1 Zeros and Singularities

Problem 1.1 \( f \in \mathcal{H}(\mathbb{C}) \) and \( |f(z)| \leq 1 + |z|^\alpha \) with \( 0 < \alpha < 1 \), then \( f \) is a constant.

Proof. For any \( r > 0 \), let \( C(0, r) = \{z = re^{i\theta}, 0 \leq \theta \leq 2\pi, r > 0\} \) be oriented counter clockwise. Then \( f \in \mathcal{H}(\mathbb{C}) \) implies that

\[
f'(z) = \frac{1}{2\pi i} \int_{C(0, r)} \frac{f(\zeta) d\zeta}{(\zeta - z)^2}
\]

for any \( z \in B(0, r) \) by Cauchy’s formula. But, \( |f(z)| \leq 1 + |z|^\alpha \) implies that

\[
|f'(z)| \leq \frac{1}{2\pi} \int_{C(0, r)} \frac{|f(\zeta)||d\zeta|}{|\zeta - z|^2} \leq \frac{1}{2\pi} \int_{C(0, r)} \left(1 + |\zeta|^\alpha\right) \frac{|d\zeta|}{|\zeta - z|^2}
\]

(1.1)

\[
\leq \frac{1}{r} + \frac{1}{r^{1-\alpha}}
\]
Consequently
\[|f'(z)| = 0\]
and \(f \equiv \kappa\) where \(\kappa\) is some constant, which completes the proof. ■

**Problem 1.2** Suppose \(f \neq 0\) and \(f\) is analytic in \(B^*(a, \delta) \subseteq G\). If either
\[
\lim_{z \to a} |z - a|^s |f(z)| = 0
\]
or
\[
\lim_{z \to a} |z - a|^s |f(z)| = \infty
\]
for some \(s \in \mathbb{R}\), then \(\exists m \in \mathbb{Z}\) s.t. (19) holds for \(s > m\) and (20) holds for \(s < m\), where \(G \subseteq \mathbb{C}\) is open and \(B^*(a, \delta) = \{z \in \mathbb{C} | 0 < |z - a| < \delta, \delta > 0\}\).

Further, \(m = 0\) iff \(z = a\) is removable and \(f(a) \neq 0\); \(m < 0\) iff \(z = a\) is removable and is a zero of order \(-m\); \(m > 0\) iff \(z = a\) is a pole of order \(m\).

**Proof.** Without loss of generality, let \(a = 0\). The analyticity of \(f\) in \(B^*(0, \delta) \subseteq G\) implies that for all \(z \in B^*(0, \delta)\), \(f(z)\) admits Laurent expansion at \(z = 0\) as
\[
f(z) = \sum_{n \in \mathbb{Z}} c_n z^n
\]
where
\[
c_n = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(\zeta)d\zeta}{\zeta^{n+1}}
\]
and \(0 < r < \min(1, \delta)\).

Suppose \(\lim_{z \to 0} |z|^s |f(z)| = 0\) holds for some \(s \in \mathbb{R}\). Then there is some \(\lambda_1\) with \(0 < \lambda_1 < r\) such that
\[
|f(z)| < |z|^{-s}
\]
for all \(z \in B^*(0, \lambda_1)\). Let \(\beta\) be \(0 < \beta < \min\{\lambda_1, \delta\}\), then (2.1) implies
\[
|c_n| \leq \left| \frac{1}{2\pi} \int_{C(0,\beta)} \frac{f(\zeta)d\zeta}{\zeta^{n+1}} \right| \leq \frac{2\pi \beta^{-s}}{2\pi \beta^{n+1}} = \beta^{-(s+n)}
\]
which means \(c_n = 0\) for all \(n < -s\), that is, for \(n \leq [-s]\).

Let \(m_0 = \min\{n | n \geq [-s] + 1, c_n \neq 0\}\). If \(m_0 = \infty\), then \(c_n = 0\) for all \(n \geq [-s] + 1\) and \(f \equiv 0\) on \(B^*(0, \delta)\), by the indentity theorem, \(f \equiv 0\) on \(B(0, \delta)\), which contradicts with the hypothesis. So, \(m_0 < \infty\) and
\[
f(z) = \sum_{n=-s}^{\infty} c_n z^n = z^{m_0} \sum_{n=0}^{\infty} c_{m_0+n} z^n
\]
Obviously, \(g(z) = \sum_{n=0}^{\infty} c_{m_0+n} z^n\) is holomorphic in \(B(0, \delta)\) since \(f(z) = \sum_{n \in \mathbb{Z}} c_n z^n\) converges locally uniformly on \(B^*(0, \delta)\). Moreover, \(g(0) = c_{m_0} \neq 0\).
Now, if \( m_0 = 0 \), then \( f \) is holomorphic on \( B(0, \delta) \) and for all \( t \) with \( t > m_0 \),

\[
\lim_{z \to 0} z^t f(z) = \lim_{z \to 0} z^t \lim_{z \to 0} \left\{ \sum_{n=0} z^{m_0+n}z^n \right\} = 0
\]

and for all \( t \) with \( t < m_0 \),

\[
\lim_{z \to 0} z^t f(z) = \lim_{z \to 0} z^t \lim_{z \to 0} \left\{ \sum_{n=0} z^{m_0+n}z^n \right\} = \infty
\]

Else, if \( m_0 > 0 \), then for all \( t \) with \( t > m_0 \),

\[
\lim_{z \to 0} z^t f(z) = \lim_{z \to 0} z^{t+m_0} \lim_{z \to 0} \left\{ \sum_{n=0} z^{m_0+n}z^n \right\} = 0
\]

and for all \( t \) with \( t < m_0 \), there are two cases: for \( t < -m_0 \), \( \lim_{z \to 0} z^t f(z) = \lim_{z \to 0} z^{t+m_0} \lim_{z \to 0} \left\{ \sum_{n=0} z^{m_0+n}z^n \right\} = \infty \) and for \( -m_0 < t < m_0 \),

\[
\lim_{z \to 0} z^t f(z) = \lim_{z \to 0} z^{t+m_0} = \lim_{z \to 0} \sum_{n=0} z^{m_0+n}z^n = c_{m_0} \neq 0
\]

and \( \lim_{z \to 0} \frac{1}{z^{t+m_0}} = \infty \) together imply that \( \lim_{z \to 0} |z^t f(z)| = \lim_{z \to 0} |z^{t+m_0}| = \infty \). Hence, for all \( t \) with \( t < m_0 \), \( \lim_{z \to 0} |z^t f(z)| = \infty \). Finally, similar reasoning for the case where \( m_0 > 0 \) yields the result for \( m_0 < 0 \), which completes the first part of the assertion.

If \( \lim_{z \to a} |z|^s |f(z)| = \infty \) for some \( s \in \mathbb{R} \), then \( |z|^s |f(z)| > M \) and \( |f(z)| > \frac{M}{|z|^s} > 0 \) for all \( z \in B^*(0, \lambda_1) \) for some \( \lambda_1 > 0 \). Let \( g(z) = \frac{1}{f(z)} \), then \( g \in \mathcal{H}(B^*(0, \lambda_1)) \) and

\[
\lim_{z \to a} |z|^{-s} |g(z)| = 0
\]

which reduces to what’s been discussed above.

To prove the second part of the assertion, simply notice that in (2.3)

\[
f(z) = \sum_{n=-s+1} c_n n^s = z^{m_0} \sum_{n=0} z^{m_0+n}z^n
\]

where \( m_0 = \min \{ n | c_n \neq 0, n \geq -s+1 \} \). Then, \( m_0 = 0 \) iff \( f(z) = \sum_{n=0} z^{m_0+n}z^n \) with \( c_0 \neq 0 \) iff \( f(0) = c_0 \neq 0 \) and \( z = 0 \) is a removable singularity; \( m_0 > 0 \) iff \( z = a \) is a pole of order \( m_0 \); \( m_0 < 0 \) iff \( z = 0 \) is removable and is a zero of order \( -m_0 \).

Further discussion on singularities are included here unified treatment.

**Corollary 1.3** Suppose \( f \in \mathcal{H}(B^*(a, R)) \) and \( \lim_{z \to a} (z-a)f(z) = 0 \), then \( f \in \mathcal{H}(B(a, R)) \).
Proof. This is just a corollary of what’s proved in the first section. The fact that \( \lim_{z \to a} (z - a)f(z) = 0 \) iff \( \lim_{z \to a} |(z - a)f(z)| = 0 \) implies the Laurent expansion of \( f(z) \) at \( z = a \)

\[
f(z) = \sum_{n \in \mathbb{Z}} c_n (z - a)^n
\]

where

\[
c_n = \frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}
\]

satisfies that for any given \( \varepsilon > 0 \)

\[
|c_n| \leq \frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}} \leq \frac{\varepsilon}{\rho^{n+1}}
\]

for sufficiently small \( \rho > 0 \). Then \( c_n = 0 \) for all \( n \leq -1 \) since \( \rho \) could be chosen arbitrarily small and

\[
f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \in \mathcal{H}(B(a,R))
\]

Theorem 2.1 Let \( X \) be a metric space and \( \emptyset \neq D \subseteq X \) open. Then, path connectedness is equivalent to connectedness.

2 Topological elements

Theorem 2.1 Let \( X \) be a metric space and \( \emptyset \neq D \subseteq X \) open. Then, path connectedness is equivalent to connectedness.
Proof. Suppose $D$ is connected. Pick any $z_0 \in D$ and define

$$G_1 = \{ z \in D | \exists \gamma_{z,z_0} \subseteq D \}$$

and

$$G_2 = \{ z \in D | \nexists \gamma_{z,z_0} \subseteq D \}$$

where for any $w, w' \in D$, $\gamma_{w,w'}$ is defined as $\gamma_{w,w'} \in C([\alpha, \beta])$ and $\gamma_{w,w'}(\alpha) = w, \gamma_{w,w'}(\beta) = w'$. Naturally, $G_1$ and $G_2$ are both open, $G_1 \cap G_2 = \emptyset, G_1 \cup G_2 = D$. But $G_1 \neq \emptyset$ since $z_0 \in G_1$. So the connectedness of $D$ implies that $G_2 = \emptyset$ and $G_1 = D$ and hence $D$ is path connected.

Conversely, suppose $D$ is path connectedness but not connected, then there are open subsets $G_1, G_2$ with $G_1 \cap G_2 = \emptyset, G_1 \cup G_2 = D$. Pick any $z_1 \in G_1, z_2 \in G_2$. Then there is some path $\gamma_{z_1,z_2} \subseteq D$ with $\gamma_{z_1,z_2}(\alpha) = z_1, \gamma_{z_1,z_2}(\beta) = z_2$. Bisect $[\alpha, \beta]$ with midpoint $c_1$, then $\gamma_{z_1,z_2}(c_1) \in G_1$ or $\gamma_{z_1,z_2}(c_1) \in G_2$. Keep this process to obtain $[\alpha_n, \beta_n]$ such that $[\alpha_n, \beta_n] \subseteq [\alpha_{n+1}, \beta_{n+1}]$, $\lim_{n \to \infty} \text{diam}[\alpha_n, \beta_n] = 0, \gamma_{z_1,z_2}(\alpha_n) \in G_1$ and $\gamma_{z_1,z_2}(\beta_n) \in G_2$. Eventually, $\lim_{n \to \infty} \alpha_n = c = \lim_{n \to \infty} \beta_n$ and either $\gamma_{z_1,z_2}(c) \in G_1$ or $\gamma_{z_1,z_2}(c) \in G_2$ will cause $\gamma_{z_1,z_2}(\beta_n) \in G_1$ or $\gamma_{z_1,z_2}(\alpha_n) \in G_2$ for sufficiently large $n$ since $G_1$ and $G_2$ are both open. So $\gamma_{z_1,z_2}(c) \notin D$, which is a contradiction. ■

Lemma 2.1 (Lebesgue Covering Lemma) If $(X, d)$ is sequentially compact and $\mathcal{G}$ is an open cover of $X$ then there is an $\varepsilon > 0$ such that if $x \in X$, there is a set $G$ in $\mathcal{G}$ with $B(x, \varepsilon) \subseteq G$.

Proof. Suppose open cover $\mathcal{G}$ covers $X$ but no such $\varepsilon > 0$ exists. Then in particular, for every integer $n$ there is a point $x_n \in \mathcal{G} \setminus B(x_n, 1/n)$. Since $X$ is sequentially compact, there is a point $x_0 \in X$ and a subsequence $\{x_{n_k}\}$ such that $\lim_{n \to \infty} x_{n_k} = x_0$. Let $G_0 \in \mathcal{G}$ such that $x_0 \in G_0$ and choose $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq G_0$. Now let $n_0$ be such that $d(x_0, x_{n_k}) < \varepsilon/2$ for all $n_k > n_0$. Let $n_k \geq \max(n_0, \varepsilon/2)$ and let $y \in B(x_{n_k}, 1/n_k)$. Then $d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \varepsilon/2 + 1/n_k = \varepsilon$. That is, $B(x_{n_k}, 1/n_k) \subseteq B(x_0, \varepsilon) \subseteq G_0$, contradicting with the choice of $x_{n_k}$. ■

Definition 2.2 A path $\gamma : [a, b] \to \mathbb{C}$ is rectifiable is it’s of bounded variance.

Lemma 2.3 (Polygon Path) Let $\gamma : [a, b] \to \mathbb{C}$ be rectifiable and $\gamma \subseteq G$ with $G$ open. Then $\gamma$ has a polygon path refinement $\rho$ such that $||\gamma - \rho|| < \varepsilon$ for any given $\varepsilon > 0$. (supremum norm)
Proof. Since \( \gamma \) is compact, it’s admissible to intersect \( G \) with \( B(R, 0) \) such that \( \gamma \subseteq G \cap B(R, 0) \) if \( G \) is unbounded. \( G \) being, \( \gamma \subseteq G \) and \( \gamma \) being compact imply \( \delta = d(\gamma, \partial G) > 0 \). Define

\[
\Omega_1 = \{ z \in \Omega : d(z, \gamma) < \delta/2 \}
\]

then \( \Omega_1 \) is a domain and \( \gamma \subseteq \Omega_1 \subseteq \Omega_1 \subseteq \Omega \). If \( \Omega_1 \) is not connected, then 
\( \gamma = (G_1 \cap \gamma) \cup (G_2 \cap \gamma) \) where \( G_i = G_i \cap \gamma = \varnothing \), \( \cup G_i = \Omega_1 \), which is a contradiction since \( \gamma \) is connected. (Bisect \( [a, b] \) countably many times as was done in the previous theorem, then \( \gamma(c) \notin (G_1 \cap \gamma) \cup (G_2 \cap \gamma) \) where \( c \) is as defined in the previous theorem.)

Since \( \gamma \) is uniformly continuous on \( [a, b] \), then for \( \delta/8 \), there exist \( \kappa > 0 \) such that whenever \( |t - t'| < \kappa \), \( |\gamma(\tau_k) - \gamma(\tau_k')| < \delta/8 \) for all \( t, t' \in [a, b] \). Let 
\[ x_k = a + \frac{b - a}{m} k, \quad k = 0, 1, ..., m. \]

Then for sufficiently large \( m \), \( x_{k+1} - x_k < \delta/8 \) and for \( \tau_k, \tau_k' \), with \( x_k \leq \tau_k, \tau_k' \leq x_{k+1} \), sup \( |\gamma(\tau_k) - \gamma(\tau_k')| < \delta/8 \).

Define a polygon by

\[ \rho(x) = (1 - \alpha)\gamma(x_k) + \alpha\gamma(x_{k+1}) \quad \text{if} \quad x_k \leq x \leq x_{k+1} \]

where \( 0 \leq \alpha \leq 1 \).

Then for all \( x \in [a, b] \), sup \( |\gamma(x) - \rho(x)| \leq \delta/8 \) and \( \rho \subseteq \bigcup_{k=1}^n B(\gamma(t_k), \delta/4) \subseteq \Omega_1 \). Hence \( \rho \) is desired refinement.

A separation of \( X \) is a pair \( U, V \) of disjoint nonempty open subset of \( X \) whose union is \( X \).

## 3 Line Integrals

**Theorem 3.1 (Existence of Riemann-Stieltjes Integral)** Let \( \gamma : [a, b] \rightarrow \mathbb{C} \) be of bounded variation and suppose that \( f : [a, b] \rightarrow \mathbb{C} \) is continuous.

Then there is a complex number \( I \) such that for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that when \( P = \{t_0 < t_1 < ... < t_n\} \) is a partition of \( [a, b] \) with \( ||P|| = \max\{t_k - t_{k-1} : 1 \leq k \leq m\} \) \( < \delta \) then

\[
|I - \sum_{k=1}^{m} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1}))| < \varepsilon
\]

With \( f(z) = u + iv \), it’s clear that
\[
\int_{\gamma} f(z) \, dz = \int_{\gamma} (u \, dx - v \, dy) + i \int_{\gamma} (u \, dy + v \, dx)
\]

which reminds us the Green’s Formula.

\(\Omega\) will always denote a domain, \(p\) and \(q\) are continuous functions on \(\Omega\), \(\gamma \subseteq \Omega\) is free to vary. General line integral of the form

\[
\int_{\gamma} p \, dx + q \, dy
\]

may possess a subclass such that an integral only depends on the end points of \(\gamma\), which is justified by

**Theorem 3.2** \(\int_{\gamma} p \, dx + q \, dy\) depends only on the end points of \(\gamma\) iff there exists a function \(U(x, y)\) in \(\Omega\) with \(\partial U/\partial x = p\) and \(\partial U/\partial y = q\), that is, iff \(dU = p \, dx + q \, dy = \frac{\partial U}{\partial x} \, dx + \frac{\partial U}{\partial y} \, dy\) (exact differential)

**Proof.** Sufficiency. Suppose the condition is fullfilled, then

\[
\int_{\gamma} p \, dx + q \, dy = \int_{\gamma} dU = \int_{a}^{b} \frac{\partial U}{\partial x} \, x'(t) \, dt + \frac{\partial U}{\partial y} \, y'(t) \, dt
\]

\[
= \int_{a}^{b} \frac{dU(x(t), y(t))}{dt} \, dt = \int_{a}^{b} U_i \, dt = U(x(t), y(t))|_{a}^{b}
\]

that is

\[
\int_{\gamma} dU = \int_{[a,b]} dU
\]

Necessity. Pick any \(z_0 = (x_0, y_0) \in \Omega\). By Theorem 3, \(\Omega\) being a domain is equivalent to being path connected. So, it’s admissible to connect any point \(z = (x, y) \in \Omega\) with a POLYGON path \(\gamma_{z_0, z}\) whose sides are parallel to the coordinate axes and form a well-defined function by

\[
U(x, y) = \int_{\gamma} p \, dx + q \, dy
\]

By further choosing the last segment (local property) to be horizontal or vertical, we obtain

\[
U(x, y) = \int_{x_0}^{x} p(x, y) \, dx + \text{cont.}
\]

Hence, \(\partial U/\partial x = p\) and \(\partial U/\partial y = q\). ■
Remark 3.1. $p$ and $q$ could be either real or complex.

As a natural consequence, $f(z)dz = f(z)dx + if(z)dy$ is an exact differential iff there exists a function $F(z)$ in $\Omega$ with \( \frac{\partial F}{\partial x} = f(z) \) and \( \frac{\partial F}{\partial y} = if(z) \). If this happens, then $F(z)$ fulfills the Cauchy-Riemann equation and is analytic since $f(z)$ is continuous.

Here $dz = dx + idy$ is taken as $dz = (dx, dy)$ since \( \frac{\partial z}{\partial x} = 1 \); \( \frac{\partial z}{\partial y} = i \).

Theorem 3.3. The integral $\int_{\gamma} f(z)dz$, with continuous $f$, depends only on the end points of $\gamma$ iff $f$ is the derivative of an analytic function in $\Omega$.

Theorem 3.4 (Polygon Appr.). $f$ continuous on $\Omega$, $\gamma \subseteq \Omega$ being rectifiable. Then for any $\varepsilon > 0$, there exists polygon $\rho \subseteq G$ such that

$$\left| \int_{\gamma} f(z)dz - \int_{\rho} f(z)dz \right| < \varepsilon$$

Proof. Since $\gamma \subseteq \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega$ and $f$ is uniformly continuous on the compact set $\Omega_1$. For $\forall \varepsilon > 0$, $\exists \kappa > 0$ such that $|f(z) - f(z')| < \varepsilon/2l(\gamma)$ for all $z, z'$ with $|z - z'| < \kappa$. Break $\gamma$ by $z_0 = \gamma(a)$, $z_1, \ldots, z_n = \gamma(b)$, the $k$th of which $\gamma_k$ is of length $s_k$ with $s_k < \min(\kappa, \delta)$. Then polygon $\rho$ with vertices $z_k$ is such that $\rho \subseteq \Omega_1$. This is true because $d(\gamma, \partial \Omega_1) = \delta/2$ implies $\gamma_k \subseteq B(z_{k-1}, s_k) \subseteq \Omega_1$ and $[z_{k-1}, z_k] \subseteq \Omega_1$.

Finally,

$$\int_{\gamma_k} f(z_{k-1})dz = \int_{[z_{k-1}, z_k]} f(z_{k-1})dz = f(z_{k-1})(z_k - z_{k-1})$$

$\blacksquare$

Theorem 3.5. Let $f(z)$ be analytic in the region $\Omega$ obtained by omitting a finite number of points $\zeta_j$ from a open disk $D$. If $f(z)$ satisfies that $\lim_{z \to \zeta_j} (z - \zeta_j)\overline{f}(z) = 0$ for all $j$, then

$$\int_{\gamma} f(z)dz = 0$$

for any closed curve $\gamma \subseteq \Omega$.
Proof. Ahlfors’ book states a proof that involves showing that \( \int_{\partial R} f(z)dz = 0 \) for any rectangle \( R \subseteq \Omega \), and define indefinite integral as \( F(z) = \int_{\lambda} f(z)dz \) for p.s. (sides parallel to the axis) path \( \lambda \subseteq \Omega \), and, show that \( dF = f(z)dx + if(z)dy \).

An general approach is a combination of Ahlfors’ and Conway’s methods. First, show that these removed points are removable singularities. Then, use polygon approximation theorem. Then use triangulation of polygons and show \( \int_{\partial T} f(z)dz = 0 \) for triangle \( T \subseteq \Omega \), which reduces to the elegant Goursat’s Theorem.

Theorem 3.6 (Morera) If \( f(z) \) is defined and continuous in a domain \( \Omega \) and if \( \int_{\gamma} f(z)dz = 0 \) for all rectifiable closed curves \( \gamma \) together with its interior \( \gamma \) in \( \Omega \), then \( f(z) \) is analytic in \( \Omega \).

Proof. Being holomorphic is a local property. Pick \( z_0 \) and choose \( B(z_0, \delta) \subseteq \Omega \). Then since \( B(z_0, \delta) \) is simply connected, \( F(z) = \int_{z_0} f(z)dz \) is well defined and \( F'(z) = f(z) \).

Remark 3.2 any closed rectifiable curve could be approximated by closed polygon. any closed polygon could be decomposed into convex polygon. every convex polygon could be decomposed into triangle. Morera Theorem’s hypothesis could be reduced to \( \int_{\partial T} f(z)dz = 0 \) for triangle \( T, T \subseteq \Omega \).

Theorem 3.7 (Green) Let \( \Omega \subseteq \mathbb{R}^2 \) be a bounded, closed region whose boundary \( \partial \Omega \) is piecewise smooth, \( P(x, y), Q(x, y), \frac{\partial P(x, y)}{\partial y}, \frac{\partial Q(x, y)}{\partial x} \in C^0(\Omega) \). Then

\[
\int_{\partial \Omega} P(x, y)dx + Q(x, y)dy = \int \int_{\Omega} \left( \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dxdy
\]

Proof. Basic idea: simply connected, union of finite simply connected, and, general \( \Omega \).

Case 1: \( \Omega = \{(x, y) : a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\} \). or \( \Omega = \{(x, y) : x_1(y) \leq x \leq x_2(y), c \leq y \leq d\} \). Then

\[
\int \int_{\Omega} \frac{\partial Q(x, y)}{\partial x} dxdy = \int_{c}^{d} dy \int_{x_1(y)}^{x_2(y)} \frac{\partial Q(x, y)}{\partial x} dx
\]

\[= \int_{c}^{d} dy \left( P(x, y)dy - \int_{\gamma(y, x_2(y))}^{\gamma(y, x_1(y))} P(x, y)dy dx \right) \]

\[= \int_{\partial \Omega} Q(x, y)dy
\]
If $\Omega$ is simply connected, then $\Omega$ could be represented in both ways given above, which is the key to the fact that for any closed $p$ inside $\Omega$, $\int_{\gamma} f(z) dz = 0$ for $f = p + iq$ iff $p$ and $q$ satisfy the Cauchy-Riemann equations, since with such an $\Omega, \int_{\partial \Omega} P(x, y) dx + Q(x, y) dy = \int \int_{\Omega} \left( \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dx dy$

For general $\Omega$. Suppose $\partial \Omega$ is composed of finitely many piecewise smooth curves. Choose an open domain $G$ that contains $\Omega$ and extend $P, Q, P_y, Q_x$ on $G$. Form sets $H_n, F_n$ which are both unions of finitely many standard $\mathbb{R}^2$ rectangles such that

$$F_n \subseteq \tilde{\Omega} \subseteq \Omega \subseteq \tilde{H}_n \subseteq H_n \subseteq G$$

and

$$\lim_{n \to \infty} |H_n \setminus F_n| = 0$$

(Here $|H_n \setminus F_n|$ means the area of the difference and standard rectangle is taken as a compact set)

Define a partition sequence $\{T_n\}$ which reserves the orientation of $\partial \Omega$ but splits $\partial \Omega$ into finitely many pairwise disjoint, closed polygons $\Gamma_n$ with $\Gamma_n \subseteq H_n \setminus F_n$ and $\lim d(T_n) = 0$. Then by definition of line integral of the second type

$$\lim_{n \to \infty} \int_{\Gamma_n} P(x, y) dx + Q(x, y) dy = \int_{\partial \Omega} P(x, y) dx + Q(x, y) dy$$

and

$$\lim_{n \to \infty} \int \int_{\Omega_n} \left( \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dx dy = \int \int_{\Omega} \left( \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dx dy$$

where $\Omega_n$ is the simply connected region with boundary $\Gamma_n$. ■

**Remark 3.3** Green’s Formula holds for any closed, bounded region $\Omega$ and its generic boundary $\partial \Omega$. But for a closed, piecewise smooth curve $\gamma$ inside $\Omega$, the interior (the set enclosed by $\gamma$) $\hat{\gamma}$ of $\gamma$ is usually bigger than the interior of $\Omega \cap \gamma$. That is, $f(z)$ may not be defined on $\hat{\gamma}$ though it’s defined on $\Omega$. So, Green’s formula can’t be applied directly on the region enclosed by $\gamma$ to induce $\int_{\gamma} f(z) dz = 0$ unless $\hat{\gamma} \subseteq \Omega$.

### 4 Argument Principle

A typical application of Rouche’s theorem would be the following. Suppose that we wish to find the number of zeros of a function $f(z)$ in the disk $|z| \leq R$. 


Using Taylor’s theorem we can write

\[ f(z) = P_{n-1}(z) + z^n f_n(z) \]

where \( \deg P_{n-1}(z) = n - 1 \). For a suitably chosen \( n \) it may happen that we can prove the inequality \( R^n |f_n(z)| \leq P_{n-1}(z) \) on \( |z| = R \). Then \( f(z) \) has the same number of zeros in \( |z| \leq R \) as \( P_{n-1}(z) \), and this number can be determined by approximate solution of the polynomial equation \( P_{n-1}(z) = 0 \).

**Lemma 4.1** If the piecewise smooth closed curve \( \gamma \) does not pass through the point \( a \), then the value of the integral

\[ \int_{\gamma} \frac{dz}{z - a} \]

is a multiple of \( 2\pi i \).

**Proof.** If the equation of \( \gamma \) is \( z = z(t), \alpha \leq t \leq \beta \), let us define

\[ h(t) = \int_{\alpha}^{t} \frac{z'(t)dt}{z(t) - a} \]

whose derivative is

\[ h'(t) = \frac{z'(t)}{z(t) - a} \]

whenever \( z'(t) \) is continuous. From this equation it follows that the derivative of \( e^{-h(t)}(z(t) - a) \) vanishes except perhaps at a finite number of points, and since this function is continuous it must reduce to a constant, that is

\[ e^{-h(t)} = \frac{z(t) - a}{z(\alpha) - a} \]

since \( z(\alpha) = z(\beta) \) we have \( e^{h(\beta)} = 1 \) and therefore \( h(\beta) \) must be a multiple of \( 2\pi i \).

Hence, the index or winding number

\[ n(\gamma, a) = \int_{\gamma} \frac{dz}{z - a} \]

is well defined.

**Remark 4.2** Consider a closed, rectifiable path \( \gamma \) inside an open disk \( \Delta \) and a point \( a \in B \) not lying on \( \gamma \). Let \( f \) be analytic in \( \Delta \). Define

\[ F(z) = \frac{f(z) - f(a)}{z - a} \]
then $F \in \mathcal{H}(\Delta \setminus \{a\})$ and $\lim_{z \to a} (z - a)F(z) = \lim_{z \to a} (f(z) - f(a)) = 0$, which implies
\[
\int_{\gamma} F(z)dz = 0
\]
and
\[
\int_{\gamma} \frac{f(z)dz}{z - a} = f(a) \int_{\gamma} \frac{dz}{z - a} = f(a)n(\gamma, a)
\]

**Theorem 4.1 (Homologous version)** Let $G$ be an open and $f : G \to \mathbb{C}$ be analytic. If $\gamma$ is a close rectifiable curve in $G$ such that $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus G$, that is, $G$ is open and $f : G \to \mathbb{C}$ be analytic, $\gamma \subseteq G$ and $\gamma \approx 0$, then for all $a \in G \setminus \gamma$,
\[
n(\gamma, a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)dz}{z - a}
\]

**Proof.** Define
\[
g(z, w) = \begin{cases} 
\frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\
f'(z) & \text{if } z = w
\end{cases}
\]
then $g$ is analytic in $G$.

Let $H = \{w \in \mathbb{C} : n(\gamma, w) = 0\}$. Then $H$ is open since $n(\gamma, w)$ is a continuous integer-valued function of $w$. Moreover, $\mathbb{C} = G \cup H$ by the hypothesis.

Define
\[
h(z) = \begin{cases} 
\int_{\gamma} g(z, w)dw & \text{if } z \in G \\
\int_{\gamma} f(w)dw & \text{if } z \in H
\end{cases}
\]
If $z \in G \cap H$, then
\[
\int_{\gamma} g(z, w)dw = \int_{\gamma} \frac{f(z) - f(w)}{z - w}dw = \int_{\gamma} \frac{f(w)dw}{w - z} - f(z)n(\gamma, z)2\pi i
\]
\[
= \int_{\gamma} \frac{f(w)dw}{w - z}
\]
hence, $h$ is well-defined.

It’s easy to see that $h$ is analytic on $H$ and $G$, (in fact, $h_z = \int_{\gamma} g_z(z, w)dw$ since $\gamma$ is compact and $g(z, w)$ is uniformly continuous on $\bar{B}(z, \delta) \times \gamma$ so limiting process of $z$ is uniform in $w$) that is, $h$ is entire. But $H$ contains a
neighborhood of $\infty$ in $\mathbb{C}_\infty$. Since $f$ is bounded on $\gamma$ and \( \lim_{z \to \infty} (w - z)^{-1} = 0 \) uniformly for $w \in \gamma$,

\[
\lim_{z \to \infty} h(z) = \lim_{z \to \infty} \int_\gamma \frac{f(w)dw}{w - z} = 0 \tag{4.1}
\]

which means $h$ is bounded on $\mathbb{C}$ and hence is a constant. But (4.1) implies that $g \equiv 0$, that is, for all $a \in G \setminus \gamma$,

\[
0 = \int_\gamma \frac{f(z) - f(a)}{z - a} dz = \int_\gamma \frac{f(z)dz}{z - a} - f(a) \int_\gamma \frac{dz}{z - a}
\]

\[\blacksquare\]

**Theorem 4.2 (Cauchy: 1st Version)** Let $G$ be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. If $\gamma_i$ are closed rectifiable curves in $G$ such that $\sum_{i=1}^{m} n(\gamma_i; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for all $a \in G - \cup_{i=1}^{m} \gamma_i$,

\[
f(a) \sum_{i=1}^{m} n(\gamma_i; a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z - a} = \frac{1}{2\pi i} \sum_{i=1}^{m} \int_{\gamma_i} \frac{f(z)dz}{z - a}
\]

**Proof.** Easy. Let $H = \{ w : \sum_{i=1}^{m} n(\gamma_i; w) = 0 \}$ and follow the same lines.

\[\blacksquare\]

**Corollary 4.3** With the same hypothesis, then

\[
\frac{1}{2\pi i} \int_\gamma f(z)dz = 0
\]

**Proof.** Easy. Substitute the $f$ above with $f(z)(z - a)$. This is a very important result which could be used to prove generalized argument principle.

\[\blacksquare\]

**Definition 4.4** Let $\gamma$ be a closed rectifiable curve in $G$. Then $\gamma$ is homotopic to zero ($\gamma \sim 0$) if $\gamma$ is homotopic to a constant curve.

**Definition 4.5** Let $G$ be open. $\gamma$ is homologous to zero, $\gamma \approx 0$, if $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$.

**Definition 4.6** An open set $G$ is simply connected iff $G$ is connected and every closed curve in $G$ is homotopic to zero.

**Theorem 4.3 (Cauchy)** Let $G$ be open, $f : G \to \mathbb{C}$, $\gamma \sim 0$ ($\gamma$ closed, rectifiable), then $\int_\gamma f(z)dz = 0$
Theorem 4.4 (Cauchy) Let \( G, f, \gamma_0, \gamma_1 \) be as given above, but with \( \gamma_0 \sim \gamma_1 \), then
\[
\int_{\gamma_0} f(z) \, dz = \int_{\gamma_1} f(z) \, dz
\]

Theorem 4.5 (Cauchy) If \( G \) is simply connected then \( \int_{\gamma} f = 0 \) for every closed rectifiable curve \( \gamma \) and every analytic function \( f \)

Proof. Suppose an easy case where \( \lambda(s, t) \in C^2(I^2) \). The idea is to define
\[
g(t) = \int_{0}^{1} f(\lambda(s, t)) \frac{\partial \lambda(s, t)}{\partial s} \, ds
\]
where \( \lambda(s, t) \) is the homotopic function and \( I^2 = [0, 1] \times [0, 1] \). Then \( g(0) = \int_{\gamma_0} f \) and \( g(1) = \int_{\gamma_1} f \). By the Leibniz’s rule \( g \) has a continuous derivative
\[
g'(t) = \int_{0}^{1} \left( f'(\lambda(s, t)) \frac{\partial \lambda(s, t)}{\partial t} \frac{\partial \lambda(s, t)}{\partial s} + \int_{0}^{1} f(\lambda(s, t)) \frac{\partial^2 \lambda(s, t)}{\partial t \partial s} \right) \, ds
\]
But
\[
\frac{\partial}{\partial s} \left( (f \circ \lambda) \frac{\partial \lambda}{\partial t} \right) = (f' \circ \lambda) \frac{\partial \lambda}{\partial t} \frac{\partial \lambda}{\partial s} + (f \circ \lambda) \frac{\partial^2 \lambda}{\partial t \partial s}
\]
so
\[
g'(t) = \int_{0}^{1} \frac{\partial}{\partial s} \left( (f \circ \lambda) \frac{\partial \lambda}{\partial t} \right) \, ds = f(\lambda(1, t)) \frac{\partial \lambda(1, t)}{\partial t} - f(\lambda(0, t)) \frac{\partial \lambda(0, t)}{\partial t}
\]
since \( \lambda(1, t) = \lambda(0, t) \) for all \( t \) by the definition of homotopy, \( g' \equiv 0 \) for all \( t \), that is, \( \int_{\gamma_0} f = \int_{\gamma_1} f \).

Use the uniform continuity of \( \lambda(s, t) \) and approximate \( \gamma_0 - \gamma_1 \) with closed polygon. (Polygon are piecewise smooth, which enable us to use the fundamental theorem) Exploit the convexity of open balls.

The idea is to form a grid of \( I^2 \) which induce a \( \varepsilon \)-net for the family of homotopic curves, then cover the orginal curves with small balls and inscribe polygons touching the curve but contained in these small balls, finally sweep the balls, i.e., \( \varepsilon \)-net across the homotopic family. Hereunder are the details.

\( \lambda(I^2) \) being compact implies \( r = d(\lambda(I^2), \partial G) \geq d(\lambda(I^2), C - G) > 0 \).

The uniform continuity of \( \lambda(s, t) \) on \( I^2 \) admits a grid on \( I^2 \) with \( T_{j,k} = \left[ \frac{j}{n}, \frac{j+1}{n} \right] \times \left[ \frac{k}{n}, \frac{k+1}{n} \right] \) and an inscription \( z_{jk} = \lambda \left( \frac{j}{n}, \frac{k}{n} \right) \) for all \( 0 \leq j, k \leq n \). For sufficiently large \( n \), \( \lambda(T_{j,k}) \subseteq B(z_{jk}, r) \), so the polygon \( P_{jk} \) with vertices \( [z_{jk}, z_{j+1,k}, z_{j+1,k+1}, z_{j,k+1}, z_{jk}] \subseteq B(z_{jk}, r) \) and \( \int_{P_{jk}} f = 0 \).
The magic is that, if $Q_k = [z_{0,k}, z_{1,k}, \ldots, z_{nk}]$, then

$$\int_{\gamma_0} f = \int_{Q_0} f = \int_{Q_1} f = \ldots = \int_{Q_n} f = f \int_{\gamma_1} f$$

one step on the ladder at a time. To see that $\int_{\gamma_0} f = \int_{Q_0} f$ observe that if $\sigma_j(t) = \gamma_0(t)$

for $j \leq t \leq j + \frac{1}{n}$, then $\sigma_j + [z_{j+1,0}, z_{j,0}] \subseteq B(z_{j0}, r)$ and $\int_{\sigma_j} f = -\int_{[z_{j+1,0}, z_{j,0}]} f = \int_{[z_{j,0}, z_{j+1,0}]} f$. Adding both sides of this equation for $0 \leq j \leq n$ yields $\int_{\gamma_0} f = \int_{Q_0} f$. Similarly, $\int_{Q_n} f = f \int_{\gamma_1} f$.

To see $\int_{Q_k} f = \int_{Q_{k+1}} f$. Just note that $\int_{P_{jk}} f = 0$ which means $\sum_{j=0}^{n-1} \int_{P_{jk}} f = 0$. But $P_{jk}$ and $P_{j+1,k}$ have intersecting sides and $z_{0,k} = \lambda \left(0, \frac{k}{n}\right) = \lambda \left(1, \frac{k}{n}\right) = z_{n,k}$, so $[z_{0,k+1}, z_{0,k}] = [-z_{1,k}, z_{1,k+1}]$. These cancellations reduces the sum into $0 = \int_{Q_k} f - \int_{Q_{k+1}} f$.

Really nice proof. ■

Notation 1 for a rectifiable Jordan (simple, closed) curve $\gamma$, its interior, denoted by $\dot{\gamma}$, is defined to be the set $\dot{\gamma} = \{z : n(\gamma, z) = 1\}$. This is quite natural, think it for a while.

Theorem 4.6 (Jordan) Let $\gamma \subseteq \mathbb{C}$ be a Jordan curve. Then $\gamma$ divides $\mathbb{C}$ into two simply connected regions, one of which is bounded and called the interior of $\gamma$ and the other of which is unbounded and is called the exterior of $\gamma$. Moreover, $\gamma$ is the mutual boundary of these two regions.

Theorem 4.7 (Residue) Let $\gamma$ be a rectifiable Jordan curve and $K = \{z_i, i = 1, \ldots, n\} \subseteq D = \dot{\gamma}$. If $f \in \mathcal{H}(D - K)$ and $f \in C^0(D - K)$, then

$$\frac{1}{2\pi i} \int_{\gamma} f = \sum_{i=1}^{n} \text{Res}(f, z_i)$$

Proof. $K = \{z_i, i = 1, \ldots, n\} \subseteq D = \dot{\gamma}$ implies it’s possible to draw $n$ disjoint, counter clockwise circles $\rho(z_i, \delta)$.

$f \in \mathcal{H}(D - K)$ and $f \in C^0(D - K)$ implies

$$\frac{1}{2\pi i} \int_{\gamma} f = \frac{1}{2\pi i} \int_{\rho(z_i, \delta)} f(z)dz = \sum_{i=1}^{n} \text{Res}(f, z_i)$$

■

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**Definition 4.7** If \( z = \infty \) is an isolated singularity of \( f \), then
\[
\text{Res}(f; \infty) = -\frac{1}{2\pi i} \int_{\gamma(0,R)} f(z) \, dz
\]

**Remark 4.8** Some book use \( \gamma \approx 0 \) (i.e., \( n(\gamma, w) = 0 \) for all \( w \in \mathbb{C} - G \)). This condition is more general since closed Jordan curve is homologous to zero. The difference is that for Jordan curve, \( z \in \gamma \) iff \( z \) satisfies \( n(\gamma, z) = 1 \). While for general closed curves, \( n(\gamma, z) \) may not be even if \( z \in \gamma \), or that the interior of a general curve should be precisely defines, as done by:

**Definition 4.9** A contour \( \Gamma \) is defined as
\[
\Gamma = \sum_{j=1}^{p} n_j \gamma_j
\]
where \( n_j \in \mathbb{Z} \) and each \( \gamma_j \) is piecewise smooth and closed.

**Definition 4.10** The interior of \( \Gamma \) is defined as \( \text{Int} \Gamma = \{ z \in \mathbb{C} - \Gamma : n(\Gamma, z) \neq 0 \} \) and the exterior is defined as \( \text{Ext} \Gamma = \{ z \in \mathbb{C} - \Gamma : n(\Gamma, z) = 0 \} \). Each of these sets is a union of certain connected (is "connected" needed here?) components of \( \mathbb{C} - \Gamma \) since \( n(\Gamma, z) \) is a constant on each component. (A component is by definition connected)

**Definition 4.11** A contour \( \Gamma \) is called simple if \( n(\Gamma, z) = 0 \) or 1 for all \( z \in \mathbb{C} - \Gamma \)

**Theorem 4.8 (Argument Principle) Homologous Version**
Let \( f \) be meromorphic in the region \( G \) with zeros \( a_i \) and poles \( b_i \) counting to multiplicity. If \( g \) is analytic in \( G \) and \( \gamma \) is closed rectifiable curve in \( G \) with \( \gamma \approx 0 \) and not passing through any \( a_i \) or \( b_j \), then
\[
\frac{1}{2\pi i} \int_{\gamma} g(z) \frac{f'(z)}{f(z)} \, dz = \sum_{i=1}^{m} g(a_i) n(a_i; \gamma) - \sum_{j=1}^{n} g(b_j) n(b_j; \gamma)
\]

**Theorem 4.9 (Simple Version)** Let \( f \) be meromorphic in the domain \( D \) and \( \gamma \) be a rectifiable Jordan curve with interior \( \Omega \subseteq D \). Suppose \( a_i, b_i \in \Omega \) are zeros and poles of \( f \) and that \( f \) has no zeros or pole on \( \gamma \). Further, \( g \) be holomorphic in \( D \), then
\[
\frac{1}{2\pi i} \int_{\gamma} g(z) \frac{f'(z)}{f(z)} \, dz = \sum_{i=1}^{m} g(a_i) s_i - \sum_{j=1}^{n} g(b_j) t_j
\]
where \( s_i \) and \( t_j \) are orders of \( a_i \) and \( b_j \).
Proof. Easy. For a zero $a_i$ of order $s_i$, $f(z) = (z - a_i)^{s_i}h(z)$ where $h(a) \neq 0$, then
\[
\frac{f'(z)}{f(z)} = \frac{s_i(z - a_i)^{s_i-1}h(z)}{(z - a_i)^{s_i}h(z)} + \frac{(z - a_i)^{s_i}h'(z)}{(z - a_i)^{s_i}h(z)} = \frac{s_i}{z - a_i} + \frac{h'(z)}{h(z)}
\]
while for a pole $b_j$ of order $t_j$, $(z - b_j)^{t_j}f(z) = h(z)$ and
\[
\frac{f'(z)}{f(z)} = -\frac{t_j}{z - b_i} + \frac{h'(z)}{h(z)}
\]
in both cases $\frac{h'(z)}{h(z)}$ is analytic. So
\[
\frac{f'(z)}{f(z)} = \sum_{i=1}^{m} \frac{s_i}{z - a_i} - \sum_{j=1}^{n} \frac{t_j}{z - b_i} + g(z)\frac{h'(z)}{h(z)}
\]
and
\[
\frac{1}{2\pi i} \int_{\gamma} g(z)\frac{f'(z)}{f(z)} \, dz = \sum_{i=1}^{m} \frac{1}{2\pi i} \int_{\gamma} g(z)s_i - \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma} g(z)t_j + \frac{1}{2\pi i} \int_{\gamma} g(z)\frac{h'(z)}{h(z)}
\]
\[
= \sum_{i=1}^{m} g(a_i)n(a_i; \gamma) - \sum_{j=1}^{n} g(b_j)n(b_j; \gamma)
\]
It should be noted that $\gamma \subseteq G$, $\gamma \approx 0$, $\gamma(\alpha) = \gamma(\beta)$ (closed) ensures that $\gamma$ could be decomposed into finitely many rectifiable Jordan curves. So, the contour $\Gamma = \sum \rho_k$ with $\rho_k$ being sufficiently small, disjoint circle around each zero and pole is homologous to $\gamma$, that is,
\[
\int_{\gamma} = \int_{\Gamma} = \int_{\sum \rho_k}
\]
which is pretty hard and unrealistic. □

Remark 4.12 Since no zero or pole of $f$ lies on $\gamma$ there is a disk $B(a; \varepsilon)$ for each $a \in \gamma$, such that a branch of log $f(z)$ can be defined on $B(a; \varepsilon)$. By Lebesgue Converging lemma, there is an $\varepsilon > 0$ such that on each $B(a; \varepsilon)$ with $a \in \gamma$ a branch of log $f(z)$ can be defined. Using the uniform continuity of $\gamma$, there is a partition such that $\gamma(t) \in B(\gamma(t_j-1); \varepsilon)$ on $I_{j-1} = \{ t : t_j-1 \leq t \leq t_j \}$ and $1 \leq j \leq k$, that is, $\gamma(I_{j-1}) \subseteq B(\gamma(t_j-1); \varepsilon)$ such that $\gamma(t_j) \in B(\gamma(t_j); \varepsilon) \cap B(\gamma(t_{j+1}); \varepsilon)$. (Intuitively, roll balls ahead of the underlying interval). Let $\lambda_j$ be a branch of log $f(z)$ defined on $B(\gamma(t_j-1); \varepsilon)$ such that $\lambda_{i-1}(\gamma(t_{j-1})) = \lambda_i(\gamma(t_{j-1}))$. If $\gamma_j$ is the path $\gamma$ restricted to $I_{j-1}$, then $\lambda'_j = f'/f$ implies
\[
\int_{\gamma_j} \frac{f'}{f} = \lambda_j(\gamma(t_j)) - \lambda_j(\gamma(t_{j-1}))
\]
Summing the sides of this equation yields
\[
\int_{\gamma} \frac{f'}{f} = \lambda_k(\gamma(1)) - \lambda_1(\gamma(0)) = \lambda_k(\gamma(0)) - \lambda_1(\gamma(0))
\]
since \( \gamma \) is closed. The fact that any two branches of \( \log \) should defer by a multiple of \( 2\pi i \) implies that
\[
\int_{\gamma} \frac{f'}{f} = \lambda_k(\gamma(1)) - \lambda_1(\gamma(0)) = 2\pi k
\]

**Theorem 4.10** If \( f \) is locally univalent and analytic, then for \( \zeta = f(z) \) it’s clear that
\[
z = \frac{1}{2\pi i} \int_{|z - a| = r} \frac{w f'(w)}{f(w) - f(z)} dw
\]

**Theorem 4.11** (Rouché) Suppose \( f \) and \( g \) are meromorphic in a neighborhood of \( B(a; R) \) with no zeros or poles on the circle \( \gamma = \{ z : |z - a| = R \} \). If \( q_f, q_g, (p_f, p_g) \) are zeroes (poles) of \( f \) and \( g \) inside \( \gamma \) counted according to their multiplicities and if
\[
|f(z) + g(z)| < |f(z)| + |g(z)|
\]
on \( \gamma \), then
\[
q_f - p_f = q_g - p_g
\]

**Proof.** From the hypothesis
\[
\left| \frac{f(z)}{g(z)} + 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1
\]
on \( \gamma \). If \( \frac{f(z)}{g(z)} > 0 \) then the above inequality fails. So \( \frac{f}{g} \) maps \( \gamma \) into \( \Omega = \mathbb{C} - [0, \infty) \) and there is a well defined branch of logarithm on \( \Omega \) with \( \lambda = \log \frac{f}{g} \) being the primitive of \( \frac{(f/g)'}{f/g} \) in a neighborhood of \( \gamma \). Thus
\[
0 = \log(f/g)_{\gamma(0)}^{\gamma(1)} = \frac{1}{2\pi i} \int_{\gamma} \frac{(f/g)'}{f/g} = \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} - \frac{g'}{g} = (q_f - p_f) - (q_g - p_g)
\]
It should be noted that the hypothesis here is much weaker. ■
**Theorem 4.12 (Branched Covering)** Let a non-constant \( f \in \mathcal{H}(D) \) and \( w_0 = f(z_0) \), \( z_0 \in D \) be a zero of order \( m \) of \( f(z_0) - w_0 \). Then there exists \( \varepsilon > 0 \), \( \delta > 0 \) such that for any \( w' \in \Delta(w_0, \varepsilon) \) the function \( f(z) - w' \) has exactly \( m \) roots \( z_i, i = 1, \ldots, m \) with \( z_i \in \Delta^*(z_0, \delta) \subseteq D \).

**Proof.** \( f \) being a non-constant implies \( z_0 \) is an isolated zero of \( f(z_0) \) and \( f'(z) \). Then there is some \( \delta > 0 \) such that \( \Delta^*(z_0, \delta) \subseteq D \) and \( f(z) - w_0 \neq 0 \), \( f'(z) \neq 0 \) for all \( \Delta^*(z_0, \delta) \). Let

\[
\varepsilon = \min \{|f(z) - f(z_0)| : z \in C(z_0, \delta)\} > 0
\]

Then for any \( w' \in \Delta(w_0, \varepsilon) \) and \( z \in C(z_0, \delta) \),

\[
|w' - w_0| < |f(z) - f(z_0) |
\]

By Rouche’s Theorem, \( f(z) - f(z_0) \) and

\[
f(z) - f(z_0) - w' + w_0 = f(z) - w'
\]

have the same number of zeros on \( B(z_0, \delta) \), which is \( z_i, i = 1, \ldots, m \). Further \( f'(z) \neq 0 \) for \( z \in B^*(z_0, \delta) \) implies each \( z_i \) is simple. ■

### 5 Various Interesting Problems

**Problem 5.1** Can you map the open unit disk conformally onto the punctured open unit disk?

**Solution.** Define

\[
w = \frac{1 + z}{1 - z}
\]

which maps \( \Delta \) onto the right half plane \( H_+ = \{ z : \text{Re} \ z > 0 \} \). Now rotate \( H_+ \) counterclockwise by \( \pi \), that is

\[
v = e^{\pi i} w
\]

Finally, exponentiate \(-H_+ = H_-\), that is,

\[
f(z) = \exp v = \exp \left(e^{\pi i} \frac{1 + z}{1 - z} \right) = \exp \left(-\frac{1 + z}{1 - z} \right)
\]

which will wrap the punctured unit disk infinitely many time while keeping the modulus from 0 to 1 as desired. ■
Problem 5.2 Map analytically $G = \mathbb{C}\setminus \{z : -1 \leq z \leq 1\}$ onto the open unit disk $\Delta$. Can the map be injective?

Solution. Function

$$w = f(z) = \frac{1}{2} \left( z + \frac{1}{z} \right)$$

expands $\Delta$ onto $\mathbb{C} - I$ and $\mathbb{C} - \Delta$ also onto $\mathbb{C} - I$. So the inverse of $f(z)$ would do. But there can’t be any such bijection, since such a map preserves simply connected region. ■

Problem 5.3 Let $G$ be a domain and $f : G \rightarrow \mathbb{C}$ be analytic and $f(G)$ be a subset of a circle. Then $f$ is a constant.

Problem 5.4 Show that

$$f(z) = \exp \left\{ -i \log \left[ i \left( \frac{1 + z}{1 - z} \right) \right]^{1/2} \right\}$$

maps $D = \{z : |z| < 1\}$ conformally onto an annulus $G$. Find all Mobius transformations $\mu(z)$ that map $D$ onto $D$ such that $f(\mu(z)) = f(z)$ on $D$.

Solution. $w = \frac{1 + z}{1 - z}$ maps $D$ onto $H_+$, then $H_+$ is rotated counterclockwise by $\frac{\pi}{2}$ to be $U_+$ then it is squeezed into $U_+/2$ since on $U_+$ there is a branch of $w_1 = (iw)^{1/2}$. Also there is a branch of log and log $w_1$ squeezes $U_+/2$ into an infinite, horizontal stip of width $\frac{\pi}{2}$. Finally, $\exp(-i \log w_1)$ rolls the rotated (clockwise by $\pi/2$) stip into an annulus of inner radius 1 and outer radius $e^{\pi/2}$. ■

A simple inequality: $\frac{\sin x}{x} \geq \frac{2}{\pi}$ for $0 \leq x \leq \frac{\pi}{2}$