STAT 528   Fall 2015   Dr. DasGupta
Second Midterm

1. Please show your work in detail, and legibly and clearly to receive credit.
2. Answer all questions; 5 bonus points are possible.

Good Luck and Happy Thanksgiving!
1. a) Suppose $Y \sim N(\theta, 1)$. But we only observe whether $Y > 0$ or $Y \leq 0$. Suppose also that $\theta$ has the $N(0, 1)$ prior. If the value of $Y$ was 2, what is the value of the posterior mean of $\theta$ based on what we actually observed? Give a numerical value to two decimals for your answer.

9 points

b) Consider the same scenario, but with $n$ iid observations. That is, suppose $Y_1, Y_2, \ldots, Y_n$ are iid $N(\theta, 1)$. But we only observe whether the $Y_i$ are $> 0$ or $\leq 0$.

Give an exact analytical formula for the MLE of $\theta$ based on what we actually observed.

5 points

Let $X_i = I_{Y_i > 0}$, then $X_1, \ldots, X_n \sim \text{iid } \text{Bern}(\theta)$, with $p = P(Y > 0) = P(Y - \theta > -\theta)$

If $n = 1$, $f(x|\theta) = \Phi(\theta) (1 - \Phi(\theta))$, $x = 0, 1, = \Phi(\theta)$.

If $y = 2$, then $x = 1$, and

$$E(\hat{\theta} \mid X = x) = -\int^0_{-\infty} \frac{\phi(\theta)}{\Phi(\theta)} \phi(\theta) d\theta$$

$$= -\int^0_{-\infty} \phi'(\theta) \Phi(\theta) d\theta$$

$$= \frac{1}{\sqrt{\pi}} = \sqrt{\frac{1}{\pi}}$$

For general $n$, $\Phi(\theta) = \frac{\sum X_i}{n}$, if $\sum X_i \neq 0$.

$$\Rightarrow \hat{\theta} = \Phi\left( \frac{\sum X_i}{n} \right)$$

If $\sum X_i \neq 0$, $n$.

An MLE of $\theta$ based on $X_i$ does not exist if $\sum X_i = 0$ or $n$. 

2
2. Let $X_1, \ldots, X_n$ be iid $N(\mu, 1), -\infty < \mu < \infty$. Classify the following estimators of $\mu$ as admissible or inadmissible under squared error loss, with clear proofs for each case:

$\bar{X} + .5; .5\bar{X} + .5; M_n$ (the sample median); the constant estimate 10,

14 points

$R(\mu; \bar{X} + .5) = \frac{1}{n} + .5^2 < \frac{1}{n} = R(\mu; \bar{X}) \quad \forall \mu$.

$\therefore \bar{X} + .5$ is inadmissible.

$R(\mu; 0X + 0) = 0 \leq 0 < 1$, being a Bayes estimator

$0X + 0$ is admissible.

$R(\mu; .5\bar{X} + .5) \text{ is admissible}$.

$E(M_n) = \mu + \frac{1}{n} \Rightarrow M_n$ is an UBE.

$\therefore R(M_n, M_n) = \text{Var}_\mu(M_n) < \text{Var}_\mu(\bar{X}) = R(\mu; \bar{X}) \quad \forall \mu$.

$\bar{X}$ being the UMVUE; $M_n$ is inadmissible.

Any constant is Bayes estimator with a point prior

and hence admissible.
3. Calculate in closed form the Fisher information function for the Beta(\(\alpha, \alpha\))
density
(Note that the two Beta parameters are assumed to be EQUAL)

7 points

\[
    f(x|\alpha) = x^{\alpha-1}(1-x)^{\alpha-1} \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \\
    \text{I.o.r.} 0 < x < 1.
\]

\[
    \therefore \text{for } 0 < x < 1, \log f = (\alpha-1)[\log x + \log (1-x)]
    + \log \Gamma(2\alpha) - 2 \log \Gamma(\alpha)
\]

\[
    \frac{\partial}{\partial x} \log f = \frac{\log x + \log (1-x) + 2 \frac{\Gamma'(2\alpha)}{\Gamma(2\alpha)} - 2 \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}}{\Gamma(2\alpha)}
\]

\[
    \frac{\partial^2}{\partial x^2} \log f = \frac{\log \Gamma(2\alpha) - 4 \left[ \frac{\Gamma'(2\alpha)}{\Gamma(2\alpha)} \right]^2 - 2 \frac{\Gamma''(\alpha)}{\left( \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} \right)} + 2 \left[ \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right]^2}{\Gamma(2\alpha)}
\]

\[
    \therefore I(\alpha) = \text{negative of (X)}.
\]
4. Give an example, precisely stated with detail, but without proofs, of each of the following phenomena:

a. The MLE is also the UMVUE.
b. A unique MLE exists, is also the UMVUE, but not minimax, with squared error loss.
c. A unique MLE exists, is not the UMVUE, and is not admissible.
d. Infinitely many MLEs exist for all \( n \) and all sample values \( X_1, \ldots, X_n \).
e. Infinitely many MLEs exist for even \( n \), but a unique MLE exists for odd \( n \).
f. A location parameter distribution for which a unique MLE always exists, but it is different from \( \bar{X} \).
g. An estimator which is admissible and minimax, with squared error loss, but biased.

21 points

a) \( N(\mu, 1) \), estimate \( \mu \)
b) \( \text{Ber}(p) \), estimate \( p \)
c) \( \text{U}[0, b] \), estimate \( b \) or \( N(\mu, \sigma^2) \), estimate \( \sigma^2 \)
d) \( \text{U}[\theta - c, \theta + c] \), estimate \( \theta \)
e) \( \text{DoubleExp}(\mu, 1) \), estimate \( \mu \)
f) \( \text{C}(\mu, 1) \)
g) \( \text{Ber}(\theta) \), the estimator \( \frac{\sum X_i + \sqrt{n}}{n + 1} \)
5. Suppose $X$ is uniform on $[\theta, 2\theta]$ and $\theta$ has a Gamma prior with density $c\theta^2 e^{-\theta}$, where $c$ is the normalizing constant of that Gamma density. Find in closed form the posterior mean of $\theta$; NOTE THAT THERE IS ONLY OBSERVATION $X$, THAT IS $n = 1$.

7 points

\[ f(x|\theta) = \frac{1}{\theta} I_{\theta < x < 2\theta} . \]

\[ \therefore E(\theta | x = x) = \frac{\int_{0}^{\infty} \frac{1}{\theta} I_{\frac{x}{2} < \theta < x} c \theta^2 e^{-\theta} d\theta}{\int_{0}^{\infty} \frac{1}{\theta} I_{\frac{x}{2} < \theta < x} c \theta^2 e^{-\theta} d\theta} \]

\[ = \frac{\int_{\frac{x}{2}}^{x} \theta^2 e^{-\theta} d\theta}{\int_{\frac{x}{2}}^{\infty} \theta^2 e^{-\theta} d\theta} \]

\[ = \frac{e^{-\frac{x}{2}} (x^2 + x + 2) - 4(x^2 + 2x + 2)}{2 e^{-\frac{x}{2}} (x + 2) - 4(x + 1)} \]

by integration by parts.
6. State, without proof, the exact expression for the Cramér-Rao lower bound on the variance of an arbitrary unbiased estimate of $\mu^3$ if you have $n$ iid observations form $N(\mu, 1)$.

2 points

If $T$ is an unbiased estimate of $\mu^3$, $E(T) = \mu^3$, $\frac{d}{d\mu} E(T) = 3\mu^2$, and so $\text{Var}(T) > \frac{(3\mu^2)^2}{n \text{I}(\mu)} = \frac{9\mu^4}{n}$. 