1. Let \( \{a_t\}_{t=-\infty}^{\infty} \) be a white noise process with \( E[a_t] = 0 \) and \( \text{var}[a_t] = \sigma_a^2 \). Define \( z_t = \mu + a_t - 5a_{t-1} \). Find the mean, autocovariance, and autocorrelation of \( z_t \), and verify that \( \{z_t\}_{t=-\infty}^{\infty} \) is stationary.

**Solution:**
\[
E[z_t] = \mu, \quad \text{var}[z_t] = 1.25\sigma_a^2, \quad \text{cov}[z_t, z_{t-1}] = -5\sigma_a^2, \quad \text{cov}[z_t, z_{t-k}] = 0, \quad k > 1.
\]
\( \rho_k = 0, k > 1 \). All are independent of \( t \).

2. Let \( \{y_t\} \) be a stationary process with mean \( \mu_y \) and autocovariance \( \gamma_y(s) = \text{cov}[y_t, y_{t-s}] \). Define \( z_t = y_t - y_{t-1} \). Obtain the mean and autocovariance of \( \{z_t\}_{t=-\infty}^{\infty} \) in terms of those of \( y_t \) and verify that it is stationary.

**Solution:**
\[
E[z_t] = 0, \quad \text{cov}[z_t, z_{t-k}] = \text{cov}[y_t - y_{t-1}, y_{t-k} - y_{t-k-1}] = 2\gamma_y(k) - \gamma_y(k-1) - \gamma_y(k+1).
\]
All are independent of \( t \).

3. Let \( \{y_t\}_{t=-\infty}^{\infty} \) and \( \{z_t\}_{t=-\infty}^{\infty} \) be two stationary processes with means \( \mu_y \) and \( \mu_z \) and autocovariances \( \gamma_y(s) \) and \( \gamma_z(s) \), independent of each other. Find the mean and autocovariance of \( w_t = ay_t + bz_t \), where \( a \) and \( b \) are constants, and show that \( \{w_t\}_{t=-\infty}^{\infty} \) is stationary.

**Solution:**
\[
E[w_t] = a\mu_y + b\mu_z, \quad \text{cov}[w_t, w_{t-k}] = a^2\gamma_y(k) + b^2\gamma_z(k).
\]
All are independent of \( t \).

4. Let \( a_i, b_i \) be independent r.v.’s with \( E[a_i] = E[b_i] = 0 \) and \( \text{var}[a_i] = \text{var}[b_i] = \sigma_i^2 \). Compute the mean and autocovariance of \( z_t = \sum_{i=1}^{\infty}(a_i \cos 2\pi\omega_i t + b_i \sin 2\pi\omega_i t) \) and show that it is stationary. [Hint: you may want to use the trigonometric identity \( \cos x \cos y + \sin x \sin y = \cos(x-y) \).]

**Solution:**
\[
E[z_t] = 0, \quad \text{cov}[z_t, z_{t-k}] = \sum_{i=1}^{\infty}\sigma_i^2 \cos 2\pi\omega_i k.
\]
All are independent of \( t \).

5. Let \( \{a_t\}_{t=1}^{\infty} \) be a white noise process with mean 0 and variance \( \sigma_a^2 \). Define \( z_0 = 0, z_t = \phi z_{t-1} + a_t, \ t = 1, 2, \ldots, \) where \( |\phi| < 1 \).

(a) Express \( z_t \) explicitly in terms of \( a_t \).

(b) Calculate the autocovariance \( \text{cov}[z_t, z_{t+s}] \) for \( s > 0 \), and show that for large \( t \), \( z_t \) is approximately stationary.

**Solution:**
\[
(a) \quad z_t = \sum_{i=0}^{t-1}\phi^i a_{t-i}.
\]
\[
(b) \quad \text{cov}[z_t, z_{t+s}] = \sum_{i=0}^{t-1}\phi^{2i+s}\sigma_a^2 = \sigma_a^2\phi^s(1 - \phi^{2t})/(1 - \phi^2) \rightarrow \sigma_a^2\phi^s/(1 - \phi^2)
\]

6. Observing \( z_1, \ldots, z_N \) from a stationary process with autocovariance \( \gamma_k \) and autocorrelation \( \rho_k = \gamma_k/\gamma_0 \). It is known that \( \text{var}[\bar{z}] = (\gamma_0/N)[1 + 2\sum_{k=1}^{N-1}(1 - k/N)\rho_k] \).

(a) If \( \rho_k \to 0 \) as \( k \to \infty \), show that \( \text{var}[\bar{z}] \to 0 \) as \( N \to \infty \).

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(b) Compare $\text{var}[\bar{z}]$ with the following autocorrelations: (i) $\rho_k = 0$, $k \neq 0$; (ii) $\rho_1 = .8$, $\rho_2 = .55$, $\rho_k = 0$, $k > 2$.

Solution:

(a) Only need to prove $\sum_{k=1}^{N-1} (1 - k/N)\rho_k/N \to 0$. For any $\delta > 0$, there exists $M$ such that $|\rho_k| < \delta$, $k > M$. Let $K > M/\delta$. For $N > K$, one has

$$\left| \sum_{k=1}^{N-1} (1 - k/N)\rho_k/N \right| \leq \sum_{k \leq M} N^{-1} + \sum_{k > M} \delta/N < 2\delta.$$ 

(b) For (i), $\text{var}[\bar{z}] = \gamma_0/N$. For (ii),

$$\text{var}[\bar{z}] = (\gamma_0/N)(1 + 2(1 - 1/N)(.8) + 2(1 - 2/N)(.55)) = (\gamma_0/N)(3.7 - 3.8/N).$$

7. Problem 2.1 in the text (p. 569 in 3rd ed; p.701 in 4th ed.), plus

(d) After inspecting the graphs in (a)-(c), do you think the series is stationary?

(e) Calculate and plot the sample ACF for lags up to 6.

(f) Assume $\rho_k = 0$, $k > 2$. Obtain approximate standard errors for $r_1$, $r_2$, and $r_k$, $k > 2$.

(g) Assume $\rho_k = 0$, $k > 2$. Obtain approximate correlation between $r_4$ and $r_5$.

Solution:

Read the data into $x$ in R.

(a) `plot(ts(x)).`

(b) `plot(x[-36],x[-1]).`

(c) `plot(x[-(35:36)],x[-(1:2)]).`

(d) It appears to be stationary as there is no obvious pattern suggesting otherwise.

(e) `x.acf<-acf(x,lag.max=6); x.acf` gives

$$(r_1, \ldots, r_6) = (0.4910, 0.1639, -0.0486, -0.1729, -0.2921, -0.5113).$$

(f) Substituting $r_1$, $r_2$ for $\rho_1$, $\rho_2$ in the approximate formula for $\text{var}[r_k]$, one has

$$\text{s.e.}[r_1] \approx \sqrt{\left[ (1 + 2r_1^2)(1 + 2r_1^2 + 2r_2^2) + 2r_2 + r_1^2 - 8r_1^2(1 + r_2) \right]/N} = 0.1292,$$

$$\text{s.e.}[r_2] \approx \sqrt{\left[ (1 + 2r_2^2)(1 + 2r_1^2 + 2r_2^2) + r_2^2 - 4r_2(2r_2 + r_1^2) \right]/N} = 0.1880,$$

$$\text{s.e.}[r_k] \approx \sqrt{(1 + 2r_1^2 + 2r_2^2)/N} = 0.2066, k > 2.$$ 

(g) $\text{cov}[r_4, r_5] \approx 2(r_1r_2 + r_1)/N = 0.0317$. $\text{corr}[r_4, r_5] \approx 0.03175/0.0317 \approx 0.975$. 

Due January 23, 2017