ISOMETRIC APPROACH TO STOCHASTIC INTEGRAL
OF CERTAIN RANDOM MEASURES

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To Myself, My Family

ABSTRACT.

Conventions and Notations
(1) The measure space \((M, \mathcal{M}, \mu)\) is interpreted as having \(M\) as the underly set, \(\mathcal{M}\) the sigma algebra on \(M\), and \(\mu\) the measure on \(M\). If \(M\) is a metric space then \(\mathcal{M}\) is taken as the sigma algebra generated by all its metric-open sets.
(2) For two measure spaces \((M, \mathcal{M}, \mu)\), \((M_1, \mathcal{M}_1, \mu_1)\) the measure space \((M \times M_1, \mathcal{M} \otimes \mathcal{M}_1, \mu \otimes \mu_1)\) is their product measure space. All measures in the paper are considered complete.
(3) Let the triple \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\mathcal{R} = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda)\) the Euclidean-metric measure space where \(\lambda\) is the Lebesgue measure on \(\mathbb{R}^n\). Let \(S = (K, \mathcal{U}, \nu)\) be their product space, that is, \(K = [0, \infty) \times \Omega, \mathcal{U} = \mathcal{B}([0, \infty)) \otimes \mathcal{F}, \nu = \lambda \otimes \mathbb{P}\).
(4) A filtration of a measure space \((M, \mathcal{M}, \mu)\) is a non-decreasing family of sub-sigma algebras \(\{\mathcal{M}_t : t \in T\}\) of \(\mathcal{M}\) where \(T\) is some index set. Filtrations in this paper are augmented with \(\mu\)-null sets of \(M\).
(5) As a basic assumption, all stochastic process are \(\mathbb{R}^n\)-valued, measurable random functions \(\{X(t, \omega)\}\) from \(S\) to \(\mathcal{R}\) and are adapted to the filtration \(\{\mathcal{F}_t : t \in [0, \infty)\}\) of \((\Omega, \mathcal{F}, \mathbb{P})\).
(6) Conventions and Notations
(7) "Return" means "log-Return"

1. INTRODUCTION

When modelling asset returns by continuous time models, the key is to model the fluctuations (or volatility) of the asset price for small durations of time. To allow for uncertainty in asset price, a stochastic part is integrated in each such model by appropriately defining the notion of "differentiation of a random process" or in other words the notion of "stochastic integral of a random process". Highly dependent on both on the path properties of the trajectory of process of integration and the regularity of the integrand, there are usually four ways to define stochastic integral with respect to a random process, namely, part-wise integration, isometric integration, multiple Ito

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calculus, and Malliavin calculus. While for a stochastic integral to make financial significance by capturing the price process and trading strategy, the stochastic integral usually takes the isometric approach or Mallian calculus approach since most processes of integral have infinite variation.

Ever since the publication of the pioneering work of Bachelier, standard brownian motion (BM) has been the cornerstone in defining stochastic integral whose resultant diffusion model integrates the efficient market hypothesis. But it turns out the such diffusion models fail to generate heavy tails or long-range dependence or sufficient discontinuities in the trajectories due to the intrinsic property of standard BM. As an effort to incorporate long-range dependence, in 196? Mandelbrot defined the famous fractional brownian motion and applied it to model the long-range dependence of ???, which still can not generate heavy tails. Recently, Bibby (1997) proposed the so-called hyperbolic diffusion model which is able to generate fat tails in the marginal distribution of the resultant stochastic process but fails to generate long-range dependence. Some researchers (eberlin) also tried generalized hyperbolic Levy process which are able to generate heavy tails but still fails to generate long-term dependence and as the number of parameters involved is 5 it is usually hard to estimate the marginal densities. Few researchers tried the so called fractional Poisson process which integrates both heavy tails and long-range dependence, but there is almost no paper on its application in constructing continuous time financial models is available.

A careful study would reveal that the above mentioned failure is due to the subtle balance and to some extent the exclusivity of tail behavior and correlation structure of the processes of integration, as clearly demonstrated by the stable self-similar process which are able to generate both heavy tails and self-similarity except that the tails are so heavy that the resulting process has infinite variance, violating the fact the most assert returns have finite variance.

Consequently a stochastic integral which allows both a flexible correlation structure and the use of heavy-tailed distribution as the marginal distribution seems to be of paramount importance and attempts to define such an integral is the aim of this paper. By examining the essence of the classic Ito’s stochastic integral with respect to standard Brownian motion, the isometric approach to stochastic integral with respect to fractional Brownian motion and the stochastic integral with respect to stable distributions, and by combining the techniques of isometric approach and random measures, it seems possible to define such an integral, thus extending the current scheme of stochastic integral for financial applications and incorporates more market reality into the resultant models.

2. Quasi-random Measures

To define the continuity needed later, we introduce a variation which extends the classic Hausdorff dimension to bounded set of any metric space.

Definition 1. The Hausdorff dimension $d_H$ of two bounded subsets $A, B \subseteq \mathbb{R}^n$ is defined to be

$$d_H (A, B) = \max \{ d (x, \bar{A}), d (y, \bar{B}) : x \in B, y \in B \}$$
where \( d(x, C) = \inf \{\|x - z\| : z \in C\} \) and \( \bar{C} \) denotes the metric closure of \( C \) in \( \mathbb{R}^n \) for any \( x \in \mathbb{R}^n \) and \( C \subseteq \mathbb{R}^n \).

Note that for Euclidean \( n \)-space \( \mathbb{R}^n \), the above definition makes

**Proposition 1.** \( d_H(A, B) = 0 \) implies \( \lambda(A \Delta B) = 0 \) for any bounded, closed \( A, B \subseteq \mathbb{R}^n \)

**Proof.** It suffices to show \( \lambda(A \setminus B) = \lambda(B \setminus A) = 0 \). Take any countable open covering \( O = \{D_i = (a_i, b_i)\} \) such that
\[
B \setminus A \subseteq \bigcup_i (a_i, b_i)
\]
Then by the assumption that \( d_H(A, B) = 0 \), it’s clear that for any \( x \in A, y \in B \)
\[
d(x, y) = 0
\]
Consequently, for the adjusted open covering
\[
\hat{O} = \left\{D_{i, \varepsilon} = (a_i, a_i + \frac{\varepsilon}{2^i}) : i \in \mathbb{N}\right\}
\]
it holds that
\[
B \setminus A \subseteq \bigcup_i D_{i, \varepsilon}
\]
which means
\[
\lambda^* (B \setminus A) \leq \sum_{i=1}^{\infty} \left| \left( a_i, a_i + \frac{\varepsilon}{2^i} \right) \right| = \varepsilon
\]
where \( \lambda^* \) is the outer measure induced by usual metric.

Since \( \varepsilon > 0 \) is arbitrary, it’s clear that
\[
\lambda^* (B \setminus A) = \lambda(B \setminus A) = 0
\]
Similary, we can show that
\[
\lambda(A \setminus B) = 0
\]
and the assertion is proved.

Let \( K^\alpha \) be the space of all random functions from \( \Omega \) to \( \mathbb{R}^n \), i.e.,
\[
K^\alpha = \{Y \in L^\alpha (\Omega, \mathcal{F}, P)\}, \alpha > 0
\]
Let
\[
B_0 (\mathbb{R}_+) = \{A \in B (\mathbb{R}_+) : \lambda(A) < \infty\}
\]

**Definition 2.** A set function
\[
\Phi : B_0 (\mathbb{R}_+) \to K^\alpha
\]
is a quasi-random mapping iff \( \{\Phi(A) : A \in B_0 (\mathbb{R}_+)\} \) is a random process.

For notational simplicity, we will interchangeably write \( \Phi(A) \) as \( \Phi_A \). We introduce the following definitions on degrees stochastic continuity of \( \Phi \)

**Definition 3.** The quasi-random mapping \( \Phi \) is
weakly continuous if
\[ \lim_{d_H(A,B) \to 0} \int \Omega |\Phi_A(\omega) - \Phi_B(\omega)| P(d\omega) = 0 \]
or
\[ \lim_{d_H(A,B) \to 0} P(\Phi_A(\omega) \neq \Phi_B(\omega)) = 0 \]

ii) Strong continuous if
\[ P\left( \left\{ \omega \in \Omega : \lim_{d_H(A,B) \to 0} \Phi_A(\omega) = \Phi_B(\omega) \right\} \right) = 1 \]

iii) Uniform continuous if
\[ \lim_{d_H(A,B) \to 0} \sup_{\omega \in \Omega} \{|\Phi_A(\omega) - \Phi_B(\omega)| : \omega \in \Omega\} = 0 \]

for \( \forall A, B \in \mathcal{B}_0(\mathbb{R}_+) \).

We adopt the algebraic notion and define
\[ (\mathcal{B}_0(\mathbb{R}_+))^I = \sum_{i \in I} \mathcal{B}_0(\mathbb{R}_+) = \{ p : I \to \mathcal{B}_0(\mathbb{R}_+) \} \]

where \( I \) is a non-empty index set and for \( n \in \mathbb{N} \) define
\[ \mathbb{N}_n = \{ J \subseteq \mathbb{N} : |J| = n \} \]

Let’s define
\[ \delta_n(\sigma(\Phi_{A_n})) = \sup \{ |P(\bigcap_{i=1}^n H_i) - \prod_{i=1}^n P(H_i)| : H_i \in \sigma(\Phi_{A_n}), 1 \leq i \leq n \} \]

where for a random variable \( x \), \( \sigma(x) \) denotes the sigma-algebra generated by \( x \).

Definition 4. The quasi-random mapping \( \Phi \) is independently scattered iff
\[ \delta\left( \sigma\left(\Phi\left(A^{(n)}\right)\right) \right) = 0 \]

for any \( A^{(n)} \in (\mathcal{B}_0(\mathbb{R}_+))^n, n \in \mathbb{N} \). Otherwise it is called "dependently scattered".

Now let’s recall the definition of fractional brownian motion (fBm) so that we can introduce the so-called quasi-random measure.

Definition 5 ([4]). The (one-sided, normalized) fractional Brownian motion (fMb) with Hurst index \( H \in (0, 1) \) is a Gaussian process \( B^H = \{ B^H_t : 0 \leq t < \infty \} \) on \( (\Omega, \mathcal{F}, P) \) such that

a1-1) \( B^H_0 = 0 \) a.s.

a1-2) \( B^H_t \sim N(0, t) \), \( 0 < t \)

a1-3) Dependent increments \( E\left(B^H_t B^H_s\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), 0 \leq t, s < \infty \)

Note that for Gaussian process uncorrelation is equivalent to independence, we can see that when \( H = \frac{1}{2} \) a fBm reduces to a standard BM.

Now let us re-define fBm in term of quasi-random measure as
Proposition 2. Let $\Phi$ satisfy
\[
E \left( \exp \left( \sqrt{-1} \sum_{i=1}^{n} a_i \Phi_{A_i} \right) \right) = \exp \left( -\frac{1}{2} \sum_{i,j=1}^{n} a_i \Cov(\Phi_{A_i}, \Phi_{A_j}) a_j \right)
\]
for any $A_i, A_j \in \mathcal{B}_0(\mathbb{R}_+), a_i \in \mathbb{R}, n \in \mathbb{N}, 1 \leq i, j \leq n$, where
\[
\Cov(\Phi_{A_i}, \Phi_{A_j}) = \frac{1}{2} \left( \lambda(A_i)^{2H} + \lambda(A_j)^{2H} - (\lambda(A_i \Delta A_j))^{2H} \right)
\]
for some $H \in (0, 1)$. Then $\Phi_{\mathcal{B}_0} = \{ \Phi_A : A \in \mathcal{B}_0(\mathbb{R}_+) \}$ is a one-side, normalized, generalized fBm.

Further, let
\[
\mathcal{A} = \{ A_t = [0, t] : 0 \leq t < \infty \}
\]
then the restriction
\[
\Phi|_{\mathcal{A}} = \{ \Phi_{A_t} : A_t \in \mathcal{A} \}
\]
is a one-sided normalized fBm (in classic sense).

Proof. The condition implies $\{ \Phi_{A_i} : 1 \leq i \leq n \}$ for each $n \in \mathbb{N}$ and hence $\Phi_{\mathcal{B}_0}$ is a Gaussian process. \hfill $\square$

We can such a $\Phi$ a fBm random measure and denote it by $\Phi^{fBm}$. Next we prove the existence of $\Phi^{fBm}$

Theorem 1. Do we have to use Kolmogorov Consistency Theorem?

Proposition 3 (Samarodintsky94). Let
\[
\Phi_A \sim S_\alpha \left( (\lambda(A))^{1/\alpha}, \int A \beta^\alpha(x) \lambda(dx), 0 \right)
\]
Then $\Phi$ is independently scattered and sigma-additive in the sense that
\[
\Phi \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \Phi(A_i), a.s.
\]

Now let’s restate the path-wise stochastic integral and Malliavin calculus approach to stochastic integral with respect to fBm.

\[
E(\Phi_A \Phi_B) = f(\lambda(A), \lambda(B)) \text{ for some } f : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ such that }
\]

$\in L$ such that
\[
E \left( \Phi \left( \bigcup_{p=m}^{n} A_i \right) \right)^2 = E \left( \sum_{p=m}^{n} \Phi(A_i) \right)^2
\]
\[
\leq E \left( \sum_{p=m}^{n} |\Phi(A_i)| \right)^2 = \sum_{i=1}^{\infty} E(\Phi(A_i))
\]
\[
= \sum_{i=1}^{\infty} \lim f
\]
whenever $A, B \in \mathcal{B}(\mathbb{R}_+)$, where $\lambda$ is the Lebesgue measure and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is some measurable function which might depend on $A, B$
One first problem is to prove that
\[ \sum_{i=1}^{\infty} \Phi(A_i) \]
converges in some sense.

Besides the properties listed in (\()), a random measure may satisfy some of the properties given below:
\[ c5): \delta(\sigma(\Phi(A)), \sigma(\Phi(B))) = \sup_{A \in \mathcal{A}_Y, B \in \mathcal{A}_Z} \{ P(A \cap B) - P(A) P(B) \} \approx g(Y, Z), \quad \text{where} \quad \mathcal{A}_Y = \sigma(\Phi(A)), \mathcal{A}_Z = \sigma(\Phi(B)) \text{ and the symbol} \]
"\(\approx\)" \text{is interpreted as "consummate" as in (Frederi) and} \(\sigma(\cdot)\) \text{denotes}
the sigma-algebra generated by the argument.

Now if we let \(C_t = [0, t] \forall t \geq 0\) and define a continuous random measure (since we want the trajectory of \(X_t\) to have a.s. version, we need
to define the continuity if the random measure. The metric for measuring
the distance between two sets need to be the Hausdorff dimension so that for
\(C_t = [0, t]\) and \(C_s = [0, s]\) it will be true that
\[ \lim_{s \to t} \Phi(C_s) = \Phi(C_t), \text{a.s.} \]
\[ d1): \Phi(\{0\}) = X_0 = 0 \]
\[ d2): X_t \sim N(0, \sqrt{t}) \]
\[ d3): E(\Phi(C_t)\Phi(C_s)) = \frac{1}{2} \left( \lambda(C_t)^{2H} + \lambda(C_s)^{2H} - (\lambda(C_t \setminus C_s)^{2H}) \right) \text{ and} \]
set \(H \in (0, 1)\)
Then \(X = \{X_t = \Phi(C_t) : t \geq 0\}\) is a fBm which includes standard BM as
a subclass.

such that \(X_t \sim N(0, \sqrt{t-s})\) i.e, then \(B_t^H\) is a normal random variable
with mean 0 and standard deviation \(\sqrt{t}\), then it is clear that
\(\Phi(C_t) = \Phi(C_t \setminus C_s) + \Phi(C_s)\) implies \(\Phi(C_t \setminus C_s) = B_t^H - B_s^H\). This forces \(B_t^H\)
is the summation of \(B_s^H\) for some \(s < t\) and the corresponding increments.
where the convergence is in \(L^2\)-norm.

**Remark 1.** For how to construct \(\alpha\)-stable processes, please see (, [8]) for
\(\alpha\)-stable random measures

**Remark 2.** Sigma additive" or aif
\[ \Phi \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \Phi(A_i) \in L^2(\Omega, \mathcal{F}, P) \]

3. **Define the General Stochastic Integral**

3.1. **Isometric Approach of Stochastic integration Revisited.** Before
constructing the unified model, let’s review some of the key ideas of isometric
approach of defining stochastic integrals. (see Yulila Mishura, 2000, isometric
approach)

Let \(\{X_t : 0 \leq t \leq T\}\) be a continuous stochastic process define on a com-
plete probability space \((\Omega, \mathcal{F}, P)\). Let \(\mathcal{F}_t = \mathcal{F}_t^X\) be the sigma field generated
by \(X\) on \([0, t]\). Let \(\Delta X_t = X_{t_i} - X_{t_{i-1}}\). Assume that the integrand \(f\) is a sim-
ple predictable process: \(f_t = \sum f_i 1_{[t_{i-1}, t_i)}(t)\) where \(f_i \in \mathcal{F}_{t_{i-1}}\) and \(t_i \in \pi_n\).
Simple predictable processes are denoted by $L_0$. For any $f \in L_0$ and any (continuous $X$) define the stochastic integral of $f$ with respect to $X$ as

$$(f, X) = \sum_i f \Delta X_i$$

Now assume $\| \pi_n \| \to 0$ If the process $X$ is a standard brownian motion, and $f = \lim f_n$ in $L^2 (P \otimes \mu)$, one can define

$$(f, B) = L^2- \lim (f_n, B)$$

using the classical Ito isometry

$$E (f_n, B)^2 = E \int (f_n)^2 \, ds$$

Now for any continuous stochastic process $X$ and simple predictable process $f$, use

$$E (f_n, B)^2 = E \int (f_n)^2 \, ds$$

to define a semi-norm for $(f, X)$. For any arbitrary integrator $X$ the semi-norm becomes a norm and we can pass to the limit in the completion of $(f, X)$ under the norm $E (f_n, B)^2 = E \int (f_n)^2 \, ds$ for all $X$ and simple predictable $f$.

In particular, we show that if $X$ is a fractional Brownian motion $B^H$ then we can define a norm by putting

$$\| (f, B^H) \|_G = \left( E \int f_n^2 \, ds \right)^{1/2}$$

in the space $G$ of random variables of the form $\{ g \in G : G = (f, B^H), f \in L_0 \}$

3.2. Mathematical Framework for Unified Model. Inspired by the above idea and knowing the stylized properties of empirical distributions of asset returns processes (Rama Cont, Mills), we would like to re-examine the key to model a return process as follows. We interchangeably write $X_t$ as $X(t, \omega)$. When constructing a continuous time model for such a process $X_t$ for $t$ in some interval, the essence is to efficiently and practically describe the small variation

$$\Delta X(t, \Delta t) = X_{t+\Delta t} - X_t$$

at every timing $t$ with given time span $\Delta t$. It’s widely acknowledged that the probability law for $\Delta X(t, \Delta t)$ depends both on $t$ and $\Delta t$ changes and it’s unreasonable to assume the for all $t, \Delta t$ the variation $\Delta X(t, \Delta t)$ follows the same probability law or follows the same family of unimodal laws. To incorporate this reality, we attempt to construct a mechanism that is capable of allowing an abundance of laws for $\Delta X(t, \Delta t)$ at each $(t, \Delta t)$ and then we "piece" these variations together by introducing an appropriate type of stochastic integral which involves the so call "multifactal mapping" in the following.

Essentially, we will define a random measure and for such a definition we let (inspired from Stable Non-gaussian random processes)

$$\mathcal{K} = \left\{ Y : [0, \infty) \times \Omega \to \mathbb{R}^n : Y \in (\mathcal{G}, \mathcal{U}, d\nu) / (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), dm) \right\}$$

$$Y := Y_t(\omega) \text{ is progressively measurable}$$
We need a norm on $K$ such that convergence makes sense. We also define a random measure

$$\Phi : (\mathbb{R}_+, \mathcal{B} (\mathbb{R}_+), dx) \mapsto (K, ||\cdot||_1)$$

which satisfies

4.1 $\Phi (\emptyset) = 0$

4.2 Sigma Additivity: $\Phi (\sum A_i) = \sum \Phi (A_i)$ for mutually disjoint $\{A_i \in \mathcal{B} (\mathbb{R}_+)\}$

4.3 Independently Scattered: $\Phi (A), \Phi (B)$ are independent whenever $A \cap B = \emptyset$

or

4.4 Dependently Scattered: $E (\Phi (A) \Phi (B)) = f (\mu (A), \mu (B))$ whenever $A, B \in \mathcal{B} (\mathbb{R}_+)$, where $\mu$ is the Lebesgue measure and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is some measurable function which might depend on $A, B$

For such a random measure, we need the so called continuity or convergence. Thus we define another distance $d$ on $(\mathbb{R}_+, \mathcal{B} (\mathbb{R}_+), \mu)$ which is

$$d (A, B) = \mu (A \Delta B)$$

and define the quotient space

$$\mathcal{B} (\mathbb{R}_+)/\sim = \{ \{B \in \mathcal{B} (\mathbb{R}_+) : d (A \Delta B) = 0 \} : A \in \mathcal{B} (\mathbb{R}_+) \}$$

To be notationally efficient, the quotient space $\mathcal{B} (\mathbb{R}_+)/\sim$ is still denoted by $\mathcal{B} (\mathbb{R}_+)$ and the resultant metric space is denoted by $(\mathcal{B} (\mathbb{R}_+), d)$. Note that this metric is complete. Now let’s impose the famous Skorohod topology $\mathcal{S}$ (See Patric Billingsly) on $K$, and define continuity of the random measure $\Phi$ by

**Definition 6.** The random measure $\Phi$ is continuous at $A$ iff

$$\lim_{d (B, A) \rightarrow 0} \mathcal{S} (\Phi (A), \Phi (B)) = 0$$

This is a nice idea. Now suppose $\Phi$ is continuous (actually $\Phi (A), \Phi (B)$ can differ very much)

With continuity, it seems we can define integrability. But this can not be hard since we want

$$\lim_{\Delta t \rightarrow 0} \Delta X (t, \Delta t) = \lim_{\Delta t \rightarrow 0} (X_{t+\Delta t} - X_t)$$

be as irregular as possible for each fix $\omega \in \Omega$, while for a pair of can $(t, \Delta t)$

$$\Delta X (t, \Delta t) \sim d \Phi (t + \Delta t) - \Phi (t)$$

for each fixed $\omega \in \Omega$.

Now we define stochastic integral with respect to $\Phi (\cdot)$. Obviously, the Lebesgue measure $\mu$ has something to do with $\Phi (\cdot)$. The strongest condition will be

$$\frac{d \Phi}{d \mu} = \varphi \in K$$

Note that this is something like Girsanov’s Theorem
For each $t$, we let $C_t = [0, t]$ and try to describe $\Delta X (t, \Delta t) = X (t + \Delta t) - X (t)$ by incorporating the dependence of the law for $\Delta X (t, \Delta t)$ on $t, \Delta t$ into $X (t + \Delta t) - X (t) = a (t, X_t) \Delta t + b (t, X_t) (\Phi (C_t) - \Phi (C_{t+\Delta t}))$

for some appropriately regular functions $a, s$, we take the isometric approach to stochastic integral.

Note that Let $\{ \Phi (C_t) : 0 \leq t \leq T \}$ is (how to call it)a continuous "stochastic" process with underlying complete probability space $(\Omega, \mathcal{F}, P)$.

**Definition 7.** The sigma-algebra generated $\mathcal{F}^\Phi_t$ by $\Phi (C_s)$ for $0 \leq s \leq t$ is defined as

$$\mathcal{F}^\Phi_t = \sigma (\Phi (\mathcal{B} ([0, t])))$$

Let $\Delta \Phi_i = \Phi (C_{t_i}) - \Phi (C_{t_{i-1}})$. Assume that the integrand $f$ is a simple predictable process: $f_i = \sum f_i 1_{[t_{i-1}, t_i)} (t)$ where $f_i \in \mathcal{F}_{t_{i-1}}$ and $t_i \in \pi_n$. Simple predictable processes are denoted by $L_0$. For any $f \in L_0$ we define the stochastic integral of $f$ with respect to $\Phi$ as

$$(f, X) = \sum_i f \Delta \Phi_i$$

Now assume $\| \pi_n \| \to 0$ If the process $\Phi$ is a standard brownian motion, and $f = \lim f_n$ in $L^2 (P \otimes \mu)$, one can define

$$(f, B) = L_2 \text{-} \lim (f_n, B)$$

using the classical Ito isometry

$$\| (f_n, B) \| = E (f_n, B)^2 = E \int (f_n)^2 \, ds$$

Now for $\Phi$ and simple predictable process $f$, define

$$G = \left\{ (f, \Phi) = \sum_i f \Delta \Phi_i : f \in L_0 \right\}$$

and defined the normed space $(G, \| \|_G)$ where

$$\| (f, \Phi) \|_G = \left( E \int f^2 \, ds \right)^{1/2}$$

Finally, take the completion of $(G, \| \|_G)$ which is denoted by

$$ (H, \| \|_G) = \left\{ L^2 \text{-} \lim (f_n, \Phi) : (f_n, \Phi) \in G, L_2 \text{-} \lim f_n \text{ exists} \right\}$$

Let can establish a well-defined and meaningful stochastic integral as

**Definition 8.** Given an adapted (probably, progressively measurable) stochastic process $g = \{ g_t (\omega) \}$ and any $t \in [0, \infty)$, the stochastic integral of $g$ with respect to $\Phi$ defined as

$$\int_0^t g_s (\omega) \, d\Phi (s) := L_2 \text{-} \lim \sum_{\| \pi_n \| \to 0} g_{s_i} (\omega) 1_{[s_{i-1}, s_i)} (t) (\Phi (C_{s_i}) - \Phi (C_{s_{i-1}}))$$

where $\pi_n = \{ t_i : 0 = s_0 < s_1 < \cdots < s_n = t \}$ and

$$g_s (\omega) = L_2 \text{-} \lim g_{t_i} (\omega) 1_{[t_{i-1}, t_i)} (t), 0 \leq s \leq t$$

and the limit is taken in $(H, \| \|_G)$
The above definition is actually of the Lebesgue-Ito type. That being done, we then have a well defined

$$X_t = X_{t_0} + \int_{t_0}^t a(s, X_s) \, ds + \int_{t_0}^t b(s, X_s) \, d\Phi(C_s)$$

For any $T > 0$, define

$$F_T^+(x) = 1 - \Pr \left( \cup_{t \in [0,T]} \{ X_t(\omega) > x \} \right)$$

and

$$F_T^-(x) = \Pr \left( \cup_{t \in [0,T]} \{ X_t(\omega) > x \} \right)$$

Then we can ask this following question which is key to financial modeling: For what random measure $\Phi$ and for what functions $a, b$ does there exist a stochastic process $X = \{ X_t(\omega) : (t, \omega) \in [0, \infty) \times \Omega \}$ such that

1. **Controllable tails**

   $$\lim_{x \to +\infty} \frac{F_T^+(x)}{C^\alpha x^{-\alpha} R(x)} = 1$$

   and

   $$\lim_{x \to -\infty} \frac{F_T^-(x)}{D^\beta |x|^\beta L(x)} = 1$$

   at least for some $\alpha, \beta$ with $0 < \alpha, \beta < 5$, where $R(x), L(x)$ are (slow varying) positive functions.

   Define

   $$\tilde{T}^+ = \inf \left\{ T \in [0, \infty) : \lim_{x \to +\infty} \frac{F_T^+(x)}{C^\alpha x^{-\alpha} R(x)} = 1 \right\}$$

   and

   $$\tilde{T}^- = \inf \left\{ T \in [0, \infty) : \lim_{x \to -\infty} \frac{F_T^+(x)}{C^\alpha x^{-\alpha} R(x)} = 1 \right\}$$

   Note that $\tilde{T}^+, \tilde{T}^-$ might be $+\infty, -\infty$.

2. **Controllable correlations**

   $$E \left( (X_t - X_s)(X_u - X_v) | X_r \right) = h(t, s, u, v; r)$$

   for $0 \leq K \leq s \leq t \leq u \leq v < \infty$, and for where for fixed $r$, $h(t, s, u, v; r)$ is a real-valued function that has some semi-definiteness on the pair $s, t$ and/or $u, v$.

3. **A stable central part**

   $$\Pr \left( \bigcup_{t \in [0, T_1]} \{ X_{t}^{\lambda \wedge T} (\omega) > x \} \right) \asymp S_\rho (\lambda, \vartheta, \mu)$$

   for some $T_1 \geq 0$, where $S_\rho (\lambda, \beta, \mu)$ a stable distribution whose interpretation is given in (Taqqu: Stable Non-Gaussian Random Processes)
A continuously concatenated characteristic function
\[ \varphi_t(z;p) = \int_{-\infty}^{+\infty} \exp(izX_t) dF_t(X_t) \]
where \( p \) are related parameters and \( F_t(x) \) is the cumulative distribution function (cdf) of \( X_t(\omega), t \in [0, \infty) \).

It should be clear that the existence of such a process \( \{X_t(\omega) : (t, \omega) \in [0, \infty) \times \Omega\} \) will give a unified model for returns processes since this process has a trajectory that captures all main stylized features of empirical distribution function of returns processes.

4. Necessary Constraints on e-pdf

Now let’s study some necessary properties of such an \( F_T(x) \) if the solution stochastic process \( X = \{X_t\} \) does exist.

**Proposition 4.** If
\[ \lim_{x \to +\infty} \frac{F_T^+(x)}{x^\alpha R(x)} = 1 \]
and
\[ \lim_{x \to -\infty} \frac{F_T^-(x)}{|x|^\beta L(x)} = 1 \]
where \( R(x), L(x) \) are not power functions. Then
\[ R(x) < \]
and
\[ L(x) < \]

**Proof.** Since \( F_T(x) \) is differentiable a.s. mod Lebesgue measure \( \mu \), we let
\[ f(x) = \frac{d}{dx} F_T(x) \]
Obviously \( f \in L^1(\mathbb{R}, \mu) \) since \( F_T(x) \) is bounded. Now for sufficiently small \( \frac{1}{M} \), we have for all sufficiently large \( x \), \( R(x) > 0 \) and
\[ x^\alpha R(x) \left( 1 - \frac{1}{M} \right) \leq 1 - \int_{-\infty}^{x} f_T(x) dx \leq x^\alpha R(x) \left( 1 + \frac{1}{M} \right) \]
So it’s clear that
\[ \frac{d}{dx} (\alpha x R(x)) = \alpha x^{\alpha-1} R(x) + x^\alpha R'(x) < 0 \]
By Gronwall’s Theorem
\[ R(x) < x \int \frac{-e_R(x)}{x^{\alpha-1}} dx \]

Question is: For what \( f, g \) and \( \Phi \) does there exist such an \( X_t(\omega) \) What will its characteristic function be like?
5. Appendix

\[ \mathcal{K} = \left\{ Y : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n : Y \in (\mathcal{G}, \mathcal{U}, d\nu) / (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \right\} \]

The following is the gradient of the log-likelihood function

References


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