STAT 528       Spring 2015       Dr. DasGupta
Second Midterm

NOTE a. Please show your work in detail, legibly, and clearly to get credit. Incomplete work will get very little partial credit.
b. If your answer to a problem is obviously wrong, it is better not to leave it there.
c. The test would be graded out of 60 points. The points add to 62, and so, you can get up to 2 bonus points.

Good Luck and Have a Safe Break!
Group A

1. Define identifiability of a family of distributions and prove that any distribution in the canonical one parameter exponential family is identifiable under some conditions. State precisely these conditions. 7 points

\[ \rho = \{ \theta : \theta \in \Theta \} \]

\[ \theta_1, \theta_2 \in \Theta \implies \rho_{\theta_1} \neq \rho_{\theta_2} \]

\[ \rho_{\theta_1}(A) \neq \rho_{\theta_2}(A) \]

The canonical one parameter exp. family with density

\[ f(x|\theta) = e^{\theta t(x) - \eta(\theta) \lambda(x)} , \theta \in \Theta \]

is identifiable whenever \( \{ f(x|\theta) : \theta \in \Theta \} \) is non-singular.

For, \( \rho_{\theta_1} = \rho_{\theta_2} \implies E_{\theta_1}(T) = E_{\theta_2}(T) \implies \eta'(\theta_1) = \eta'(\theta_2) \)

\[ \implies \theta_1 = \theta_2 \]

\( \implies \eta' \) is one-to-one

\[ \eta''(\theta) \text{ being } V_{\theta}(T) \]
2. Precisely define a minimal sufficient statistic.

3 points

A statistic $T(x_1, \ldots, x_n)$ is minimal sufficient if

a) $T$ is sufficient.

b) Given any other sufficient statistic $S(x_1, \ldots, x_n)$,
   $T = h(S)$ for some function $h(\cdot)$. 

3. Let $X_1, X_2, \ldots, X_n$ be iid $U[-\theta, \theta], \theta > 0$. Derive with a rigorous proof the MLE of $\theta$ and show that it is unique.

6 points

The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} I_{|X_i| \leq \theta} = \frac{1}{(2\theta)^n} \prod_{i=1}^{n} I_{\theta > |X_i|}$$

$$= \frac{1}{(2\theta)^n} I_{\theta > \max_i |X_i|}$$

$$\therefore \max_i |X_i| \text{ is the MLE and being strictly}$$
4. True or False? (No proofs needed; write out ONE of True, False. DO NOT write T, F).
   a) An MLE is a root of the likelihood equation.
   b) If \( T(X_1, \cdots, X_n) \) is sufficient, then \( T^2(X_1, \cdots, X_n) \) is also sufficient.
   c) \( Beta(\alpha, \alpha), \alpha > 0 \) belongs to the curved exponential family.
   d) For estimating a univariate normal mean under squared error loss, the sample mean is a Bayes estimator.
   e) \( N(\theta, \theta) \), \( \theta > 0 \), is a scale parameter family.
   15 points

\[
\text{a) False, b) False, c) True, d) False, e) False.}
\]
5. Suppose $X_1, \ldots, X_n$ are iid $C(\mu, 1)$; here $C$ stands for Cauchy.

Find explicitly $\lim_{n \to \infty} P$(The likelihood function is unimodal).

5 points

Let $N_\mu$ be the number of local maxima and minima
of the likelihood function. Then, $N_\mu$ is, a.s.,
odd, when $n \geq 1$.

Denote $N_\mu = 2k_n + 1$, $k_n \geq 0$, $n \geq 1$.

(It was stated in lecture) that $k_n$ is asymptotically
$\text{Poisson} \left( \frac{1}{n} \right)$.

$\lim \frac{k_n}{n}$ (The likelihood $f_\mu$ is unimodal)

$\Rightarrow \lim_{n} P(k_n = 1) = e^{-\frac{1}{n}}$

$\Rightarrow \lim_{n} P(N_\mu = 1) = e^{-\frac{1}{n}}$

(Nice interpretation of $e^{-\frac{1}{n}}$).
Group B

6. You are given three independent observations $X_1, X_2, X_3$, of which $X_1 \sim Bin(3, p), X_2 \sim Bin(4, 2p),$ and $X_3 \sim Bin(3, p^2)$. The values of the actual observations are $x_1 = 2, x_2 = 3, x_3 = 1$.

a) Plot carefully the likelihood function. Your plot must have the correct shape, scaled accurately, and all points of maxima and minima have to be correctly shown.

b) Discuss with enough mathematical detail maximum likelihood estimation of $p$ based on these specific data, but do not compute an MLE.

10 points

For general $x_1, x_2, x_3$, the likelihood function is

$$L(p) = e^p (1-p)^{x_1} \cdot p^{1-x_1} \cdot \frac{1}{b} \left(1-\frac{1}{b}\right)^{x_2} \cdot p^{1-x_2} \cdot \left(1-\frac{1}{b}\right)^{x_3} \cdot p^{1-x_3}$$

For the given data, $L(p) = e^p \left(1-p\right)^3 \left(1+p\right) \cdot \left(1-\frac{1}{b}\right)^3 \cdot \left(1-\frac{1}{b}\right)^3 \cdot \left(1-\frac{1}{b}\right)^3$

$\rightarrow 0$, $p \rightarrow 0, 1$

Since $L(p)$ must be at a critical point in $(0, 1)$,

$$\frac{d}{dp} \log L(p) = \frac{1}{p} - \frac{3}{1-p} + \frac{3}{1+p} - \frac{3}{1-2p} = 0$$

$\Leftrightarrow 2b^3 - 10b^2 - 17b + 7 = 0$.

There are 2 roots in $(0, 1)$: $\frac{429}{27}$ and $\frac{435}{20}$

$L(\frac{429}{27}) = \approx 0.0001$

1 mle
7. Suppose $X_1, \ldots, X_n$ are iid observations from the density $f(x|\mu) = e^{\mu-x}, x \geq \mu$. Suppose $\mu$ has a standard exponential prior distribution. Assuming squared error loss, find the Bayes risk of the MLE of $\mu$.

14 points

The MLE is $e^{\mu} = \frac{1}{e} \mathbb{E} I_{\mu \leq X(1)}$.

$\therefore$ the MLE of $\mu$ is $X(1)$.

$n(X(n) - \mu) \sim \text{Exp}(1)$.  \[ \therefore X_i - \mu \stackrel{iid}{\sim} \text{Exp}(1) \]

$R(\lambda, \lambda(1)) = \mathbb{E}_\mu \frac{(X(n) - \mu)^2}{n^2} = \frac{1}{n^2} \mathbb{E} \left[ \frac{n(X(1) - \mu)^2}{n} \right]$

$= \frac{\frac{2}{n^2}}{1 + \frac{1}{n^2}}$.  \[ \therefore R(\lambda, \lambda(1)) = \frac{2}{n^2} \left[ 1 + \frac{1}{n^2} \right] \]

$\therefore \mathbb{R}(\bar{X}(n), \lambda(n)) = \frac{2}{n^2}$.  \[ \therefore \mathbb{R}(\bar{X}(n), \lambda(n)) \approx \frac{2}{n^2} \]
8. Write your instructor's full legal name.
2 points

Mr. Buffalo