STAT 417    Fall 2008    Dr. DasGupta

First Midterm

**NOTE** a. Please explain your work legibly and clearly to get credit.

b. The exam is open notes, but closed book.

c. You may get a maximum of 5 free bonus points. Answer as much as you can.
1. Suppose two independent observations $X_1, X_2$ are available, of which $X_1 \sim N(\mu, 1)$, and $X_2 \sim N(\mu, 16)$.

   a) Find, with proof, a sufficient statistic.

   b) Give an unbiased estimate of $\mu$ based on the sufficient statistic. What is its variance?

   $5+5 = 10$ points

   The likelihood function is

   \[
   l(\theta) = e^{-\frac{1}{2} (x_1 - \mu)^2} e^{-\frac{1}{32} (x_2 - \mu)^2}
   \]

   \[
   = e^{-\frac{x_1^2}{2} - \frac{x_2^2}{32}} e^{\frac{-x_1^2}{2} - \frac{x_2^2}{32}} \cdot \mu (\frac{x_1 + x_2}{16})
   \]

   \[
   \therefore T(X_1, X_2) = X_1 + \frac{X_2}{16} \text{ is a sufficient stat.}
   \]

   by the Neyman-Fisher theorem.

   \[
   E\left( X_1 + \frac{X_2}{16} \right) = \frac{17}{16} \mu
   \]

   \[
   \therefore E\left( \frac{16}{17} \left( X_1 + \frac{X_2}{16} \right) \right) = \mu
   \]

   \[
   \sqrt{\text{Var}\left( \frac{16}{17} \left( X_1 + \frac{X_2}{16} \right) \right)} = \frac{16}{17} \cdot \frac{1}{17} + \frac{1}{17^2} \cdot \frac{16}{17} = \frac{16}{17} \cdot \frac{17}{17} = \frac{16}{17}
   \]
2. Suppose \( n = 4 \) observations with values 0.2, 1, 5 are available from a Poisson(\( \lambda \)) distribution.

a. Give an approximate plot of the likelihood function. The plot should be correct about the main features, but NEED NOT be drawn to exact scale.

b. Find the numerical value of the MLE of \( \lambda \).

c. Find the MSE of the MLE of \( \lambda \), and plot the MSE.

\[ 4 + 4 + 3 = 11 \text{ points} \]

\[
\text{likelihood function} \quad l(\lambda) = e^{-4\lambda} \lambda^{0+2+1+5} = e^{-4\lambda} \lambda^8.
\]

\[
\log l(\lambda) = 8 \log \lambda - 4\lambda.
\]

\[
\frac{d}{d \lambda} \log l = \frac{8}{\lambda} - 4 = 0 \quad \Rightarrow \quad \lambda = 2.
\]

\( \log l(\lambda) \) has one stationary point, goes to zero as \( \lambda \to 0, \infty \), and is everywhere differentiable. \( \lambda = 2 \) is the global max.

The value of MLE of \( \lambda \) is \( \hat{\lambda} = 2 \).

\[
\text{MSE of } \hat{\lambda} = \overline{X} = \frac{1}{n} (\overline{X} - \lambda)^2 = \frac{1}{n} = \frac{\lambda}{4}
\]

\[ 0 \quad \rightarrow \hat{\lambda} \]
3. Give an example of each of the following phenomena (no proofs needed):
   a. An MLE which is biased, but asymptotically unbiased.
   b. A concept invented by Fisher.
   c. A density for which the Cramér-Rao lower bound is not applicable.
   d. An unbiased estimate of $\theta$ in the $U[\theta, 2\theta]$ distribution.
   e. A distribution which has only one free parameter, but sufficient statistics are two dimensional.
   f. A function of $p$ in the $Ber(p)$ distribution which is not unbiasedly estimable.

$6 \times 5 = 30$ points

a) $\mathbb{N}(\mu, \sigma^2)$; $\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$

b) Sufficiency / Likelihood

c) $U[0, \theta]$

d) $\frac{2}{x}$

e) $\chi^2(\mu, \mu^1), \mu > 0$

f) $\frac{1}{1-x}$
4. Let \( X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \theta(1-\theta)), 0 < \theta < 1 \).

a. Derive an expression for the score function.

b. Derive the Fisher information function.

c. Find the CRLB for the variance of an unbiased estimate of \( \theta \).

d. Give an explicit unbiased estimate of \( \theta \).

e. Does your estimate of part d) attain the CRLB? Prove your answer.

\[ 3+6+2+1+2 = 14 \text{ points} \]

\[
\ell(\theta) = \frac{1}{(\theta(1-\theta))^{\frac{n}{2}}} \cdot e^{-\frac{1}{2\theta(1-\theta)} \sum (X_i - \theta)^2}
\]

\[
\Rightarrow \log \ell(\theta) = -\frac{n}{2} \log \left( \frac{\theta}{1-\theta} \right) - \frac{1}{2\theta(1-\theta)} \sum (X_i - \theta)^2
\]

\[
= -\frac{n}{2} \log \left( \frac{\theta}{1-\theta} \right) - \frac{\sum X_i^2}{2\theta(1-\theta)} + \frac{\sum X_i}{1-\theta} - \frac{n\theta}{2(1-\theta)}
\]

i. Score function

\[
= \frac{d}{d\theta} \log \ell(\theta) = \frac{\sum X_i^2 - 2\theta \sum X_i - \theta \left[ n - 2\theta(\sum X_i + n) + 2n\theta^2 \right]}{2\theta^2 (1-\theta)^2}
\]

\[
\frac{d^2}{d\theta^2} \log \ell(\theta) = \frac{\sum X_i^2 (2 - 6\theta(1-\theta)) + \theta \left[ -4\theta^2 \sum X_i + n(3\theta - 1 - 2\theta^2 + \theta) \right]}{2\theta^3 (1-\theta)^3}
\]

\[
\therefore \quad I(\theta) = \frac{\left[ 2 - 6\theta(1-\theta) \right] \left( \theta^2 + \theta(1-\theta) \right) + \theta \left[ -4\theta^3 + 3\theta - 1 - 2\theta^2 + 2\theta^3 \right]}{2\theta^3 (1-\theta)^3}
\]

\[
\Rightarrow \quad I(\theta) = \frac{\theta \left[ 2 - 6\theta(1-\theta) \right] + \theta \left[ 3\theta - 2\theta^2 - 2\theta^3 - 1 \right]}{2\theta^3 (1-\theta)^3}
\]
The use of $\theta$ is simply $\bar{X}$. Its variance is $\frac{\theta(1-\theta)}{n}$. Evidently, $\frac{\theta(1-\theta)}{n} \neq \frac{1}{n \bar{X}(\theta)}$. Therefore, $\frac{1}{n \bar{X}(\theta)}$ is the CRLB.