1. For an AR(1) process \( z_t = \phi z_{t-1} + a_t, |\phi| < 1 \), its spectral density can be shown to be
\[
 f(\omega) = \left( \sigma_a^2/\gamma_0 \right) |1 - \phi e^{-i2\pi\omega}|^{-2}.
\]
Verify that \( f(\omega) = (1 - \phi^2)/(1 - 2\phi \cos 2\pi\omega + \phi^2) \).

**Solution:** Note that \( \gamma_0 = \sigma_a^2/(1 - \phi^2) \), and
\[
|1 - \phi e^{-i2\pi\omega}|^2 = 1 + \phi^2 - 2\phi \cos 2\pi\omega + \phi^2 e^{i2\pi\omega}.
\]

2. For an MA(1) process \( z_t = a_t - \theta a_{t-1} \), its spectral density can be shown to be
\[
 f(\omega) = \left( \sigma_a^2/\gamma_0 \right) |1 - \theta e^{-i2\pi\omega}|^{-2}.
\]
Verify that \( f(\omega) = 1 - 2\theta \cos 2\pi\omega/(1 + \theta^2) \).

**Solution:** Note that \( \gamma_0 = \sigma_a^2(1 + \theta^2) \), and
\[
|1 - \theta e^{-i2\pi\omega}|^2 = 1 + \theta^2 - 2\theta \cos 2\pi\omega + \theta^2 e^{i2\pi\omega}.
\]

3. Let \( \{y_t\} \) be a stationary process with the power spectrum \( p_y(\omega) \). Define \( z_t = y_t - y_{t-1} \).

Obtain the power spectrum of \( z_t \) in terms of \( p_y(\omega) \).

**Solution:**
\[
p_z(\omega) = \sum_k \gamma_z(k)e^{-i2\pi kw} = \sum_k (2\gamma_y(k) - \gamma_y(k-1) - \gamma_y(k+1))e^{-i2\pi kw} = p_y(\omega)(2 - e^{-i2\pi\omega} - e^{i2\pi\omega}) = p_y(\omega)|1 - e^{-i2\pi\omega}|^2
\]

4. Let \( \{y_t\}_{-\infty}^\infty \) and \( \{z_t\}_{-\infty}^\infty \) be two stationary processes, independent of each other, with power spectra \( p_y(\omega) \) and \( p_z(\omega) \). Find the power spectrum of the stationary process \( w_t = ay_t + bz_t \) in terms of \( p_y(\omega) \) and \( p_z(\omega) \), where \( a \) and \( b \) are constants.

**Solution:** Since \( \gamma_w(k) = a^2\gamma_y(k) + b^2\gamma_z(k), p_w(\omega) = a^2p_y(\omega) + b^2p_z(\omega) \).

5. Let \( a_i, b_i \) be independent r.v.'s with \( E[a_i] = E[b_i] = 0 \) and \( \text{var}[a_i] = \text{var}[b_i] = \sigma^2_i \). Find the spectral distribution of the stationary process \( z_t = \sum_{i=1}^m (a_i \cos 2\pi\omega_i t + b_i \sin 2\pi\omega_i t) \).

Note that the spectral density does not exist since the spectral distribution is discrete.

**Solution:** Since \( \gamma_k = \sum_{i=1}^m \sigma^2_i 2\pi\omega_i k, F(\omega) \) has jumps of size \( \sigma^2_i/2 \sum_{j=1}^m \sigma^2_j \) at \( \omega = \pm \omega_i \).

Due January 28, 2019