Problem 1. Let \( f \) be analytic in \( D = \{ z : |z| < 1 \} \) and suppose that \( |f(z)| \leq M \) for all \( z \in D \). (a) If \( f(z_k) = 0 \) for \( 1 \leq k \leq n \) show that

\[
|f(z)| \leq M \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - z \bar{z}_k} \right|
\]  

(1)

for \( |z| < 1 \). (b) If \( f(z_k) = 0 \) for \( 1 \leq k \leq n \), each \( z_k \neq 0 \) and \( f(0) = Me^{ia}(z_1z_2, \ldots, z_n) \), find a formula for \( f \).

Solution 2. (a) To be notionally efficient, define

\[
S_D = \{ h \in \mathcal{H}(D) : h(D) \subseteq D, h(0) = 0 \}
\]

Without loss of generality, suppose \( M = 1 \).

Attempt 1: Induction on \( n \). Let \( \varphi_k(\zeta) = \varphi_{z_k}(\zeta) = \frac{\zeta - z_k}{1 - \bar{z}_k \zeta} \). Suppose (1) is valid for \( m = n - 1 \), that is, if \( f \) has only \( n - 1 \) zeros \( z_k, 1 \leq k \leq (n - 1) \) in \( D \), then

\[
|f(z)| \leq \prod_{k=1}^{m} \left| \frac{z - z_k}{1 - z \bar{z}_k} \right|
\]

Define

\[
g(z) = (z - z_n)^{r_n} f(z)
\]

where \( z_n \neq z_k \) for \( 1 \leq k \leq (n - 1) \) and \( r_n \) is the multiplicity of \( z_n \). Then

\[
|g(z)| = |(z - z_n)^{r_n} f(z)| \leq |(z - z_n)^{r_n}| |f(z)|
\]

\[
\leq |(z - z_n)^{r_n}| \prod_{k=1}^{m} \left| \frac{z - z_k}{1 - z \bar{z}_k} \right|
\]

Let

\[
K = \max_{|z|=1} |(z - z_n)^{r_n}| > 0
\]

and

\[
h(z) = \frac{(z - z_n)^{r_n}}{K}
\]

Then

\[h \circ \varphi^{-1}_n \in S_D\]

and

\[
\left| \frac{(z - z_n)^{r_n}}{K} \right| \leq \left| \frac{z - z_n}{1 - z \bar{z}_n} \right|
\]

Thus

\[
|g(z)| \leq K \left| \frac{z - z_n}{1 - z \bar{z}_n} \right| \prod_{k=1}^{m} \left| \frac{z - z_k}{1 - z \bar{z}_k} \right| = K \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - z \bar{z}_k} \right|
\]

Attempt 2: use recursive compositions, get lost :(