Notes in Stochastic Calculus

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Abstract

Brownian Motion and Stochastic Calculus
Going to Infinity :)  

Dates shown are final data of compiling and solutions to textbook problems may contained in lemma or propositions or ...

1 Invariance Properties of (sub/super)martingales w.r.t stopping times

The reason we need to study such invariance properties is that we want to control the process while sampling or monitoring and under certain conditions we can push a finite stopping time/optional time to its infinity while at the same time keeping the sub/super-martingale properties. Also its combination with uniformly integrable sequences gives powerful discretization and decomposition results in stochastic analysis.

Theorem 1 Let \( X = (X_n, \mathcal{F}_n); 1 \leq n \leq N \), be a supermartingale. Then for any two stopping times \( \tau, \sigma \) with respect to \( \mathcal{F}_n \), such that \( P(\tau \leq N) = P(\sigma \leq N) = 1 \),

\[
x_\sigma \geq E(x_\tau | \mathcal{F}_\sigma)
\]

on \( \{\tau \geq \sigma\} \), or, equivalently

\[
x_{\sigma \wedge \tau} \geq E(x_\tau | \mathcal{F}_\sigma)
\]
Proof. First of all we have
\[ E(|x_\tau|) = \sum_{\tau=1}^{N} \int_{\{\tau=n\}} x_n dP \leq \sum_{\tau=1}^{N} |x_n| < \infty \]
Then, take \( \{\sigma = n\} \) for each \( 1 \leq n \leq N \) and consider the set \( \{\sigma = n\} \cap \{\tau \geq n\} \) where \( x_\sigma = x_n \) and by the previous lemma
\[ E\left(\left|1_{\{\sigma = n\}} x_\sigma \right| \mathcal{F}_\sigma\right) = E\left(\left|1_{\{\sigma = n\}} x_n \right| \mathcal{F}_n\right) \text{ P-a.s.} \]
So it’s sufficient to show
\[ x_n \geq E\left(x_\tau \big| \mathcal{F}_n\right) \]
on \( \{\sigma = n\} \cap \{\tau \geq n\} \). To this end, pick any \( A \in \mathcal{F}_n \) and consider
\[
\int_{A \cap \{\sigma = n\} \cap \{\tau \geq n\}} (x_n - x_\tau) dP = \int_{A \cap \{\sigma = n\} \cap \{\tau > n\}} (x_n - x_\tau) dP \\
= \int_{A \cap \{\sigma = n\} \cap \{\tau \geq n+1\}} (x_n - x_\tau) dP \geq \int_{A \cap \{\sigma = n\} \cap \{\tau \geq n+1\}} (x_{n+1} - x_\tau) dP \\
\geq \cdots \geq \int_{A \cap \{\sigma = n\} \cap \{\tau \geq N\}} (x_N - x_\tau) dP = \int_{A \cap \{\sigma = n\} \cap \{\tau = N\}} (x_N - x_N) dP \\
\geq 0
\]
since \( X = (X_n, \mathcal{F}_n) ; 1 \leq n \leq N \), is a supermartingale and the (intersection) set is appropriately measurable.

Finally, from \( P\left(\Omega - \bigcup_{n=1}^{N} \{\sigma = n\}\right) = 0 \), it’s clear that
\[ x_{\tau \wedge \sigma} \geq E\left(x_\tau \big| \mathcal{F}_\sigma\right) \text{ P-a.s. on } \{\tau \geq \sigma\} \]

## 2 Uniformly Integrable Sequences

The reason for exploiting uniformly integrable sequences is that we can enable the passage of limit thought the integral sign so that we can study the continuity properties of the trajectories and other things.

It’s natural to "close" a uniformly integrable sub/super-martingale at its end, to make it "regular" or "closed" and recover the original process via conditioning on the sequence of sigma-algebras. Undoubtedly, this is of paramount importance to the convergence theory with the help of up/down-crossing inequalities.

**Lemma 2** Let \( \{\mathcal{F}_n\}_n=1^\infty \) be an arbitrary sequence of sub-\( \sigma \)-fields of \( \mathcal{F} \) and let \( \xi \in L(\Omega, \mathcal{F}, P) \). Then \( E(\xi|\mathcal{F}_n) \) is uniformly integrable.

Proof. Just let
\[ x_n = E(\xi|\mathcal{F}_n) \]
Then it’s clear that \( E(|x_n|) < \infty \) and
\[ \int_{\{|x_n| > c\}} |x_n| dP \leq \int_{\{|x_n| > c\}} |\xi| dP \]
But
\[ P\{|x_n| > c\} = \int_{\{|x_n| > c\}} dP \leq \frac{1}{c} \int_{\Omega} |x_n| dP \leq \frac{1}{c} \int_{\Omega} |\xi| dP \]
Proposition 5

Under the hypothesis of the previous problem and with additionally

\[ \lim_{c \to \infty} \int_{\{|x_n| > c\}} |x_n| \, dP \leq \lim_{c \to \infty} \int_{\{|x_n| > c\}} |\xi| \, dP \]

\[ \leq \lim_{c \to \infty} \int_{\Omega} |\xi| \, dP \left( \frac{1}{c} \int_{\Omega} |\xi| \, dP \right) = 0 \]

which justifies the assertion. ■

Proposition 3 (Doob) Let \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) be a monotone sequence of sub-\( \sigma \)-fields of \( \mathcal{F} \) and let \( \xi \in L(\Omega, \mathcal{F}, P) \).

Then

\[ \lim_n \mathbb{E}(\xi|\mathcal{F}_n) \to \mathbb{E}(\xi|\lim_n \mathcal{F}_n) \]

both a.s. and in the mean.

Proof. Since \( \{\mathbb{E}(\xi|\mathcal{F}_n) : n \geq 1\} \) is a uniformly integrable martingale, we have that \( \{\mathbb{E}(\xi|\mathcal{F}_n) : \infty \geq n \geq 1\} \) is also a martingale. Suppose \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) is nondecreasing. Take \( A \in \bigcup \mathcal{F}_n \), it’s clear that \( A \in \mathcal{F}_n \) for some \( n \)

\[ E(\xi 1_A) = E(\xi|\mathcal{F}_n) 1_A = E(\mathbb{E}(\xi|\mathcal{F}_n) 1_A) = E(\mathbb{E}(X_\infty|\mathcal{F}_n) 1_A) = E(X_\infty 1_A) \]

where \( X_\infty = \lim E(\xi|\mathcal{F}_n) \). Since all sets \( A \) that satisfy the above identity is a monotone class containing \( \bigcup \mathcal{F}_n \), it’s true that for any \( A \in \bigvee \mathcal{F}_n \), holds the identity \( X_\infty = E(\xi|\mathcal{F}_\infty) \).

In case \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) is non-increasing, \( \{E(\xi|\mathcal{F}_n), \mathcal{F}_n : n \geq 1\} \) is a uniformly integrable backward martingale and for any \( A \in \bigwedge \mathcal{F}_n \subseteq \mathcal{F}_n \) it’s clear that \( E(X_\infty 1_A) = E(E(\xi|\mathcal{F}_n) 1_A) = E(\xi 1_A) \), that is \( X_\infty = E(\xi|\bigwedge \mathcal{F}_n) \) ■

Problem 4 Let \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) be a decreasing sequence of sub-\( \sigma \)-fields of \( \mathcal{F} \), i.e., \( \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \mathcal{F} \) and let \( \{X_n, \mathcal{F}_n\} \) be a backward submartingale, i.e., \( E|X_n| < \infty \) and \( E(X_n|\mathcal{F}_{n+1}) \geq X_{n+1} \) a.s. \( P \) for all \( n \geq 1 \). Then

\[ l = \lim E(X_n) > -\infty \] implies \( \{X_n\}_{n=1}^{\infty} \) is uniformly integrable.

Instead we prove a strong version

Proposition 5 Under the hypothesis of the previous problem and with additionally

\[ EX_+^1 < \infty \]

then \( X_\infty := \lim_n X_n \) is well define and \( E(X_\infty^+) < \infty \). Moreover, with the previous assumption, the following are equivalent

a) \( \{X_n, \mathcal{F}_n; 1 \leq n < \infty\} \) is uniformly integrable

b) \( \{X_n, \mathcal{F}_n; 1 \leq n < \infty\} \) is fundamental is \( L^1 \)

c) \( X_\infty = \lim_n X_n \) a.s. exists and \( \{X_n, \mathcal{F}_n; 1 \leq n \leq \infty\} \) is a backward submartingale, where \( \mathcal{F}_\infty = \bigcap_{n=1}^{\infty} \mathcal{F}_n \)

d) \( EX_+^1 < \infty, \liminf E(X_n) > -\infty \)

Proof. Let \( \beta(I;n) \) be the number of up-crossings of the interval \((a, b)\). Then from the fact that

\[ \{X_k, \mathcal{F}_k; n \geq k \geq 1\} \]

is a submartingale, it’s clear that

\[ E(\beta(I;n)) \leq \frac{E(X_1 - a)^+}{b - a} < \infty \]
and so
\[ E(\beta(I;n)) = E \left( \lim_{n \to \infty} \beta(I;n) \right) < \infty \]
and
\[ \lim_{n \to \infty} \beta(I;n) < \infty \text{ P-a.s.} \]

Let \( a, b \) range over \( \mathbb{Q} \), the above inequality implies
\[ X_\infty := \lim_n X_n \]
is well-defined. By Fatou’s lemma
\[ E(X^\infty_\infty) \leq \liminf E(X^+_n) \leq \liminf E(X^-_1) < \infty \]
since \( \{X^+_n, \mathcal{F}_k; n \geq k \geq 1\} \) is also a backward submartingale.

(a) = (b): By the previous part, \( X_\infty := \lim_n X_n \) is the almost sure limit needed. Thus from Note 3, for the sequence \( \{X_n, \mathcal{F}_n; 1 \leq n \leq \infty\} \), the equivalence is established.

(b) to (c): It suffices to show that
\[ \sup_{fj} [\sup_{X^n} (\mathbb{E} X^n)] \leq \sup_{fj} \lim_{n \to \infty} \mathbb{E} X_n \]
and that\( \exists \), \( \sup \mathbb{E} X_n \) is the almost sure limit needed. Thus from Note 3, for\( n = 1 \) is finite. So for any \( \varepsilon > 0 \), there is some \( n_0 \) such that
\[ 0 \leq \mathbb{E}(X_{n_0}) - \mathbb{E}(X_n) < \varepsilon \]
for all \( n \geq n_0 \). Also, from the backward submartingale property, we have
\[ -\mathbb{E}(X_n 1_{\{X_n < -c\}}) = -\mathbb{E}(X_n) + \mathbb{E}(X_n 1_{\{X_n \geq -c\}}) \]
\[ \leq -\mathbb{E}(X_n) + \mathbb{E}(X_{n_0} 1_{\{X_n \geq -c\}}) \]
\[ = \mathbb{E}(X_{n_0}) - \mathbb{E}(X_n) - \mathbb{E}(X_{n_0} 1_{\{X_n < -c\}}) \]
\[ \leq \varepsilon - \mathbb{E}(X_{n_0} 1_{\{X_n < -c\}}) \]

Thus
\[ \sup_n \mathbb{E}(1_{\{|X_n| > c\}}) \leq \sup_n \mathbb{E}(X_n 1_{\{X_n > c\}}) + \sup_n \left[ -\mathbb{E}(X_n 1_{\{X_n < -c\}}) \right] \]
\[ \leq \sup_n \mathbb{E}(X_n 1_{\{X_n > c\}}) + \sup_{n > n_0} \left[ -\mathbb{E}(X_n 1_{\{X_n < -c\}}) \right] + \sup_{n \leq n_0} \left[ -\mathbb{E}(X_n 1_{\{X_n < -c\}}) \right] \]
\[ \leq \sup_n \mathbb{E}(1_{\{|X_n| > c\}}) + \sup_{1 \leq n \leq n_0} \left[ \mathbb{E}(1_{\{|X_n| > c\}}) \right] + \varepsilon + \sup_{n > n_0} \mathbb{E}(X_{n_0} 1_{\{X_n < -c\}}) \]
But
\[ \sup_n E \left( 1_{\{|X_n| > c\}} \right) \leq \frac{1}{c} \sup E (|X_n|) \to 0 \]

Consequently
\[ \lim_{c \to +\infty} \sup_n E \left( |X_n| 1_{\{|X_n| > c\}} \right) = 0 \]
which completes the proof. \(\blacksquare\)

**Proposition 6** For a submartingale \( \{X_n, \mathcal{F}_n; 1 \leq n\} \). The following are equivalent

a) There is some \( X_\infty \in L(\Omega, \mathcal{F}, P) \) such that \( E(X_\infty | \mathcal{F}_n) \geq X_n \)
b) \( \{X_n^+: n \geq 1\} \) is uniformly integrable
c) \( \{X_n^+: n \geq 1\} \) is foundamental in \( L^1 \)
d) \( X_\infty := \lim X_n \) exists \( P \)-a.s. and \( X_\infty^+ \in L^1 \) and \( \{X_n, \mathcal{F}_n; 1 \leq n, X_\infty, \mathcal{F} = \bigvee \mathcal{F}_n\} \) is a submartingale

**Proof.** Use the relation
\[ E \left( X_\infty^+ | \mathcal{F}_n \right) \geq E^+ \left( X_\infty | \mathcal{F}_n \right) \geq X_n^+ \]
We only prove (2), (3) to (4). Letting
\[ X_n^{(k)} = X_n \lor (-k) \]
induces a submartingale \( \{X_n^{(k)}, \mathcal{F}_n; 1 \leq n\} \). So from \( E \left( X_n^{(k)} 1_A \right) \geq E \left( X_n 1_A \right) \) for all \( m \geq n \geq 1 \) and \( A \in \mathcal{F}_n \). Notice that for all \( m \geq 1, -k \leq X_m^{(k)} \leq X^+_m \) and \( \{X_n^{(k)}\} \) is uniformly integrable, we have \( \lim X_n^{(k)} = X_{\infty}^{(k)} := X_\infty \lor (-k) \) and
\[ E \left( X_n^{(k)} 1_A \right) = \lim E \left( X_m^{(k)} 1_A \right) \geq E \left( X_n^{(k)} 1_A \right) \]
that is,
\[ E \left( X_n^{(k)} | \mathcal{F}_n \right) \geq X_n^{(k)} \quad (1) \]
Finally from that fac that \( X_\infty^+ - X_n^{(k)} \uparrow X_\infty^+ - X_\infty \), it’s clear that for each \( n \geq 1 \)
\[ \lim_k E \left( X_\infty^+ - X_n^{(k)} | \mathcal{F}_n \right) = \lim_k E \left( X_\infty^+ - X^{(k)} | \mathcal{F}_n \right) \]
\[ = E \left( X_\infty^+ - X_\infty | \mathcal{F}_n \right) = E \left( X_\infty^+ | \mathcal{F}_n \right) - E \left( X_\infty | \mathcal{F}_n \right) \]
and
\[ \lim_k E \left( X_n^{(k)} | \mathcal{F}_n \right) = E \left( X_\infty | \mathcal{F}_n \right) \]
Thus together with (1), we have
\[ E \left( X_\infty | \mathcal{F}_n \right) \geq X_n \]
by letting \( k \to \infty. \blacksquare \)

**Corollary 7** For a submartingale \( \{X_n, \mathcal{F}_n; 1 \leq n\} \). The following are equivalent

a) \( \{X_n; n \geq 1\} \) is uniformly integrable
b) \( X_\infty := \lim X_n \) exists \( P \)-a.s. and \( X_\infty^+ \in L^1 \) and a potential \( \{\pi_n, \mathcal{F}_n\} \) such that
\[ X_n = E \left( X_\infty | \mathcal{F}_n \right) - \pi_n \]

**Proof.** Let \( \pi_n = E \left( X_\infty | \mathcal{F}_n \right) - X_n \) \(\blacksquare\)
Remark 8 A potential is a non-negative supermartingale such that \( E(\pi_n) \to 0 \). It’s easily seen that a potential is uniformly integrable.

Theorem 9 For a uniformly integrable submartingale \( \{X_n, \mathcal{F}_n; 1 \leq n\} \) with \( X_\infty = \lim X_n \) a.s. and for any stopping time \( \tau, \sigma \) it holds that
\[
X_{\tau \wedge \sigma} \geq E(X_\sigma | \mathcal{F}_\tau)
\]

Proof. By the decomposition
\[
\pi_n = E(X_\infty | \mathcal{F}_n) - X_n
\]
and \( \pi_\infty = 0 \) we have a potential \( \{\pi_n, \mathcal{F}_n; 1 \leq n \leq \infty\} \). So for every \( A \in \mathcal{F}_\tau \)
\[
E(\pi_{\sigma \wedge \tau}) A = E(\pi_{\sigma \wedge \tau}) A = \lim_k E(\pi_{\sigma \wedge k}) A \leq \lim inf E(\pi_{\sigma \wedge k}) A
\]
\[
\leq \lim inf E(\pi_{\tau \wedge \sigma} A) = E(\pi_{\tau \wedge \sigma} A)
\]
The second inequality is obtained by using bounded stoppint time \( k \wedge \sigma \) on the submartingale \( \{-\pi_n, \mathcal{F}_n; 1 \leq n \leq \infty\} \). Thus
\[
\pi_{\tau \wedge \sigma} \geq E(\pi_\sigma | \mathcal{F}_\tau) \text{ a.s. } \{\sigma \geq \tau\}
\]
Let \( X_\infty = \lim X_n \) a.s. and define \( Y_n = E(X_\infty | \mathcal{F}_n) \) then \( \{Y_n, \mathcal{F}_n; 1 \leq n\} \) is a uniformly integrable martingale which induces \( Y_\infty = \lim Y_n \) a.s. and \( E(Y_\infty | \mathcal{F}_\tau) = Y_\tau \text{, and} \)
\[
X_\infty = \lim X_n = \lim Y_n - \lim \pi_n = Y_\infty - \pi_\infty = Y_\infty \text{ a.s.}
\]
So
\[
E(X_\sigma | \mathcal{F}_\tau) = E(Y_\sigma - \pi_\sigma | \mathcal{F}_\tau) \geq (Y_\tau - \pi_\tau) = X_n
\]
also by \( X_\infty = Y_\infty \) a.s. ■

Remark 10 For martingales, just decompose \( \Omega \) into \( \bigcup_{n=1}^{\infty} \{\sigma = n\} \bigcup \{\sigma = \infty\} \) and use the fact that \( \{X_n, \mathcal{F}_n; 1 \leq n \leq \infty\} \) is a martingale

3 Doob and Riesz Decomposition for Discrete Time

It’s so natural that when we consider the difference between a sequence and its corresponding shifted conditioned sequence, we’ll get a decomposition, which is the classical Doob decomposition.

Moreover, with an \( L^1 \)-bounded martingale, we can even push the decomposition to its infinity index to get a potential by a small change, which is what Riesz decomposition says.

Lemma 11 Extended Dominated Convergence. Let \( \{f_k\}, \{g_k\} \) be two sequences of functions such that \(|f_k| \leq |g_k|, f_k \to f, g_k \to g, \text{ and } \int g_k = \int g < \infty. \) Then \( \int f_k = \int f \)

Proof. The basic idea is to use Fatou’s lemma. From the hypothesis it’s clear that
\[
g_k + f_k, g_k - f_k \geq 0
\]
So
\[
\int g + \int f = \int \lim inf g_k + \int \lim inf f_k \leq \int \lim inf (g_k + f_k)
\]
\[
\leq \lim inf \int (g_k + f_k) = \int g + \lim inf \int f_k
\]
\[
\int g - \int f = \int \lim \inf g_k - \int \lim \sup f_k \leq \int \lim \inf (g_k - f_k)
\]
\[
= \int g - \lim \sup \int f_k
\]

(since \(g, f\) are convergent.) Consequently

\[
\lim \inf \int f_k \geq \int f \geq \lim \sup \int f_k
\]

Theorem 12 (Doob) For an integrable adapted sequence \(\{X_n, \mathcal{F}_n : n \geq 1\}\). There is an integrable martingale \(\{X_n, \mathcal{F}_n : n \geq 1\}\) and an integrable adapted sequence \(\{\eta_n, \mathcal{F}_{n-1} : n \geq 1\}\) with \(\eta_0 = 0 \text{ a.s.}\) such that

\[
X_n = M_n + \eta_n
\]

and this decomposition is stochastically unique.

Proof. Let \(M_1 = X_1, \eta_0 = 0\) and

\[
\eta_n = E (X_n | \mathcal{F}_{n-1}) - X_{n-1}
\]

and

\[
M_n = X_n - \eta_n
\]

which is the required decomposition.

Also,

\[
E (X_n | \mathcal{F}_{n-1}) = E (M_n + \eta_n | \mathcal{F}_{n-1})
\]
\[
= E (M_n | \mathcal{F}_{n-1}) + \eta_n
\]
\[
= M_{n-1} + \eta_{n-1} + (\eta_n - \eta_{n-1})
\]
\[
= X_{n-1} + (\eta_n - \eta_{n-1})
\]

Thus is \(\{X_n, \mathcal{F}_n : n \geq 1\}\) is a submartingale if and only if \(\eta_n\) is non-decreasing a.s. 

Theorem 13 (Riesz) For a submartingale \(\{X_n, \mathcal{F}_n : n \geq 1\}\) with

\[
\sup_{n \geq 1} E (|X_n|) < \infty
\]

there is an integrable martingale \(\{M_n, \mathcal{F}_n : n \geq 1\}\) and a potential \(\{\pi_n, \mathcal{F}_n : n \geq 1\}\) such that

\[
X_n = M_n - \pi_n
\]

and such a decomposition is stochastically unique.

Proof. As can be seen, there is only a sign change in the decomposition from Doob’s decomposition. The key is to use

\[
E (X_{n+p+1} | \mathcal{F}_n) = E (E (X_{n+p+1} | \mathcal{F}_{n+p}) | \mathcal{F}_n)
\]
\[
\geq E (X_{n+p} | \mathcal{F}_n)
\]

to define

\[
M_n := \lim_{p} E (X_{n+p} | \mathcal{F}_n)
\]
Then by Fatou’s lemma and Jessen’s inequality

\[ E(|M_n|) = E \left( \lim_p E(X_{n+p} | \mathcal{F}_n) \right) \leq \liminf_p E(|E(X_{n+p} | \mathcal{F}_n)|) \leq \lim \inf_{n \geq 1} E(|X_n|) \leq \sup_{n \geq 1} E(|X_n|) < \infty \]

which is \( \{M_n\} \) is an \( L^1 \) sequence.

From \( X_n \leq E(X_{n+p} | \mathcal{F}_n) \leq M_n \) a.s. and by applying Dominated convergence theorem on the sequence \( \{E(X_{n+p} | \mathcal{F}_n) : p \geq 1\} \) we have

\[ E(M_{n+1} | \mathcal{F}_n) = E \left[ \lim_p E((X_{n+p+1} | \mathcal{F}_{n+1}) | \mathcal{F}_n) \right] = \lim_p E(E((X_{n+p+1} | \mathcal{F}_{n+1}) | \mathcal{F}_n)) = \lim_p E(X_{n+p+1} | \mathcal{F}_n) = \lim_p E(X_{n+p} | \mathcal{F}_n) = M_n \]

and so \( \{M_n, \mathcal{F}_n\} \) is a martingale.

Now let \( \pi_n = M_n - X_n \)

Then \( \{\pi_n, \mathcal{F}_n\} \) is a supermartingale such that

\[ \pi_n \geq E(\pi_{n+p} | \mathcal{F}_n) = E(M_{n+p} | \mathcal{F}_n) - E(X_{n+p} | \mathcal{F}_n) = M_n - E(X_{n+p} | \mathcal{F}_n) \geq 0 \]

Combining the fact

\[ E(\pi_n) \leq E(|M_n|) + E(|X_n|) < \infty \]

and by using the dominated convergence theorem on \( \{E(\pi_{n+p} | \mathcal{F}_n) : p \geq 1\} \) it’s clear that

\[ \lim_p E(\pi_{n+p}) = \lim_p E(E(\pi_{n+p} | \mathcal{F}_n)) = E \left( \lim_p E(\pi_{n+p} | \mathcal{F}_n) \right) = E \left( M_n - \lim_p E(X_{n+p} | \mathcal{F}_n) \right) = 0 \]

Thus \( \{\pi_n, \mathcal{F}_n\} \) is a potential.

Finally, if \( \{M'_n, \pi'_n\} \) is another such decomposition. Then

\[ M_n = \lim_p E(X_{n+p} | \mathcal{F}_n) = \lim_p E(M'_{n+p} - \pi'_{n+p} | \mathcal{F}_n) = M'_n - \lim_p E(\pi'_{n+p} | \mathcal{F}_n) \]

But by Fatou’s lemma we have

\[ 0 \leq E \left( \lim_p E(\pi'_{n+p} | \mathcal{F}_n) \right) \leq \lim_p E(\pi'_{n+p}) = 0 \]

Thus \( \lim_p E(\pi'_{n+p} | \mathcal{F}_n) = 0 \) a.s. □
4 Continuous Time Martingales

With this solid understanding of discreted time sub/supermartingales, it’s time to embark on its continuous version under the assumption on the regularity of their sample paths.

The key role right/left-continuity plays is to provide some seperability in the trajectories

**Proposition 14** For any $\lambda \in \mathbb{R}$ and a right/left continuous process $\{X_t, \mathcal{F}_t\}$

$$
\bigcup_{\sigma \leq t \leq \tau} \{X_t \geq \lambda\} = \left\{ \sup_{\sigma \leq t \leq \tau} X_t \geq \lambda \right\} = \bigcup_{\sigma \leq r \leq \tau, r \in \mathbb{Q}} \left\{ \sup_{\sigma \leq t \leq \tau} X_t \geq \lambda \right\}
$$

and

$$U_t (\alpha, \beta; X(\omega)) = \sup \{ U_F (\alpha, \beta; X(\omega)) : F \subseteq I, |F| < \infty \}
= \sup \{ U_F (\alpha, \beta; X(\omega)) : F \subseteq I \cap \mathbb{Q}, |F| < \infty \} = c$$

Thus no measurablity concerns and the limit sup is equivalent to that of a countable sequence.

**Proof.** Obviously $\bigcup_{\sigma \leq t \leq \tau} \{X_t \geq \lambda\} \subseteq \bigcup_{\sigma \leq t \leq \tau} \{X_t \geq \lambda\}$. For the inverse inclusion, we don’t have to treat the the end points. Suppose $X_s > \lambda$ for $s \in [\sigma, \tau)$, then by right-continuity there is some $r \in \mathbb{Q}$ such that $X_r > \lambda$. If $\sup_{\sigma \leq t \leq \tau} X_t = \lambda$, then there is a sequence $\{t_n \in [\sigma, \tau]\}$ such that $\lim n X_{t_n} = \lambda$. By right-continuity, we can choose $r_n \in \mathbb{Q}$ such that $|X_{r_n} - X_{t_n}| < \varepsilon/2^n$ for any preassigned $\varepsilon > 0$. Thus $|X_{r_n} - \lambda| \leq |X_{t_n} - \lambda| + |X_{r_n} - X_{t_n}| < \varepsilon$, i.e., $\lim X_{r_n} = \lambda$. Consequently, the inverse inclusion holds.

Simily reasoning apply for left-continuity.

Same trick can be used to the second claim. □

**Problem 15** Let $\{X_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a right-continuous, non-negative supermartingale; then $X_\infty (\omega) = \lim_{t \to \infty} X_t (\omega)$ exists a.s. for $\omega \in \Omega$ and $\{X_t, \mathcal{F}_t : 0 \leq t \leq \infty\}$ is a supermartingale.

**Proof.** For each $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{Q}$ let $U_{[0,n]} (\alpha, \beta; X(\omega))$ be the number of up-crossings of the process for $0 \leq t \leq n$. Then by crossing inequalities

$$\sup_{n \in \mathbb{N}} \max \left\{ E \left[ U_{[0,n]} (\alpha, \beta; X(\omega)) \right] , E \left[ D_{[0,n]} (\alpha, \beta; X(\omega)) \right] \right\}$$

$$\leq \frac{E (X_n - \alpha)^+}{\beta - \alpha} \leq \frac{E (X_n^+) + |\alpha|}{\beta - \alpha} \leq \frac{E (X_1) + |\alpha|}{\beta - \alpha}$$

we have

$$A = \bigcup_{\alpha < \beta, \alpha, \beta \in \mathbb{Q}} \{ \omega \in \Omega : U_{[0,\infty]} (\alpha, \beta; X(\omega)) = \infty \}$$

is a $P$-null set. Consequently on $\Omega \setminus A$

$$\lim_{t \to \infty} X_t = X_\infty$$

Moreover

$$E |X_\infty| \leq \lim_{t \to \infty} \inf E (|X_t|) \leq E (|X_0|) < \infty$$

To show that

$$E (X_\infty | \mathcal{F}_t) \leq X_t$$
it suffices to show \( \{X_t\} \) is uniformly integrable, which allows
\[
\mathbb{E}(X_{\infty} | \mathcal{F}_t) = \lim_{s \to \infty} \mathbb{E}(X_s | \mathcal{F}_t) \leq X_t
\]
for any \( s \geq t \). But
\[
\int_{\Omega} 1_{\{|X_t| > C\}} dP \leq \frac{1}{C} \int_{\{|X_t| > C\}} |X_t| dP \leq \frac{1}{C} \int_{\Omega} |X_0| dP \to 0
\]
as \( C \to \infty \) for all \( t \geq 0 \) implies
\[
\int_{\{|X_t| > C\}} |X_t| dP = \int_{\Omega} 1_{\{|X_t| > C\}} |X_t| dP \leq \left( \frac{1}{C} \int_{\Omega} |X_0| dP \right) \int_{\Omega} |X_t| dP \leq \left( \frac{E(|X_0|)^2}{C} \right) \to 0
\]
as \( C \to \infty \). Thus (2) is justified. 

**Problem 16** Suppose \( \{\mathcal{F}_t\} \) satisfies the usual conditions. Then every right-continuous, uniformly integrable supermartingale \( \{X_t, \mathcal{F}_t : 0 \leq t < \infty\} \) admits Riesz decomposition
\[
X_t = M_t + Z_t
\]
a.s. as the sum of a right-continuous integrable martingale \( \{M_t, \mathcal{F}_t : 0 \leq t < \infty\} \) and a potential \( \{Z_t, \mathcal{F}_t : 0 \leq t < \infty\} \)

**Proof.** Since each \( \mathcal{F}_t \) is augmented with \( P \)-null sets and is right-continuous, there is no worry about the escape of \( P \)-null sets from limit process. Let \( X_{\infty} := \lim X_t \) a.s.

**Method 1:** For \( 0 \leq u \leq s \),
\[
\mathbb{E}(X_{t+s} | \mathcal{F}_t) = \mathbb{E}(\mathbb{E}(X_{t+s} | \mathcal{F}_{t+u}) | \mathcal{F}_t) \leq \mathbb{E}(X_{t+u} | \mathcal{F}_t)
\]
implies
\[
M_t := \lim_{s \to \infty} \mathbb{E}(X_{t+s} | \mathcal{F}_t) = \mathbb{E}(X_{\infty} | \mathcal{F}_t) \tag{3}
\]
exists a.s. and
\[
\mathbb{E}(\{|M_t|\}) = \mathbb{E}\left( \lim_{s \to \infty} |\mathbb{E}(X_{t+s} | \mathcal{F}_t)| \right) \leq \liminf \mathbb{E}(\{|X_{t+s}| | \mathcal{F}_t\}) = \liminf \mathbb{E}(\{|X_{t+s}|\}) < \infty
\]
by uniformly integrability. Moreover from \( M_t \leq \mathbb{E}(X_{t+s} | \mathcal{F}_t) \leq X_t \), the dominate convergence theory (DCT) implies
\[
\mathbb{E}(M_{t+v} | \mathcal{F}_t) = \mathbb{E}\left( \lim_{s \to \infty} \mathbb{E}(X_{t+v+s} | \mathcal{F}_{t+v}) | \mathcal{F}_t \right) = \lim_{s \to \infty} \mathbb{E}(\mathbb{E}(X_{t+v+s} | \mathcal{F}_{t+v}) | \mathcal{F}_t) = \lim_{s \to \infty} \mathbb{E}(X_{t+v+s} | \mathcal{F}_t) = M_t
\]
which means \( \{M_t, \mathcal{F}_t\} \) is a martingale.

To show that \( \{M_t\} \) is right continuous, we have to show
\[
\lim_{t_n \downarrow t} M_{t_n} = \lim_{t_n \downarrow t} \lim_{s \to \infty} \mathbb{E}(X_{t_n+s} | \mathcal{F}_{t_n}) = \lim_{s \to \infty} \lim_{t_n \downarrow t} \mathbb{E}(X_{t_n+s} | \mathcal{F}_{t_n}) = \lim_{s \to \infty} \mathbb{E}(X_{t_n+s} | \mathcal{F}_{t_n}) = M_t
\]
Now let 
\[ Z_t = X_t - M_t \]
Then \( \{Z_t, \mathcal{F}_t\} \) is a supermartingale such that
\[
Z_t \mathop{\geq}_{\mathbb{F}} E(Z_{t+s} | \mathcal{F}_t) = E(X_{t+s} | \mathcal{F}_t) - E(M_{t+s} | \mathcal{F}_t) = E(X_{t+s} | \mathcal{F}_t) - M_t \geq 0
\]
and such that
\[
\lim_{s \to \infty} E(Z_{t+s}) = \lim_{s \to \infty} E(E(Z_{t+s}) | \mathcal{F}_t) = E(M_{t+s} - \lim_{s \to \infty} E(X(Z_{t+s}) | \mathcal{F}_t)) = 0
\]
a.s. Consequently \( \{Z_t, \mathcal{F}_t\} \) is a potential.

Stochastic uniqueness is easily justified.

Method 2: Take \( Q^+ = Q \cap [0, \infty) \) and for all \( r_n \in Q^+ \), we use the discrete version Rieze decomposition theorem. To get for each \( r_n \in Q^+ \), there exists martingale \( \{M_{r_n}, \mathcal{F}_{r_n}\} \) and potential \( \{Z_{r_n}, \mathcal{F}_{r_n}\} \) such that
\[
X_{r_n} = M_{r_n} - Z_{r_n}
\]
Now for each \( t \geq 0 \), take \( r^{(t)}_k \in Q^+ \) such that \( r^{(t)}_k \downarrow t \). By uniform integrability in the sense of backward supermartingale, and the right-continuity of process and the augmented sigma-algebra, we have
\[
X_{r_k^{(t)}} \to X_t, M_{r_k^{(t)}} \to M_t, Z_{r_k^{(t)}} \to Z_t
\]
both in \( L^1 \) and a.s., which is
\[ Z_t = X_t - M_t \]

Method 3: By the uniform integrability of \( \{X_t\} \) it’s clear that
\[
X_\infty := \lim_{t \to \infty} X_t
\]
holds in \( L^1 \) and a.s. and \( \{X_t, \mathcal{F}_t : 0 \leq t \leq \infty\} \) is a supermartingale.

Now define
\[
M_t = E(X_\infty | \mathcal{F}_t) \tag{4}
\]
then \( \{M_t, \mathcal{F}_t\} \) is also a uniformly integrable martingale such that
\[
M_t = E(X_\infty | \mathcal{F}_t) \leq X_t
\]
Further, by letting
\[ Z_t = X_t - M_t \]
we have the needed potential. ■

Remark 17 In the first method, no limit is taken w.r.t. the right-continuous filtration. Is something missing? Notice that (3) and (4) are essentially the same thing.

Problem 18 The following three conditions are equivalent for a non-negative right-continuous submartingale \( \{X_t, \mathcal{F}_t : 0 \leq t < \infty\} \):

a) it is a uniformly integrable family
b) it converges in $L^1$

c) it converges a.s. to an integrable $X_\infty$ such that $\{X_t, \mathcal{F}_t : 0 \leq t \leq \infty \}$ is a submartingale.

**Proof.** a) to b). For any $\varepsilon > 0$. Take $C > 0$ such that

$$\sup \left\{ \int_{\{X_t > C\}} X_t dP \right\} < \varepsilon$$

Then

$$\sup \left\{ \int_{\Omega} X_t dP \right\} \leq \sup \left\{ \int_{\{X_t \leq C\}} X_t dP \right\} + \sup \left\{ \int_{\{X_t > C\}} X_t dP \right\} \leq C + \varepsilon$$

Thus

$$A = \bigcup_{\alpha, \beta \in \mathbb{Q}, \alpha < \beta} \{ \omega \in \Omega : U_{[0, \infty)} (\alpha, \beta; X(\omega)) = \infty \}$$

is a $P$-null set ($U_{[0, \infty)} = \lim_{n \to \infty} U_{[0, n]}$) and

$$\lim_{t \to \infty} X_t := X_{\infty}$$

exists a.s. and in $L^1$. Moreover

$$\int X_{\infty} dP \leq \liminf_{t \to \infty} \int X_t dP < \infty$$

and

$$\left| \int X_t dP - \int X_{\infty} dP \right| \leq \int_{\{|X_t - X_{\infty}| \geq \varepsilon\}} (X_t - X_{\infty}) dP + \int_{\{|X_t - X_{\infty}| < \varepsilon\}} (X_t - X_{\infty}) dP \leq \varepsilon + \int_{\{|X_t - X_{\infty}| \geq \varepsilon\}} |X_t| dP + \int_{\{|X_t - X_{\infty}| \geq \varepsilon\}} |X_{\infty}| dP \leq \varepsilon + \varepsilon_1 + \int_{\{|X_t - X_{\infty}| \geq \varepsilon_2\} \cap \{ \max\{X_t, X_{\infty}\} \leq C\}} |X_t| dP + \int_{\{\max\{X_t, X_{\infty}\} > C\} \cap \{|X_t - X_{\infty}| \geq \varepsilon\}} |X_t| dP \leq \varepsilon + \varepsilon_1 + \varepsilon_2 + C \int_{\{|X_t - X_{\infty}| \geq \varepsilon\}} dP$$

which justifies b).

Actually a) and b) are equivalent for submartingales.

b) to c) Given that

$$\lim_{t \to \infty} \int X_t := K$$

in $L^1$ it’s clear that $\sup \left\{ \int_{\Omega} X_t dP \right\} < \infty$ and consequently $\lim_{t \to \infty} X_t := X_{\infty}$ a.s. with $0 \leq \int X_t dP \leq \int X_{\infty} dP < \infty$ for any $t \geq 0$. For any $A \in \mathcal{F}_t$, use DCT on $X_t \geq E(X_t|\mathcal{F}_t)$, we obtain

$$X_t \leq E(X_{\infty}|\mathcal{F}_t)$$

as desired.

c) to a). Given that $\lim_{t \to \infty} X_t := X_{\infty}$ a.s. and $X_t \leq E(X_{\infty}|\mathcal{F}_t)$, we have

$$\sup \left\{ \int X_t dP \right\} \leq \int X_{\infty} dP$$
and

\[
\int_{\{X_t > c\}} X_t dP \leq \int_{\{X_t > c\}} X_{\infty} dP \\
= \int_{\{X_t > c\} \cap \{X_{\infty} > b\}} X_{\infty} dP + \int_{\{X_t > c\} \cap \{X_{\infty} \leq b\}} X_{\infty} dP \\
\leq \int_{\{X_{\infty} > b\}} X_{\infty} dP + b \int_{\{X_t > c\}} dP \\
\leq \int_{\{X_{\infty} > b\}} X_{\infty} dP + \frac{b}{c} \int X_t dP
\]

(since \(\{X_t > c\} \in \mathcal{F}_t\)) Pick \(b\) sufficiently large and let \(c \to \infty\), it’s clear that

\[
\lim c \sup \left\{ \int_{\{X_t > c\}} X_t dP \right\} = 0
\]

establishing uniform integrability. ■

**Remark 19** Non-negativity is use in c) to a) to preserve order :) Actually this submartingale is closed such that

\[E(X_{\infty}|\mathcal{F}_t) \geq X_t\]

Suppose \(X_t\) converges in \(L^1\) to some \(Y\) then \(\lim X_t\) exists a.s. and

\[E|X_t - Y| \to 0\]

implies \(Y = X_{\infty}\) a.s.

**Problem 20** The following four conditions are equivalent for a right-continuous martingale \(\{X_t, \mathcal{F}_t: 0 \leq t < \infty\}\):

a) b) c) as in the above problem with "submartingale" changed into "martingale".

d) There is an integrable \(Y\) such that \(X_t = E(Y|\mathcal{F}_t)\) a.s. for all \(t\) and \(Y = X_{\infty}\) a.s

**Proof.** a), b), c) are easy since for a martingale inequalities becomes equalities. Let’s show b) or c) to d).

Let \(X_{\infty} = \lim X_t\) and define for each \(n\)

\[X_t^{(n)} = X_t \vee (-n)\]

Then \(\{X_t \vee (-n), \mathcal{F}_t: 0 \leq t < \infty\}\) is a submartingale such that for each \(A \in \mathcal{F}_t\)

\[\int X_t^{(n)} 1_A \geq \int X_s^{(n)} 1_A\]

for \(t \geq s\). From \(-n \leq X_t^{(n)} \leq X_t^+\) and uniformly intergrability of \(\{X_t^+\}\), it’s clear that

\[\lim_t X_t^{(n)} = X_{\infty}^{(n)} := X_{\infty} \vee (-n)\]

a.s and (by DCT)

\[\int X_{\infty}^{(n)} 1_A \geq \int X_s^{(n)} 1_A\]

i.e.,

\[E(X_{\infty}^{(n)}|\mathcal{F}_s) \geq X_s^{(n)}\] (5)
Notice that \( X_\infty^+ - X_\infty^{(n)} \uparrow X_\infty^+ - X_\infty \) as \( n \to \infty \). Then
\[
\lim_n \left[ E \left( X_\infty^+ | \mathcal{F}_s \right) - E \left( X_\infty^{(n)} | \mathcal{F}_s \right) \right] = \lim_n E \left( X_\infty^+ - X_\infty^{(n)} | \mathcal{F}_s \right) = E \left( X_\infty^+ - X_\infty | \mathcal{F}_s \right) = E \left( X_\infty^+ | \mathcal{F}_s \right) - E \left( X_\infty | \mathcal{F}_s \right)
\]
as. and consequently
\[
\lim_n E \left( X_\infty^{(n)} | \mathcal{F}_s \right) = E \left( X_\infty | \mathcal{F}_s \right)
\]
Combining the limit of (5) we have
\[
\lim_n E \left( X_\infty^{(n)} | \mathcal{F}_s \right) = E \left( X_\infty | \mathcal{F}_s \right) \geq X_s
\]
which completes the proof. Finally, it must be true that \( Y = X_\infty \) since \( E \left( Y | \mathcal{F}_s \right) = X_s = E \left( X_\infty | \mathcal{F}_s \right) \) for all \( s \). (or by \( E \left( |X_t - Y| \right) = 0 = E \left( |X_t - X_\infty| \right) \))

Now that the submartingal is majorised by on \( X_\infty \), it must be uniformly integrable, wich give a), b), c)

**Problem 21** Establish the optional sampling theorem for a right continuous submartingale \( \{X_t, \mathcal{F}_t : 0 \leq t < \infty \} \) and optional times \( S \leq T \) under either of the following tow conditions:

a) \( T \) is a bounded optional time \( (T \leq a \ for \ some \ a) \)

b) there exists an integrable \( Y \) such that \( X_t \leq E \left( Y | \mathcal{F}_t \right) \) a.s. for every \( t \geq 0 \)

**Proof.** a) Without loss of generality, let \( a = 1 \) and suppose \( 1 = \sup \{T(\omega) : \omega \in \Omega \} \). Define
\[
A_{n,k} = \left\{ \omega \in \Omega : \frac{(k-1)}{2^n} \leq T(\omega) < \frac{k}{2^n} \right\}
\]
and
\[
B_{n,k} = \left\{ \omega \in \Omega : \frac{(k-1)}{2^n} \leq S(\omega) < \frac{k}{2^n} \right\}
\]
for each \( n \) and \( 1 \leq k \leq 2^n + 1 \). With \( S \leq T \leq 1 \) we define
\[
T_n(\omega) = \sum_{k=1}^{2^n+1} \frac{k}{2^n} 1_{A_{n,k}}(\omega)
\]
and
\[
S_n(\omega) = \sum_{k=1}^{2^n+1} \frac{k}{2^n} 1_{B_{n,k}}(\omega)
\]
Then \( T_n \downarrow T, S_n \downarrow S \), and each \( T_n, S_n \) is a stopping time w.r.t to the filtration, and \( S_n \leq T_n \). So optional sampling for discrete time
\[
\int X_{S_n} 1_A \leq \int X_{T_n} 1_A
\]
for all \( A \in \mathcal{F}_{S_n} \) and for every \( A \in \mathcal{F}_{S^+} = \bigcap_{n=1}^{\infty} \mathcal{F}_{S_n} \) and for \( A \in \mathcal{F}_S \) (since \( \mathcal{F}_S \subseteq \mathcal{F}_{S_n} \) by \( S \leq S_n \)).
Now obviously \( \{X_{S_n}, \mathcal{F}_{S_n} : n \geq 1\} \) is a backward submartingale

\[
E (X_{S_n} \mid \mathcal{F}_{S_{n+1}}) \geq X_{S_{n+1}}
\]

with \( E (X_{S_n}) \uparrow \geq E (X_0) \). Therefore \( \{X_{S_n}\} \) is uniformly integrable and so is \( \{X_{T_n}\} \). By right continuity and uniform integrability

\[
\int_A X_S = \lim_n \int_A X_{S_n} \leq \lim_n \int_A X_{T_n} = \int_A X_T
\]

for all \( A \in \mathcal{F}_{S+} \), which justifies

\[
E (X_T \mid \mathcal{F}_{S+}) \geq X_S
\]

b) We will show that \( \{X_t\} \) is uniformly integrable so that we have a last element equal to \( Y \) a.s., so the original proof in the Theorem can be applied. By the assumption

\[
\sup t \left\{ \int \left| X_t \right| \right\} \leq \int |Y| < \infty
\]

and

\[
\int_{\{\left| X_t \right| > c\}} |X_t| \leq \int_{\{\left| X_t \right| > c\}} |Y| \leq \frac{1}{c} \left( \int \left| X_t \right| \right) \left( \int Y \right)
\]

Thus

\[
\limsup \frac{1}{c} \int_{\{\left| X_t \right| > c\}} |X_t| = 0
\]

and \( \{X_t\} \) is uniformly integrable, which implies \( X_\infty := \lim X_t \) a.s. and \( E (|X_t - X_\infty|) = 0 \) and \( \{X_t, \mathcal{F}_t : 0 \leq t \leq \infty\} \) is a submartingale. By the Theorem on the textbook

\[
E (X_T \mid \mathcal{F}_S) \geq X_T
\]

for all \( S \leq T \). ■

**Remark 22** I don’t see exactly why \( T \) being bounded helps. It might be because if \( T \) is unbounded then \( X_\infty \), the last element of the submartingale is NEVER defined. So

\[
E (X_\infty \mid \mathcal{F}_{S+}) \geq X_S
\]

may be an impossibility.

**Remark 23** If \( \{X_t\} \) is a uniformly integrable submartingale and if there is some integrable \( Y \) such that

\[
E (Y \mid \mathcal{F}_t) \geq X_t
\]

Could we show that

\[
Y = X_\infty = \lim X_t
\]

a.s.? It seems that this need not to hold.

**Problem 24** Suppose that \( \{X_t, \mathcal{F}_t : 0 \leq t < \infty\} \) is a right-continuous submartingale and \( S \leq T \) are stopping times of \( \{\mathcal{F}_t\} \). Then

a) \( \{X_t \wedge T, \mathcal{F}_t : 0 \leq t < \infty\} \) is a submartingale

b) \( E (X_t \wedge T \mid \mathcal{F}_S) \geq X_{S \wedge T} \) a.s. for all \( t \geq 0 \)
Proof. One nice thing is that \( t \land S \leq t \land T < \infty \) a.s.. So no worries about \( \infty \) time. Thus by optional sampling theorem for bounded stopping times

\[
E(X_{t\land T}|\mathcal{F}_{S+}) \geq X_{t\land S}
\]

and \( \mathcal{F}_S \subseteq \mathcal{F}_{S+} \) gives

\[
E(X_{t\land T}|\mathcal{F}_{S}) \geq X_{t\land S}
\]

\[
\square
\]

Problem 25 A submartingale of constant expectation, i.e., with \( E(X_t) = E(X_0) \) for every \( t \geq 0 \), is a martingale.

Proof. Take \( A \in \mathcal{F}_s \). Then \( E(X_{t1A}) \geq E(X_{s1A}) \) and \( E(X_{t1A^c}) \geq E(X_{s1A^c}) \) and since the expectation is constant, strict inequality cannot hold, \( X \) is a martingale. \( \square \)

Problem 26 A right continuous process \( \{X_t, \mathcal{F}_t : 0 \leq t < \infty \} \) with \( E(|X_t|) < \infty \) is a submartingale if and only if for every pair \( S \leq T \) of bounded stopping times of the filtration \( \{\mathcal{F}_t \} \) we have

\[
E(X_S) \leq E(X_T)
\]

Proof. Necessity follows from Problem 21. For sufficiency, define

\[
S = s1_A + t1_{A^c}
\]

for any \( A \in \mathcal{F}_s \). Then \( S \leq t \) and both of them are bounded. What’s more interesting is that

\[
S = \begin{cases} 
  t & \text{if } \omega \in A^c \\
  s & \text{if } \omega \in A
\end{cases}
\]

that is, \( S \) selects \( s \) and \( t \). By the hypothesis \( E(X_t) \geq E(X_S) \) we have

\[
E(X_t) = E(X_{t1A}) + E(X_{t1A^c}) = E(X_{s1A}) + E(X_{s1A^c})
\]

\[
\geq E(X_{s1A}) + E(X_{s1A^c}) = E(X_s)
\]

\( \square \)

Problem 27 Let \( \{X_t, \mathcal{F}_t : 0 \leq t < \infty \} \) be a continuous, non-negative supermartingale and

\[
T = \inf \{t \geq 0; X_t = 0\}
\]

Show that

\[
X_{T+t} = 0; 0 \leq t \text{ a.s on } \{T < \infty\}
\]

right-continuous submartingale

Problem 28 Let \( \{Z_t, \mathcal{F}_t : 0 \leq t < \infty \} \) be a continuous, non-negative supermartingale with \( Z_\infty := \lim Z_t = 0 \) a.s. Then for every \( s \geq 0, b > 0 \):

a) \( P[\sup_{t>s} Z_t \geq b | \mathcal{F}_s] = \frac{1}{b} Z_s \) a.s. on \( \{Z_s < b\} \)

b) \( P[\sup_{t\geq s} Z_t \geq b] = P[Z_s \geq b] + \frac{1}{b} E[Z_s 1_{\{Z_s < b\}}] \)
**Proof.** Define

$$T = \inf \{ t \in [s, \infty) : Z_t = b \}$$  \hspace{1cm} (6)

Then $T$ is a stopping time (since $Z_t$ is continuous by Prob 2.7 on page 7 of the text) and $\{Z_{t \wedge T}, \mathcal{F}_t : 0 \leq t < \infty\}$ is a martingale such that

$$\int_{A \cap \{Z_s < b\}} Z_s dP = \int_{A \cap \{Z_s < b\}} Z_s 1_{\{T > t\}} dP$$

$$= bP[A \cap \{Z_s < b, T \leq t\}] + \int_{A \cap \{Z_s < b\}} Z_t 1_{\{T > t\}} dP$$

Since $Z_t 1_{\{T > t\}}$ is bounded by $b$ (by the definition of $T$) and converges to zero as $t \to \hat{f}$, we have by DCT

$$\int_{A \cap \{Z_s < b\}} Z_s dP = bP[A \cap \{Z_s < b, T < \infty\}]$$

$$= b \int_{A \cap \{Z_s < b\}} E(1_{\{T < \infty\}} | \mathcal{F}_s) dP$$

i.e.,

$$\frac{1_{\{Z_s < b\}} Z_s}{b} = E(1_{\{\sup \{z_t \geq b\}\}} | \mathcal{F}_s)$$  \hspace{1cm} (7)

Now take the unconditional expectations on both sides of (7), b) is established $\blacksquare$

**Remark 29** The above problem illustrates how to use stopping times to control the process.