Key

STAT 417                Spring 2009                Dr. DasGupta
First Midterm

NOTE a. Please explain your work legibly and clearly to get credit.
   b. The exam is open notes, but closed book.
   c. You may get a maximum of 5 free bonus points. Answer as much as you can.

mean: 48.7/60
median: 49.3/60
s.d.: 8.3

≥ 53     A
≥ 40     B
≥ 30     C
1. A manufacturing process produces fibers of variable length. The pdf of the length of a fiber is

\[ f(x|\theta) = \theta^{-2} xe^{-x/\theta}, x > 0, \theta > 0. \]

Your data are the lengths \( X_1, X_2, \ldots, X_n \) of \( n \) independently sampled fibers.

(a) Find with proof a sufficient statistic.

(b) Find the MLE of \( \theta \).

(c) Prove or disprove that the MLE of \( \theta \) is unbiased.

\( 6+6+4 = 16 \) points

(a) The likelihood function is

\[ l(\theta) = \prod_{i=1}^{n} \theta^{-2} x_i^\theta e^{-x_i/\theta} = \theta^{-2n} \frac{\prod x_i^\theta}{\theta^n} \]

\[ \theta(\theta, \sum x_i) \]

\[ = \prod_{i=1}^{n} x_i \frac{1}{\theta} \]

By the Fisher-Neyman Factorization theorem, \( \sum x_i^\theta \) is a sufficient statistic.

(b) \[ \frac{d}{d\theta} \log l(\theta) = \frac{d}{d\theta} \left[ -2n \log \theta - \frac{\sum x_i^\theta}{\theta} + \sum \log x_i \right] \]

\[ = -\frac{2n}{\theta} + \frac{\sum x_i^\theta}{\theta^2} = \theta^2 \left( \frac{\sum x_i^\theta}{2n} - 2n \theta \right) \]

\[ > 0 \quad \text{if} \quad \theta < \frac{2n}{\sum x_i^\theta} \]

\[ = 0 \quad \text{if} \quad \theta = \frac{2n}{\sum x_i^\theta} \]

\[ < 0 \quad \text{if} \quad \theta > \frac{2n}{\sum x_i^\theta} \]

\[ \therefore \theta = \frac{\sum x_i^\theta}{2n} = \frac{\bar{x}}{2} \]

is the MLE of \( \theta \).

(c) \[ X_i \sim \text{Gamma}(x, \theta) \]

\[ \therefore \quad \frac{1}{2} \text{Beta}(x, \theta) = 2 \]

\[ \therefore E(\bar{x}) = \mu = x \theta = 2 \theta \Rightarrow \frac{\bar{x}}{2} \text{ is an unbiased est. of } \theta. \]
2. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, (1-\theta)^2), 0 < \theta < 1.$

a. Derive the Fisher information function.
b. Find the CRLB for the variance of an unbiased estimate of $\theta$.
c. Give an explicit unbiased estimate of $\theta$.
d. Does your estimate of part c) attain the CRLB? Prove your answer.

$8+2+1+1 = 12$ points

\[ f(x|\theta) = \frac{1}{\sqrt{2\pi} (1-\theta)^2} e^{-\frac{(x^2 - 2\theta x + \theta^2)}{2(1-\theta)^2}} \]

\[ \Rightarrow \log f(x|\theta) = -\frac{1}{2} \log 2\pi - \log (1-\theta) - \frac{(\theta - x)^2}{2(1-\theta)^2} \]

\[ \Rightarrow \frac{d}{d\theta} \log f(x|\theta) = \frac{1}{1-\theta} - \frac{\theta - x}{(\theta - 1)^2} - \frac{(\theta - x)^2}{2(\theta - 1)^3} \]

\[ = \frac{1}{1-\theta} - \frac{\theta - x}{(\theta - 1)^2} + \frac{(\theta - x)^2}{(\theta - 1)^3} \]

\[ \Rightarrow \frac{d^2}{d\theta^2} \log f(x|\theta) = \frac{1}{(1-\theta)^2} - \frac{1}{(\theta - 1)^2} - \frac{(\theta - x)(-2)}{(\theta - 1)^3} \]

\[ + \frac{2(\theta - x)}{(\theta - 1)^3} + (\theta - x)^2 \left( -\frac{3}{(\theta - 1)^4} \right) \]

\[ = \frac{4(\theta - x)}{(\theta - 1)^3} - \frac{3(\theta - x)^2}{(\theta - 1)^4} \]

\[ \therefore I(\theta) = -E_\theta \left[ \frac{d^2}{d\theta^2} \log f(x|\theta) \right] = - \left[ 4\theta - \frac{3(1-\theta)^2}{(1-\theta)^4} \right] \]

\[ = \frac{3}{n} \left( \frac{(1-\theta)^2}{(1-\theta)^4} \right) = \frac{3}{n} \left( \frac{(1-\theta)^2}{3n} \right) = \frac{(1-\theta)^2}{3n} \]

\[ (1-\theta)^2 \frac{1}{n} \stackrel{\text{one side of } \theta}{\longrightarrow} \text{Var}_\theta (\bar{X}) \]
3. Give an example of each of the following phenomena:
   a. An MLE which is biased, but asymptotically unbiased.
   b. An UMVUE which attains the Cramér-Rao lower bound.
   c. A fundamental concept invented by Fisher.
   d. A doctoral student of Fisher.
   e. A density for which the Cramér-Rao lower bound is not applicable.
   f. An unbiased estimate of $\theta$ in the $U[\theta, \theta + 1]$ distribution.
   g. A distribution for which the sample mean is not a sufficient statistic.

$7 \times 4 = 28$ points

a. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, 
   $\sigma^2 = \frac{1}{n} \sum (X_i - \bar{X})^2$ 

b. Same as a; assume $\sigma^2$ is known; take $\mu = \bar{X}$

c. Sufficiency / Maximum Likelihood

d. C. R. Rao

e. $U[0, \Theta]$

f. $\bar{X} - \frac{1}{2}$

g. $U[0, \Theta]$
4. Anirban’s tiger has a certain homerange in the Happy Hollow forest. The tiger doesn’t tell anyone what the homerange is. But he does venture the information that the homerange is circular in shape.

Prints of the tiger’s paws have been located at six coordinates: (1,0), (2,3), (2.2, 3.1), (.5, .1), (-2, -1.2), (-2.5, 1.5).

Give sensible estimates for the center and size of the tiger’s homerange using these paw prints.

4+4 = 8 points
average in x-direction of the data points

\[ \frac{1+2+2-2+5-2-2-5}{6} = 1.2 \]

average in y-direction of the data points

\[ \frac{0+3+2.1+1-1.2+1.5}{6} = 1.08 \]

so an estimate of the true center of the homogeneity is \((1.2, 1.08)\).

\[ \begin{array}{ccc}
\text{Center} & \text{Data Point} & \text{Distance from Center} \\
(1.2, 1.08) & (1.5) & \sqrt{(1.2-1.5)^2 + (1.08-1.5)^2} = 1.34 \\
(2.3, 3) & (1.2, 3.1) & \sqrt{(2.3-1.2)^2 + (3.1-3)^2} = 2.63 \\
(1.5, 1) & (-2.3, -1.2) & 1.82 \\
(2.5, 1.5) & (-2.5, 1.5) & 3.47 \\
\end{array} \]
5. Write the course number of this course.
1 point

Philosophy 999