Stat-Math 519, Fall 2016 Dr. DasGupta

First Midterm

• Please show your work in detail, legibly and clearly to be eligible for credit.

• The test will be graded out of 60 points; you can get a maximum of 5 bonus points. Answer as much as you can.

Good Luck!

mean = 45.8

median = 46.5

standard deviation = 11.4

51 = A−

30 = B−

1. In a completely dark room with 10 chairs, six people came and occupied six chairs at random.
a) What is the probability that at least one of three specific chairs got occupied?
b) What is the probability that all but one of three specific chairs got occupied?
5+5 = 10 points

a $$1 - \frac{\binom{7}{6}}{\binom{10}{6}}$$

b $$\frac{\binom{3}{2}\binom{7}{4}}{\binom{10}{6}}$$

2. n distinct balls are distributed completely at random into n distinct cells.

a) Given that exactly one cell remains empty, what is the probability that the first cell is the empty one? Prove your answer.

4 points

b) Given that at least one cell remains empty, what is the probability that the first cell is empty?

7 points

<u>a</u> Let A_i = Cell i remains empty

Then $P(A_1 / \text{Exactly one cell remains empty})$

$= P(A_2 / \text{Exactly one cell remains empty})$

$= \cdots$

$= P(A_n / \text{Exactly one cell remains empty})$.

$\therefore\ 1 = P(\bigcup_{i=1}^{n} A_i / \text{Exactly one cell remains empty})$

$= \sum_{i=1}^{n} P(A_i / \text{Exactly one cell remains empty})$

$= n\,P(A_1 / \text{Exactly one cell remains empty})$

$\Rightarrow P(A_1 / \text{Exactly one cell remains empty}) = \frac{1}{n}$.

<u>b</u> $$P(A_1 / \bigcup_{i=1}^{n} A_i) = \frac{P(A_1)}{P(\bigcup_{i=1}^{n} A_i)} = \frac{P(A_1)}{1 - P(\bigcap_{i=1}^{n} A_i^c)} = \frac{\frac{(n-1)^n}{n^n}}{1 - \frac{n!}{n^n}} = \frac{(n-1)^n}{n^n - n!}, \quad n > 1.$$

3. Coupons are drawn, independently, with replacement, one at a time, from a set of 10 coupons. Find, explicitly, the expected number of draws

a) until the first drawn coupon is drawn again;

b) until a duplicate occurs.

5+5 = 10 points

a Let X = # draws to draw the first drawn coupon

Then $X \sim \text{Geom}(p)$, $p = \frac{1}{10}$.

$\therefore E(X) = \dfrac{1}{\frac{1}{10}} = 10$.

$\therefore$ E(The draw at which the first drawn coupon is drawn again) = 11.

b Let N = The draw at which the first duplicate occurs.

Then $P(N > 11) = 0$.

For $n \le 10$,

$P(N > n)$ = P(1st n draws result in distinct coupons)

$$= \frac{10 * 9 * \cdots * (11-n)}{10^n}.$$

By the tailsum formula,

$$E(N) = 1 + \sum_{n=1}^{10} \frac{10 * 9 * \cdots * (11-n)}{10^n}$$

4. Among the patients at the coronary unit of a hospital, 20% of those with, and 35% of those without myocardial infarction, have had a stroke. If 40% of the patients in the coronary unit have myocardial infarction, what percent have had strokes ?

8 points

By the total probability formula,

$.2 * .4 + .35 * .6 = .29.$

29%

5. A fair six sided die is rolled 12 times. Find the expected value of the number of different faces that appear exactly three times.

6 points

Let A_i = Face i appears exactly three times ;

X = # faces that appear exactly three times.

Then, $X = \sum_{i=1}^{6} I_{A_i}$.

$$\therefore \quad E(X) = \sum_{i=1}^{6} P(A_i) = 6\,P(A_1) = 6\binom{12}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^9$$

6. Let $X \sim \mathrm{Bin}(15, .5)$. Compute exactly $E[(2X-15)^9]$.
5 points

Since the distribution of X is symmetric about $\frac{15}{2}$, the distribution of $2X-15$ is symmetric about zero.

$\therefore$ all odd moments of $2X-15$ are zero.

7. An urn contains four black, four white, and four blue balls. Three balls are drawn at random from the urn. Is it more likely that the balls will all be of the same color if sampling is with replacement, or without replacement? Prove your answer.
10 points

For with replacement sampling,

$$P(\text{Balls are of same color}) = \frac{\binom{3}{1} 4^3}{12^3} = \frac{1}{9}$$

For without replacement sampling,

$$P(\text{Balls are of same color}) = \frac{\binom{3}{1}\binom{4}{3}}{\binom{12}{3}} = \frac{12}{220} = \frac{3}{55}.$$

$$\frac{1}{9} > \frac{3}{55} \text{ because } 55 > 27.$$

8. Suppose X has a geometric distribution with parameter p. Find the expectation of $|X-2|$.
5 points

$$\frac{1}{p}-2 = E(X-2) = E(X-2)I_{X<2} + E(X-2)I_{X\geq 2}$$

$$= -p + E(X-2)I_{X\geq 2}$$

$$\therefore \quad E(X-2)I_{X\geq 2} = p+\frac{1}{p}-2.$$

$$\therefore \quad E|X-2| = E|X-2|I_{X<2} + E|X-2|I_{X\geq 2}$$

$$= p + E|X-2|I_{X\geq 2}$$

$$= p + E(X-2)I_{X\geq 2}$$

$$= p + p + \frac{1}{p} - 2 \qquad = \frac{1}{p} - 2q.$$