein is obtained using similar arguments.

It remains to deal with $B_{13}^{(3)}(t)$, which can be rewritten as:
\[
B_{13}^{(3)}(t) = \frac{1}{2} t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \int_{\mathbb{R}} \left( \int_{0}^{t^{\frac{1}{3}} + \frac{1}{4} \sigma |z|} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{w^2}{2\sigma^2}} \, dw \right) |z| (p_Z(1, z) - C|z|^{-Y-1}) \, dz
\]
\[
= \frac{1}{2} t^{\frac{2}{3} + \frac{1}{4} - \frac{2}{3}} \int_{0}^{1} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{w^2}{2\sigma^2}} \left| z \right|^2 (p_Z(1, z) - C|z|^{-Y-1}) \, dz \, du,
\]
(A.43)

where we change variables $u = t^{\frac{1}{3} - \frac{1}{4} w} / |z|$ and apply Fubini's theorem in the second equality. For simplicity, we write $p_Z(z)$ instead of $p_Z(1, z)$ hereafter. Next, denoting the characteristic function of $Z_1$ by $\hat{p}_Z(x)$, we have
\[
p_Z(z) = \mathcal{F} \left( \frac{1}{\sqrt{2\pi}} p_Z \right) (z), \quad z^2 p_Z(z) = \mathcal{F} \left( \frac{-1}{\sqrt{2\pi}} \hat{p}_Z'' \right) (z),
\]
(A.44)

where $\mathcal{F}(h)(z) := \int_{\mathbb{R}} e^{-izv} h(v) dv / \sqrt{2\pi}$ denotes the Fourier transformation of $h \in L_1(\mathbb{R})$. Also, regarding $|x|^{-Y-2}$ as a tempered distribution, it is known that
\[
|z|^{1-Y} = \mathcal{F} \left( K^{-1} |z|^{-Y-2} \right) (z),
\]
(A.44)
with \( K := -2\sin(\pi(Y - 2)/2)\Gamma(Y - 1)/\sqrt{2\pi} \). In particular, by definition,
\[
\int_{\mathbb{R}} |z|^{1-Y} \phi(z) \, dz = \int_{\mathbb{R}} K^{-1} |x|^{Y-2} \mathcal{F}(\phi)(x) \, dx,
\]  
(A.45)
for any Schwartz function \( \phi \). Thus, combining (A.43)-(A.45),
\[
B_{13}^{(3)}(t) = \frac{1}{2\sqrt{2\pi}} \left[ t^{\frac{1}{2} Y + \frac{3}{2}} \int_{0}^{\infty} \mathcal{F} \left( \frac{t^{\frac{1}{2} - \frac{1}{2}} - \frac{\hat{\phi}}{2\pi} \frac{e^{-t^{\frac{1}{2} - \frac{1}{2}} x^{2}}}{x^{2}} \right)(x) \right) \left( -1 + \frac{1}{\sqrt{2\pi}} \hat{\rho}'(x) - \frac{C}{K} |x|^{Y-2} \right) dx du,
\]
\[
\left. \int_{0}^{1} u^{-1} e^{-\frac{1 - \frac{1}{2} x^{2}}{2\pi} x^{2}} \left( 1 + \frac{1}{\sqrt{2\pi}} \hat{\rho}'(x) + \frac{C}{K} |x|^{Y-2} \right) dx du. \right]
\]
Recalling that \( \hat{\rho}_z(x) = e^{-c|x|^Y} \) with \( c := 2C|\cos(\pi Y/2)|\Gamma(-Y) \), we have:
\[
\frac{1}{\sqrt{2\pi}} \hat{\rho}'(x) + \frac{C}{K} |x|^{Y-2} = -\frac{cY(Y-1)}{\sqrt{2\pi}} e^{-c|x|^Y} |x|^{Y-2} + \frac{1}{\sqrt{2\pi}} (cY|x|^{Y-1}) e^{-c|x|^Y} + \frac{C}{K} |x|^{Y-2}
\]
\[
= \frac{1}{\sqrt{2\pi}} (cY|x|^{Y-1}) e^{-c|x|^Y} + \frac{cY(Y-1)}{\sqrt{2\pi}} |x|^{Y-2} \left( 1 - e^{-c|x|^Y} \right),
\]
where in the last equality we used \( C/K = cY(Y-1)/\sqrt{2\pi} \). Hence,
\[
B_{13}^{(3)}(t) = \frac{cY(Y-1)}{2\pi} t^{\frac{1}{2} Y + \frac{3}{2}} \int_{0}^{\infty} u^{-1} e^{-\frac{1 - \frac{1}{2} x^{2}}{2\pi} x^{2}} x^{2} e^{-c|x|^Y} dx du
\]
\[
+ \frac{cY(Y-1)}{2\sqrt{2\pi}} \left[ t^{\frac{1}{2} Y + \frac{3}{2}} \int_{0}^{\infty} u^{-1} e^{-\frac{1 - \frac{1}{2} x^{2}}{2\pi} x^{2}} x^{2} e^{-c|x|^Y} \right] dx du
\]
\[
:= B_{13}^{(3,1)}(t) + B_{13}^{(3,2)}(t). \quad (A.46)
\]
For \( B_{13}^{(3,1)}(t) \), changing variables \( v = t^{\frac{1}{2} - \frac{1}{2} \sigma x/u} \),
\[
B_{13}^{(3,1)}(t) = \frac{cY(Y-1)}{2\pi} t^{\frac{1}{2} Y + \frac{3}{2}} \int_{0}^{\infty} e^{-\frac{1 - \frac{1}{2} x^{2}}{2\pi} x^{2}} x^{2} e^{-c|x|^Y} \left( \frac{1}{\sigma} \right)^{Y-2} \left( \frac{1}{\sigma} \right) dx du
\]
\[
= \frac{cY(Y-1)}{2\pi \sigma^{2Y-1}} \left[ t^{1 - \frac{3}{2}} \int_{0}^{\infty} e^{-\frac{1}{2} v^{2} x^{2}} e^{-c|x|^Y} \right] dx du.
\]
Hence, by the dominated convergence theorem,
\[
\lim_{t \to 0} t^{\frac{1}{2} - \frac{1}{2}} B_{13}^{(3,1)}(t) = \frac{cY(Y-1)}{2\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \mathbb{E} \left( |W_1|^{2Y-2} \right) = \frac{2C^2\Gamma^2(-Y)}{\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \mathbb{E} \left( |W_1|^{2Y-2} \right). \quad (A.47)
\]
Similarly, for \( B_{13}^{(3,2)}(t) \),
\[
B_{13}^{(3,2)}(t) = \frac{cY(Y-1)}{2\pi} t^{\frac{1}{2} Y + \frac{3}{2}} \int_{0}^{\infty} e^{-\frac{1 - \frac{1}{2} x^{2}}{2\pi} x^{2}} \left( \frac{1}{\sigma} \right)^{Y-2} \left( \frac{1}{\sigma} \right) \left[ 1 - e^{-c(\frac{1}{\sigma} x/u)} \right] \frac{t^{1 - \frac{3}{2}}}{u} dx du
\]
\[
= \frac{cY(Y-1)}{2\pi \sigma^{2Y-1}} \left[ t^{1 - \frac{3}{2}} \int_{0}^{\infty} e^{-\frac{1}{2} v^{2} x^{2}} \left( 1 - e^{-c(\frac{1}{\sigma} x/u)} \right) \left( \frac{1}{u} \right) \right] \frac{t^{1 - \frac{3}{2}}}{u} dx du.
\]
Again, by the dominated convergence theorem,
\[
\lim_{t \to 0} t^{\frac{1}{2} - \frac{1}{2}} B_{13}^{(3,2)}(t) = \frac{-c^2Y(Y-1)}{2\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \mathbb{E} \left( |W_1|^{2Y-2} \right) = \frac{-c^2\Gamma^2(-Y)}{\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \mathbb{E} \left( |W_1|^{2Y-2} \right). \quad (A.48)
\]
Combining (A.39), (A.40) and (A.46)-(A.48),
\[
\lim_{t \to 0} t^{\frac{1}{2} - \frac{1}{2}} B_{13}(t) = \frac{-c^2\Gamma^2(-Y)}{\sqrt{2\pi}\sigma^{2Y-1}} \mathbb{E} \left( |W_1|^{2Y-2} \right) := d'_{11}. \quad (A.49)
\]
Combining (A.31), (A.33), (A.37), (A.38) and (A.49), the asymptotic behavior for \( B_1(t) \), as \( t \to 0 \), is given by

\[
B_1(t) = -\frac{1}{2} \operatorname{CT}(-Y) \left(M^Y - (M^*)^Y\right) t^{1 - \frac{3}{2}} + t^{1 - \frac{3}{2}} \left[ B_{13}(t) + \frac{C \sigma^{1-Y}}{2Y-1} \mathbb{E} \left( |W_1|^{-1-Y} \right) \right] + o(t^{\frac{3}{2}})
\]

\[
= -\frac{1}{2} \operatorname{CT}(-Y) \left(M^Y - (M^*)^Y\right) t^{1 - \frac{3}{2}} + t^{1 - \frac{3}{2}} \left[ t^{1 - \frac{3}{2}} d_{31} + o(t^{1 - \frac{3}{2}}) + \frac{C \sigma^{1-Y}}{2Y-1} \mathbb{E} \left( |W_1|^{-1-Y} \right) \right] + o(t^{\frac{3}{2}})
\]

\[
= -\frac{1}{2} d_3^{Y} t^{\frac{3}{2}} + \frac{C \sigma^{1-Y}}{2Y-1} \mathbb{E} \left( |W_1|^{-1-Y} \right) t^{1 - \frac{3}{2}} + d_{31} t^{2 - Y} + o(t^{\frac{3}{2}}),
\]  

(A.50)

setting \( d_3 := \operatorname{CT}(-Y) \left[M^Y - (M^*)^Y\right] \).

**Step 2.** Next, we analyze \( B_2(t) \) by decomposing it as:

\[
B_2(t) = e^{-(\eta t + \sqrt{\tau} \sqrt{w})} \int_{0}^{\infty} \mathbb{E} \left[ \left( e^{-t \frac{3}{2} \tilde{u}_1} - 1 \right) 1 \{ z_1 \leq -t \frac{1}{2} \tilde{u}_1, \tilde{u}_1 < 0 \} \right] \frac{1 - e^{-\sqrt{\tau} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} \frac{1 - e^{\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw
\]

\[
+ e^{-(\eta t + \sqrt{\tau} \sqrt{w})} \int_{0}^{\infty} \mathbb{P} \left( Z_1 \leq -t \frac{1}{2} \tilde{u}_1 \right) \frac{1 - e^{-\sqrt{\tau} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} \frac{1 - e^{\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw
\]

\[
:= B_{21}(t) + B_{22}(t).
\]  

(A.51)

We begin with proving that \( B_{21}(t) = o \left( t^{1/2} \right) \) as \( t \to 0 \). To this end, consider first

\[
B_{21}^{(1)}(t) := \int_{0}^{\infty} b_{21}^{(1)}(t; w) \frac{1 - e^{-\sqrt{\tau} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} \frac{1 - e^{\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw,
\]

where

\[
b_{21}^{(1)}(t; w) := \mathbb{E} \left[ \left( e^{-t \frac{3}{2} \tilde{u}_1} - 1 \right) 1 \{ z_1 \leq -t \frac{1}{2} \tilde{u}_1, \tilde{u}_1 < 0 \} \right].
\]

Note that, for any \( 0 < t < 1 \) and \( w > 0 \), by (2.18),

\[
0 \leq t^{-\frac{1}{2}} b_{21}^{(1)}(t; w) = t^{-\frac{1}{2}} \mathbb{E} \left( 1 \{ z_1 \leq -t \frac{1}{2} \tilde{u}_1, \tilde{u}_1 < 0 \} \int_{-\infty}^{0} 1 \{ t \frac{3}{2} \tilde{u}_1 \leq u \leq 0 \} e^{-u} \, du \right)
\]

\[
\leq t^{-\frac{3}{2}} \int_{-\infty}^{0} e^{-u} \mathbb{P} \left( \tilde{U}_1 \leq -t \frac{1}{2} \tilde{u}_1 \right) \, du \leq \mathbb{E} \left( e^{-\tilde{U}_1} \right) t^{-\frac{3}{2}} \int_{-\infty}^{0} e^{-u(1-t^{-\frac{3}{2}})} \, du = e^{\eta} \frac{t^{-\frac{1}{2}}}{1 - t^{\frac{3}{2}}}.
\]

Since \( Y \in (1, 2) \), by the dominated convergence theorem,

\[
\lim_{t \to 0} t^{-\frac{3}{2}} B_{21}^{(1)}(t) = \int_{0}^{\infty} \left( \lim_{t \to 0} t^{-\frac{3}{2}} b_{21}^{(1)}(t; w) \right) \frac{w e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw \leq \frac{1}{\sqrt{\pi}} e^{\eta} \lim_{t \to 0} \frac{t^{-\frac{3}{2}}}{1 - t^{\frac{3}{2}}} = 0.
\]  

(A.52)

Next, consider

\[
B_{21}^{(2)}(t) := \int_{0}^{\infty} b_{21}^{(2)}(t; w) \frac{1 - e^{-\sqrt{\tau} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} \frac{1 - e^{\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw,
\]

(A.53)

where \( b_{21}^{(2)}(t; w) \) is defined and further decomposed as:

\[
b_{21}^{(2)}(t; w) := \mathbb{E} \left[ \left( e^{-t \frac{3}{2} \tilde{u}_1} - 1 \right) 1 \{ z_1 \leq -t \frac{1}{2} \tilde{u}_1, \tilde{u}_1 \geq 0 \} \right]
\]

\[
= \mathbb{E} \left[ \left( e^{-t \frac{3}{2} \tilde{u}_1} - 1 + t \tilde{u}_1 \mathbb{E} \tilde{U}_1 \right) 1 \{ z_1 \leq -t \frac{1}{2} \tilde{u}_1, \tilde{u}_1 \geq 0 \} \right] - t \mathbb{E} \mathbb{E} \left( \tilde{U}_1 1 \{ z_1 \leq -t \frac{1}{2} \tilde{u}_1, \tilde{u}_1 \geq 0 \} \right).
\]  

(A.54)

Note that (since \( 1 < Y < 2 \)) as \( t \to 0 \),

\[
0 \leq t^{-\frac{1}{2}} \int_{0}^{\infty} t^{-\frac{3}{2}} \mathbb{E} \left( \tilde{U}_1 1 \{ z_1 \leq -t \frac{1}{2} \tilde{u}_1, \tilde{u}_1 \geq 0 \} \right) \frac{1 - e^{-\sqrt{\tau} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} \frac{1 - e^{\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw \leq t^{-\frac{1}{2}} \frac{1}{\sqrt{\pi}} \mathbb{E} \tilde{U}_1 \int_{0}^{\infty} w e^{-\frac{w^2}{2\pi \sigma^2}} \, dw \to 0.
\]  

(A.55)
Moreover, by (2.14-i) and the decomposition \( \bar{U}_1 = M^* \bar{U}_i^{(\theta)} - G^* \bar{U}_i^{(n)} \), for any \( t > 0 \) and \( w > 0 \),

\[
\mathbb{E} \left[ \left( e^{-tZ_1} - 1 + tZ_1 \right) I_{\{Z_1 \leq -t^{\frac{1}{2}} + \frac{1}{2} w, \bar{U}_1 \geq 0 \}} \right] = \mathbb{E} \left[ \int_0^{tZ_1} (1 - e^{-u}) du I_{\{Z_1 \leq -t^{\frac{1}{2}} + \frac{1}{2} w, \bar{U}_1 \geq 0 \}} \right]
\]

\[
= \int_0^{\infty} (1 - e^{-u}) \mathbb{P} \left( Z_1 \leq -t^{\frac{1}{2}} + \frac{1}{2} w, \bar{U}_1 \geq t^{-\frac{1}{2}} u \right) du
\]

\[
\leq \int_0^{\infty} (1 - e^{-u}) \mathbb{P} \left( \bar{U}_1 \geq t^{-\frac{1}{2}} u \right) du
\]

\[
\leq 2^{Y+1} K_1 \left( (M^*)^Y + (G^*)^Y \right) t \int_0^{\infty} (1 - e^{-u}) u^{-Y} du.
\]

Hence, by the dominated convergence theorem,

\[
0 \leq t^{-\frac{1}{2}} \int_0^{\infty} \mathbb{E} \left[ \left( e^{-tZ_1} - 1 + tZ_1 \right) I_{\{Z_1 \leq -t^{\frac{1}{2}} + \frac{1}{2} w, \bar{U}_1 \geq 0 \}} \right] \frac{1 - e^{-\sqrt{t} w}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi\sigma^2}} dw
\]

\[
\leq 2^{Y+1} K_1 \left( (M^*)^Y + (G^*)^Y \right) \sqrt{t} \int_0^{\infty} (1 - e^{-u}) u^{-Y} du \int_0^{\infty} \frac{w e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw \rightarrow 0, \quad t \rightarrow 0. \tag{A.56}
\]

In light of (A.53)-(A.55) and (A.56), \( B_{21}^{(2)}(t) = o(t^{1/2}) \). Together with (A.52), and since \( B_{21}(t) = e^{-(\eta t + \sqrt{t}\gamma)}(B_{21}^{(1)}(t) + B_{21}^{(2)}(t)) \), we conclude that

\[
\lim_{t \to 0} t^{-\frac{1}{2}} B_{21}(t) = 0. \tag{A.57}
\]

Finally, we analyze \( B_{22}(t) \) defined via (A.51). To this end, let

\[
\tilde{B}_{22}(t) := t^{\frac{1}{2} - 1} B_{22}(t) - \frac{C_{\sigma}^{1-Y}}{2^{Y}} \mathbb{E} \left[ |W_1^*|^{1-Y} \right]
\]

\[
= t^{\frac{1}{2} - 1} e^{-(\eta t + \sqrt{t}\gamma)} \int_0^{\infty} \mathbb{E} \left( \frac{1 - e^{-\sqrt{t} \sigma W_1^*}}{\sqrt{t}} I_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) p_z(1, z) dz
\]

\[
- t^{\frac{1}{2} - 1} \int_0^{\infty} \mathbb{E} \left( \sigma W_1^* I_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) C_z^{-Y-1} dz
\]

\[
= t^{\frac{1}{2} - 1} \left( e^{-(\eta t + \sqrt{t}\gamma)} - 1 \right) \int_0^{\infty} \mathbb{E} \left( \frac{1 - e^{-\sqrt{t} \sigma W_1^*}}{\sqrt{t}} I_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) p_z(1, z) dz
\]

\[
+ t^{\frac{1}{2} - 1} \int_0^{\infty} \mathbb{E} \left( \left( \frac{1 - e^{-\sqrt{t} \sigma W_1^*}}{\sqrt{t}} - \sigma W_1^* \right) I_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) p_z(1, z) dz
\]

\[
+ t^{\frac{1}{2} - 1} \int_0^{\infty} \mathbb{E} \left( \sigma W_1^* I_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) (p_z(1, z) - C_z^{-Y-1}) dz
\]

\[
:= B_{22}^{(1)}(t) + B_{22}^{(2)}(t) + B_{22}^{(3)}(t), \tag{A.58}
\]

where we used the symmetry of \( Z_1 \) in the second equality. As shown next, the first two terms in (A.58) are such that

\[
B_{22}^{(1)}(t) = O \left( t^{1/2} \gamma^{-Y} \right) = O(\sqrt{t}), \quad B_{22}^{(1)}(t) = O(\sqrt{t}), \quad t \to 0. \tag{A.59}
\]

Indeed, for the first relation above, note that, by (2.17),

\[
\int_0^{\infty} \mathbb{E} \left( \frac{1 - e^{-\sqrt{t} \sigma W_1^*}}{\sqrt{t}} I_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) p_z(1, z) dz
\]

\[
\leq \mathbb{E} \left( \sigma W_1^* I_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) \int_0^{\infty} \frac{e^{-w}}{\sqrt{2\pi\sigma^2}} dw \leq K_3 t^{1-Y} \int_0^{\infty} w^{1-Y} e^{-\frac{w^2}{2\sigma^2}} dw.
\]
Hence, by the dominated convergence theorem,

\[ B^{(3)}_{21}(t) = \frac{1}{2} t^{\frac{1}{2} - \frac{x}{2}} \int_{0}^{\infty} \left( \int_{0}^{\infty} F \left( \int_{0}^{\infty} \int_{0}^{\infty} F \left( \frac{t}{2\pi\sigma^2} e^{-\frac{t^2 - 2\sigma^2 u^2}{2\sigma^2}} \right) \right) \left( 1 - \frac{1}{\sqrt{2\pi}} \hat{P}(x) + \frac{C}{K} |x| \right) \right) dx du \]

where we change variables \( u = t^{\frac{1}{2}} + \frac{x}{2} w/|z| \) and apply the Fubini Theorem in the second equality. Using the same argument given after (A.43), we get

\[ B^{(3)}_{21}(t) = \frac{1}{2} t^{\frac{1}{2} + \frac{x}{2} - \frac{1}{2}} \int_{0}^{1} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-\frac{t^2 - 2\sigma^2 u^2}{2\sigma^2}} \left( 1 - \frac{1}{\sqrt{2\pi}} \hat{P}(x) + \frac{C}{K} |x| \right) dx du \]

with

\[ B^{(3)}_{21}(t) := -\frac{cY^2}{2\pi} t^{\frac{1}{2} + \frac{x}{2} - \frac{1}{2}} \int_{0}^{1} \int_{0}^{\infty} e^{-\frac{t^2 - 2\sigma^2 u^2}{2\sigma^2}} \left( 1 - \frac{1}{\sqrt{2\pi}} \hat{P}(x) + \frac{C}{K} |x| \right) dx du \]

For \( B^{(3)}_{22}(t) \), changing variables to \( v = t^{\frac{1}{2} + \frac{x}{2}} \sigma x/u \),

\[ B^{(3)}_{22}(t) = -\frac{cY^2}{2\pi} t^{\frac{1}{2} + \frac{x}{2}} \int_{0}^{1} \int_{0}^{\infty} e^{-\frac{t^2 - 2\sigma^2 u^2}{2\sigma^2}} \left( 1 - \frac{1}{\sqrt{2\pi}} \hat{P}(x) + \frac{C}{K} |x| \right) dx du \]

Hence, by the dominated convergence theorem,

\[ \lim_{t \to 0} B^{(3)}_{21}(t) = -\frac{c^2 Y}{4\sqrt{2\pi}\sigma^2 Y^{2} - 1} \frac{\pi}{\sqrt{2\pi}\sigma^2 Y^{2} - 1} \right) \right) \]

Similarly, for \( B^{(3)}_{22}(t) \),

\[ B^{(3)}_{22}(t) = -\frac{cY(Y-1)}{2}\frac{\pi^2}{\sqrt{2\pi}\sigma^2 Y^{2} - 1} \]

Again, by the dominated convergence theorem,

\[ \lim_{t \to 0} B^{(3)}_{22}(t) = -\frac{c^2 (Y-1)}{4\sqrt{2\pi}\sigma^2 Y^{2} - 1} \]

Combining (A.58)-(A.62),

\[ \lim_{t \to 0} B^{(3)}_{22}(t) = -\frac{c^2 (Y-1)^2 \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2 \left( -Y \right)}{2\pi \sigma^2 Y^{2} - 1} \]

Hence, by combining (A.51), (A.57) and (A.63),

\[ B_{2}(t) = t^{\frac{1}{2}} \left[ \frac{C a^{1-Y}}{2Y} \int_{0}^{\infty} F \left( \frac{t}{2\pi\sigma^2} e^{-\frac{t^2 - 2\sigma^2 u^2}{2\sigma^2}} \right) \left( 1 - \frac{1}{\sqrt{2\pi}} \hat{P}(x) + \frac{C}{K} |x| \right) \right] + o(t^{\frac{1}{2}}) \]

\[ = C a^{1-Y} \left( t^{\frac{1}{2}} d_{32}(t^{\frac{1}{2}}) + o \left( t^{\frac{1}{2}} \right) + \frac{C a^{1-Y}}{2Y} \int_{0}^{\infty} F \left( \frac{t}{2\pi\sigma^2} e^{-\frac{t^2 - 2\sigma^2 u^2}{2\sigma^2}} \right) \left( 1 - \frac{1}{\sqrt{2\pi}} \hat{P}(x) + \frac{C}{K} |x| \right) \right] + o(t^{\frac{1}{2}}) \]

\[ = C a^{1-Y} \left( t^{\frac{1}{2}} d_{32}(t^{\frac{1}{2}}) + d_{32}^{1-Y} + o \left( t^{\frac{1}{2}} \right) + a(t^{\frac{1}{2}}) + o(t^{2-Y}), \ t \to 0. \]
Step 3. We finally study the behavior of $B_3(t)$ by further decomposing it as:

$$B_3(t) = e^{-(\eta t + \sqrt{T_0})} \int_0^\infty \overline{E} \left[ \left( e^{-t \frac{\tilde{u}_1}{\sigma \pi^2 \sqrt{2}}} \right) 1 \{ z_{1 \geq \frac{t}{\sqrt{2}} + w} \} \right] \frac{1 - e^{\sqrt{T_w} \frac{w^2}{2 \pi \sigma^2}}}{\sqrt{2 \pi \sigma^2}} \, dw \tag{A.65}$$

$$+ e^{-(\eta t + \sqrt{T_0})} \int_0^\infty \overline{P} \left( Z_1 \geq \frac{t}{\sqrt{2}} + w \right) \frac{1 - e^{\sqrt{T_w} \frac{w^2}{2 \pi \sigma^2}}}{\sqrt{2 \pi \sigma^2}} \, dw \tag{A.66}$$

$$+ e^{-(\eta t + \sqrt{T_0})} \int_0^\infty \overline{E} \left[ 1 \{ z_{1 \geq \frac{t}{\sqrt{2}} + w} \} \left( e^{-t \frac{\tilde{u}_1}{\sigma \pi^2 \sqrt{2}}} - e^{-t \frac{\tilde{u}_1}{\sigma \pi^2 \sqrt{2}}} (z_{1 \geq \tilde{u}_1}) - t^{\frac{1}{2} - t} Z_1 \right) \right] e^{\sqrt{T_w} \frac{w^2}{2 \pi \sigma^2}} \, dw \tag{A.67}$$

$$+ e^{-(\eta t + \sqrt{T_0})} \int_0^\infty t^{\frac{1}{2} - t} \overline{E} \left( Z_1 1 \{ z_{1 \geq \frac{t}{\sqrt{2}} + w} \} \right) e^{\sqrt{T_w} \frac{w^2}{2 \pi \sigma^2}} \, dw \tag{A.68}$$

$$:= B_{31}(t) + B_{32}(t) + B_{33}(t) + B_{34}(t).$$

First, note that the term $B_{32}(t)$ is similar to the term $B_{22}(t)$ in (A.51) and, thus, using arguments similar to those leading to (A.63) gives,

$$B_{32}(t) = \frac{C \sigma^{-1-Y}}{2Y} \overline{E} \left( |W_1|^1 \right) \frac{t^{1-\frac{1}{Y}} - d_{32} t^{2-Y} + o(t^{2-Y}), \quad t \to 0. \tag{A.69}$$

Next, the term $B_{34}(t)$ is similar to the term $B_{13}(t)$ introduced in (A.31) and, thus, using arguments similar to those leading to (A.49) gives,

$$B_{34}(t) = \frac{C \sigma^{-1-Y}}{2(Y-1)} \overline{E} \left( |W_1|^1 \right) \frac{t^{1-\frac{1}{Y}} + d_{31} t^{2-Y} + o(t^{2}), \quad t \to 0. \tag{A.70}$$

It remains to analyze $B_{31}(t)$ and $B_{33}(t)$. For $B_{31}(t)$, note that the expectation appearing therein can be written as

$$\overline{E} \left[ \left( e^{-t \frac{\tilde{u}_1^+(M^* U_{1}^{(p)} - G^* U_{1}^{(n)}) - 1} \right) 1 \{ z_{1 \geq \frac{t}{\sqrt{2}} + w} \} \right] = \overline{E} \left[ \left( e^{-t \frac{\tilde{u}_1^+(G^* U_{1}^{(p)} - M^* U_{1}^{(n)}) - 1} \right) 1 \{ z_{1 \geq \frac{t}{\sqrt{2}} + w} \} \right],$$

where $(\tilde{U}_{1}^{(p)}, \tilde{U}_{1}^{(n)}) := (-\tilde{U}_{1}^{(n)}, -\tilde{U}_{1}^{(p)}) \equiv (\tilde{U}_{1}^{(p)}, \tilde{U}_{1}^{(n)})$. Thus, $B_{31}(t)$ is the same as the term $B_{21}(t)$ defined in (A.51) but with the role of the parameters $M^*$ and $G^*$ reversed. In other words, if we write $B_{21}(t; M^*, G^*) := B_{21}(t)$ to emphasize the dependence on the parameters $M^*$ and $G^*$, we have that $B_{31}(t) = B_{21}(t; G^*, M^*)$. Therefore, in view of (A.57),

$$\lim_{t \to 0} t^{-\frac{1}{2}} B_{31}(t) = 0. \tag{A.71}$$

To finish, we further decompose $B_{33}(t)$ as:

$$B_{33}(t) = e^{-(\eta t + \sqrt{T_0})} \int_0^\infty \overline{E} \left[ 1 \{ z_{1 \geq \frac{t}{\sqrt{2}} + w} \} \int_{\frac{t}{\sqrt{2}}}^{\frac{t}{\sqrt{2}} + \tilde{u}_1} \left( e^{-t \frac{u}{\sigma \pi^2 \sqrt{2}}} - 1 \right) \, du \right] e^{\sqrt{T_w} \frac{w^2}{2 \pi \sigma^2}} \, dw \tag{A.72}$$

$$= t^{\frac{1}{2}} e^{-(\eta t + \sqrt{T_0})} \int_0^\infty \left[ \int_{-\infty}^{0} (e^{-x} - 1) \, dP \left( Z_1 \geq t^{\frac{1}{2}} + w, \tilde{U}_1 \leq t^{\frac{1}{2}} x \leq Z_1 + \tilde{U}_1 \right) \right] e^{\sqrt{T_w} \frac{w^2}{2 \pi \sigma^2}} \, dw \tag{A.73}$$

$$+ t^{\frac{1}{2}} e^{-(\eta t + \sqrt{T_0})} \int_0^\infty \left[ \int_{0}^{\infty} (e^{-x} - 1) \, dP \left( Z_1 \geq t^{\frac{1}{2}} + w, \tilde{U}_1 \leq t^{\frac{1}{2}} x \leq Z_1 + \tilde{U}_1 \right) \right] e^{\sqrt{T_w} \frac{w^2}{2 \pi \sigma^2}} \, dw. \quad \text{(A.69)}$$

When $x < 0$, by (2.18), for any $t > 0$ and $w > 0$,

$$P_t(w, x) := \overline{P} \left( Z_1 \geq t^{\frac{1}{2}} + w, \tilde{U}_1 \leq t^{\frac{1}{2}} x \leq Z_1 + \tilde{U}_1 \right) \leq \overline{P} \left( \tilde{U}_1 \leq t^{\frac{1}{2}} x \right) \leq \overline{E} \left( e^{-\tilde{U}_1} \right) e^{t^{\frac{1}{2}} x} = \sigma e^{x}. \tag{A.74}$$
Hence, for $0 < t < 1$ and since $1 < Y < 2$,

$$0 \leq t^{-1}e^{-(nt+\sqrt{t})} \int_0^\infty \left[ \int_{-\infty}^t (e^{-x} - 1) \tilde{P} (Z_1 \geq t^{\frac{1}{2}} - \frac{1}{2} w, \bar{U}_1 \leq t^{\frac{1}{2}} x \leq Z_1 + \bar{U}_1) \, dx \right] e^{\sqrt{tw} \frac{\sigma^2}{\sqrt{2\pi}}} dw$$

$$\leq t^{-1}e^{-(nt+\sqrt{t})} e^t \int_{-\infty}^0 (e^{-x} - 1) e^{\frac{1}{2} t^2 x} dx \int_0^\infty w \, e^{\frac{\sigma^2}{2}} e^{\frac{\sigma^2}{\sqrt{2\pi}}} dw$$

$$= \frac{t^2}{1 - t^2} e^{-(nt+\sqrt{t})} e^t \int_0^\infty w \, e^{\frac{\sigma^2}{2}} e^{\frac{\sigma^2}{\sqrt{2\pi}}} dw \to 0, \quad t \to 0. \quad (A.70)$$

For the second integral in (A.69), using arguments similar to those leading to (2.14-i), there exists a constant $\bar{w} \in (0, \infty)$, such that

$$t^{-1} \tilde{P} \left( t^{\frac{1}{2}} x \leq Z_1 + \bar{U}_1 \right) \leq \bar{w} x^{-Y}, \quad \text{for any } x > 0,$$

and thus, by the dominated convergence theorem,

$$\lim_{t \to 0} t^{-1}e^{-(nt+\sqrt{t})} \int_0^\infty \left[ \int_{-\infty}^t (e^{-x} - 1) \tilde{P} (Z_1 \geq t^{\frac{1}{2}} - \frac{1}{2} w, \bar{U}_1 \leq t^{\frac{1}{2}} x \leq Z_1 + \bar{U}_1) \, dx \right] e^{\sqrt{tw} \frac{\sigma^2}{\sqrt{2\pi}}} dw$$

$$= \int_0^\infty \left[ \int_{-\infty}^t (e^{-x} - 1) \left( \lim_{t \to 0} \frac{1}{t} \tilde{P} (Z_1 \geq t^{\frac{1}{2}} - \frac{1}{2} w, \bar{U}_1 \leq t^{\frac{1}{2}} x \leq Z_1 + \bar{U}_1) \right) \, dx \right] e^{\frac{\sigma^2}{2}} e^{\frac{\sigma^2}{\sqrt{2\pi}}} dw. \quad (A.71)$$

It remains to compute the limit in the above integrand. For any $t > 0$, $x > 0$ and $w > 0$,

$$\frac{1}{t} P_t(w, x) = \frac{1}{t} \tilde{P} \left( \bar{U}_1^{(p)} + \bar{U}_1^{(n)} \geq t^{\frac{1}{2}} - \frac{1}{2} w, M^* \bar{U}_1^{(p)} - G^* \bar{U}_1^{(n)} \leq t^{\frac{1}{2}} x \leq M \bar{U}_1^{(p)} - G \bar{U}_1^{(n)} \right)$$

$$= \frac{1}{t} \int \tilde{P} \left( \bar{U}_1^{(p)} \geq t^{\frac{1}{2}} - \frac{1}{2} w + u, \frac{t^{\frac{1}{2}} x - Gu}{M} \leq \bar{U}_1^{(p)} \leq \frac{t^{\frac{1}{2}} x - G^* u}{M^*} \right) p_U(1, u) \, du.$$

Note that

$$\frac{t^{\frac{1}{2}} x - Gu}{M} \leq \frac{t^{\frac{1}{2}} x - G^* u}{M^*} \iff \frac{t^{\frac{1}{2}} x - Gu}{M + G} \leq \frac{t^{\frac{1}{2}} x - G^* u}{M^*} \iff u \leq \frac{t^{\frac{1}{2}} x - M^* t^{\frac{1}{2}} - \frac{1}{2} w}{M + G},$$

$$\frac{t^{\frac{1}{2}} - \frac{1}{2} w + u}{M + G} \leq \frac{t^{\frac{1}{2}} x - Gu}{M} \iff u \leq \frac{t^{\frac{1}{2}} x - Mt^{\frac{1}{2}} - \frac{1}{2} w}{M + G}.$$

Hence,

$$\frac{1}{t} P_t(w, x) = \frac{1}{t} \int_{-\infty}^{\frac{t^{\frac{1}{2}} x - M^{\frac{1}{2}} - \frac{1}{2} w}{M + G}} \tilde{P} \left( \frac{t^{\frac{1}{2}} x - Gu}{M} \leq \bar{U}_1^{(p)} \leq \frac{t^{\frac{1}{2}} x - G^* u}{M^*} \right) p_U(1, u) \, du$$

$$+ \frac{1}{t} \int_{\frac{t^{\frac{1}{2}} x - M^{\frac{1}{2}} - \frac{1}{2} w}{M + G}}^{\frac{t^{\frac{1}{2}} x - M^{\frac{1}{2}} - \frac{1}{2} w}{M^*}} \tilde{P} \left( \frac{t^{\frac{1}{2}} x + u}{M + G} \leq \bar{U}_1^{(p)} \leq \frac{t^{\frac{1}{2}} x - G^* u}{M^*} \right) p_U(1, u) \, du$$

$$:= I_1(t; w, x) + I_2(t; w, x). \quad (A.72)$$

For the first integral in (A.72), note that for any $t > 0$, $x > 0$ and $w > 0$,

$$u \leq \frac{t^{\frac{1}{2}} x - M^{\frac{1}{2}} - \frac{1}{2} w}{M + G} < \frac{t^{\frac{1}{2}} x}{M + G} < \frac{t^{\frac{1}{2}} x}{G} \Rightarrow t^{\frac{1}{2}} x - Gu > 0, \quad \frac{x - Mw^\sqrt{t}}{M + G} > 0 \iff t < \frac{x^2}{M^2 w^2}.$$
Hence, by (2.13) and the dominated convergence theorem, for any \( x > 0, w > 0 \) and \( u \leq t^{-\frac{1}{2}}(x - Mw)/(M + G) \),

\[
\lim_{t \to 0} I_1(t; w, x) = \int_{\mathbb{R}} p_{U}(1, u) \left( \lim_{t \to 0} \frac{1}{t} \mathbb{I} \left( \frac{t^{-\frac{1}{2}}x - Gu}{M} \leq U_1^t \leq \frac{t^{-\frac{1}{2}}x - G^*u}{M^*} \right) \right) \mathbb{I} \left\{ u \leq \frac{t^{-\frac{1}{2}}/x - M^* - 1/\sqrt{w}}{M + G} \right\} du
\]

\[
= \int_{\mathbb{R}} p_{U}(1, u) \left( \lim_{t \to 0} \frac{1}{t} \mathbb{I} \left( U_1^t \geq \frac{t^{-\frac{1}{2}}x - Gu}{M} \right) \right) \mathbb{I} \left\{ u \leq \frac{t^{-\frac{1}{2}}/x - M^* - 1/\sqrt{w}}{M + G} \right\} du
\]

\[
- \int_{\mathbb{R}} p_{U}(1, u) \left( \lim_{t \to 0} \frac{1}{t} \mathbb{I} \left( U_1^t \geq \frac{t^{-\frac{1}{2}}x - G^*u}{M^*} \right) \right) \mathbb{I} \left\{ u \leq \frac{t^{-\frac{1}{2}}/x - M^* - 1/\sqrt{w}}{M + G} \right\} du
\]

\[
= \frac{C}{Y} \left( M^Y - (M^*)^Y \right) x^{-Y}. \tag{A.73}
\]

For the second integral in (A.72), since for any \( x > 0 \) and \( w > 0 \), \( t^{-\frac{1}{2}}x - Mt^{-\frac{1}{2}}w > 0 \) if and only if \( t < w^2/(M^2w^2) \),

\[
0 \leq \frac{1}{t} \int_{\frac{t^{-\frac{1}{2}}x - Mt^{-\frac{1}{2}}w + tw}{M + G}} \mathbb{P} \left( t^{-\frac{1}{2}}x + u \leq U_1^{t(\rho)} \leq \frac{t^{-\frac{1}{2}}x - G^*u}{M^*} \right) p_{U}(1, u) du
\]

\[
\leq \frac{1}{t} \mathbb{P} \left( U_1^{t(\rho)} \geq \frac{t^{-\frac{1}{2}}x + tw}{M + G} \right) \mathbb{P} \left( -\frac{t^{-\frac{1}{2}}x - Mt^{-\frac{1}{2}}w}{M} \right) \to 0, \quad t \to 0. \tag{A.74}
\]

Combining (A.71)-(A.74), and using (A.2),

\[
\lim_{t \to 0} \frac{e^{-(\gamma t + \sqrt{t}t_1)}}{t} \int_{0}^{\infty} \left( e^{-x - 1} \mathbb{I} \left( Z_t \geq t^{-\frac{1}{2}}x, U_1^t \leq t^{-\frac{1}{2}}x \leq Z_1 + U_1^t \right) \right) dx \left( e^{\sqrt{t}w} e^{-\frac{w^2}{2t^2}} \right) dw = - \frac{CT(-Y)}{2} \left[ M^Y - (M^*)^Y \right],
\]

which, together with (A.70), leads to:

\[
\lim_{t \to 0} t^{-\frac{1}{2}}B_{33}(t) = - \frac{d_3}{2}, \tag{A.75}
\]

where \( d_3 = CT(-Y) \left( M^Y - (M^*)^Y \right) \). Hence, by combining (A.65)-(A.68) and (A.75), we get

\[
B_3(t) = - \frac{d_3}{2} t^2 + \frac{C\sigma^1 - Y}{2(Y - 1)} \mathbb{E} \left( |W_1^*|^{1 - Y} \right) t^{1 - \frac{Y}{2}} + (d_3' - d_3'') t^{2 - Y} + o(t^2) + o(t^{2 - Y}), \quad t \to 0. \tag{A.76}
\]

We are now in possession of obtaining the higher-order asymptotic expansion. First, by combining (A.30), (A.50), (A.64) and (A.76),

\[
A_1(t) = - \frac{d_3}{2} t^2 + \frac{C\sigma^1 - Y}{Y(Y - 1)} \mathbb{E} \left( |W_1^*|^{1 - Y} \right) t^{1 - \frac{Y}{2}} + 2(d_3' - d_3'') t^{2 - Y} + o(t^2) + o(t^{2 - Y}), \quad t \to 0.
\]

By combining the previous expression with (A.27)-(A.28),

\[
\Delta_0(t) = \left( \frac{\gamma - e_1}{2} - \frac{\sigma^2}{4} - d_3' \right) t^2 + \frac{C\sigma^1 - Y}{2(Y - 1)} \mathbb{E} \left( |W_1^*|^{1 - Y} \right) t^{1 - \frac{Y}{2}} + 2 \left( d_3' - d_3'' - \frac{e_2}{2} \right) t^{2 - Y} + o(t^2) + o(t^{2 - Y}),
\]

which yields (3.12), by noting that the coefficient of the first term above reduces to the expression \( d_{31} \) in (3.10) and that \( d_{32} = 2(d_3' - d_3'') \).

**Proof of Proposition 3.5.** When the diffusion component is present, [15, Proposition 5] implies that \( \hat{\sigma}(t) \to \sigma \) as \( t \to 0 \). In particular, \( \hat{\sigma}(t)t^{1/2} \to 0 \) as \( t \to 0 \) and, thus, (A.22) above still holds. Let \( \tilde{\sigma}(t) := \hat{\sigma}(t) - \sigma \), then \( \tilde{\sigma}(t) \to 0 \) as \( t \to 0 \) and, (A.22) can be written as

\[
C_{BS}(t, \tilde{\sigma}(t)) = \frac{\sigma}{\sqrt{2\pi}} t^2 + \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}} t^2 - \frac{\tilde{\sigma}(t)^3}{24\sqrt{2\pi}} t^2 + O \left( t^2 \right), \quad t \to 0. \tag{A.77}
\]
By comparing (3.7)-(3.8) and (A.77), and since the third term in (A.77) is \(O\left(\frac{t^3}{2}\right)\), we have

\[
\frac{C^2_{21\frac{Y}{2}}Y^{1-Y}}{\sqrt{n}} \Gamma \left(1 - \frac{Y}{2}\right) t^{\frac{3-Y}{2}} \sim \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}} t^{\frac{1}{2}}, \quad t \to 0,
\]

and, therefore,

\[
\tilde{\sigma}(t) \sim \frac{C^2_{21\frac{Y}{2}}Y^{1-Y}}{\sqrt{n}} \Gamma \left(1 - \frac{Y}{2}\right) t^{\frac{1}{2}} := \sigma t^{\frac{1}{2}}, \quad t \to 0. \quad \text{(A.78)}
\]

Next, set \(\bar{\sigma}(t) := \tilde{\sigma}(t) - \sigma - \sigma t^{1-Y}\), then (A.77) can be rewritten as:

\[
C_{BS}(t, \bar{\sigma}(t)) = \sigma \sqrt{\frac{2}{\pi}} t^{\frac{3}{2}} + \frac{C^2_{21\frac{Y}{2}}Y^{1-Y}}{\sqrt{n}} \Gamma \left(1 - \frac{Y}{2}\right) t^{\frac{3-Y}{2}} + \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}} t^{\frac{1}{2}} - \frac{\tilde{\sigma}(t)^3}{24\sqrt{2\pi}} t^{\frac{3}{2}} + O \left(\frac{t^5}{2}\right), \quad t \to 0. \quad \text{(A.79)}
\]

We can finally deduce (3.12) by comparing the first three terms in (3.11) with (A.79). \(\square\)

References


