Two-Factor Mixed Effects Model

- One factor random and one factor fixed
- Assume A fixed and B random
- Parameter assumptions/restrictions are now:
  \[ \sum \alpha_i = 0 \text{ and } \beta \sim N(0, \sigma^2_{\beta}) \]
  
  \[ (\alpha\beta)_{ij} \sim N(0, \sigma^2_{\alpha\beta}) \]

- Known as the **unrestricted** mixed model
- SAS uses this model in its procedures

Unrestricted Mixed Model

- Same partition of total sum of squares as two-way random
- Different assumptions/restrictions alter EMS

\[
E(\text{MSE}) = \sigma^2 \\
E(\text{MSA}) = \sigma^2 + bn \sum \alpha_i^2 / (a - 1) + n\sigma^2_{\alpha\beta} \\
E(\text{MSB}) = \sigma^2 + a\sigma^2_{\beta} + n\sigma^2_{\alpha\beta} \\
E(\text{MSAB}) = \sigma^2 + n\sigma^2_{\alpha\beta}
\]
Hypothesis Tests

- Tests require different MS in denom
  \( H_0: \alpha_1 = \alpha_2 = \ldots = 0 \rightarrow \text{MSA/MSAB} \)
  \( H_0: \sigma^2_\beta = 0 \rightarrow \text{MSB/MSAB} \)
  \( H_0: \sigma^2_{\alpha\beta} = 0 \rightarrow \text{MSAB/MSE} \)
- Variance Estimates (Using ANOVA method)
  \[ \hat{\sigma}^2 = \text{MSE} \]
  \[ \hat{\sigma}^2_\beta = (\text{MSB} - \text{MSAB})/an \]
  \[ \hat{\sigma}^2_{\alpha\beta} = (\text{MSAB} - \text{MSE})/n \]

Multiple Comparisons

\[ Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \]
\[ Y_{i..} = \mu + \alpha_i + \bar{\beta} + (\bar{\alpha}\bar{\beta})_i + \bar{\varepsilon}_i \]
\[ \text{Var}(Y_{i..}) = \sigma^2_\beta/b + \sigma^2_{\alpha\beta}/b + \sigma^2/bn \]
\[ Y_{i..} - Y_{i.} = \alpha_i - \alpha_i' + (\bar{\alpha}\bar{\beta})_i - (\bar{\alpha}\bar{\beta})_i' + \bar{\varepsilon}_i - \bar{\varepsilon}_i' \]
\[ \text{Var}(Y_{i..} - Y_{i.}) = 2\sigma^2_{\alpha\beta}/b + 2\sigma^2/bn \]
\[ = 2(n\sigma^2_{\alpha\beta} + \sigma^2)/bn \]
- For pairwise comparison, use \( 2\text{MSAB}/bn \)
- For \( Y_{i..} \), use approximate method

Other Two-Way Mixed Model

- Consider interaction a hybrid of random and fixed effect
- Parameter assumptions/restrictions are now:
  1. \( \sum \alpha_i = 0 \) and \( \beta \sim N(0, \sigma^2_\beta) \)
  2. \((\alpha\beta)_{ij} \sim N(0, (a-1)\sigma^2_{\alpha\beta}/a)\)
  3. \( \sum (\alpha\beta)_{ij} = 0 \) for \( \beta \) level \( j \)
- Known as restricted mixed effects model

Restricted Mixed Model

- The \((a - 1)/a\) is used to simplify the EMS
  \[ \text{E(MSE)} = \sigma^2 \]
  \[ \text{E(MSA)} = \sigma^2 + bn \sum \alpha^2_i/((a-1) + n\sigma^2_{\alpha\beta}) \]
  \[ \text{E(MSB)} = \sigma^2 + an\sigma^2_\beta \]
  \[ \text{E(MSAB)} = \sigma^2 + n\sigma^2_{\alpha\beta} \]
- Because of (3), not all \((\alpha\beta)_{ij}\) indep

\[ \text{Cov}((\alpha\beta)_{ij}, (\alpha\beta)_{ij'}) = -\frac{1}{a}\sigma^2_{\alpha\beta} \]

NOTE: If \( X_i \sim N(0, \sigma^2) \) then
\[ \text{Cov}(X_i - \bar{X}, X_j - \bar{X}) = -\frac{1}{n}\sigma^2 \]
Hypothesis Tests

- Tests require different MS in denom
  \[ H_0 : \alpha_1 = \alpha_2 = \ldots = 0 \rightarrow \text{MSA/MSAB} \]
  \[ H_0 : \sigma^2 = 0 \rightarrow \text{MSB/MSE} \]
  \[ H_0 : \sigma^2_{\alpha\beta} = 0 \rightarrow \text{MSAB/MSE} \]
- Variance Estimates (Using ANOVA method)
  \[ \hat{\sigma}^2 = \text{MSE} \]
  \[ \hat{\sigma}^2_\beta = (\text{MSB} - \text{MSE})/an \]
  \[ \hat{\sigma}^2_{\alpha\beta} = (\text{MSAB} - \text{MSE})/n \]

Multiple Comparisons

\[
Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}
\]

\[
\bar{Y}_{.i} = \mu + \alpha_i + \bar{\beta} + (\alpha\bar{\beta})_i + \bar{\varepsilon}_{i.}
\]

\[
\text{Var}(\bar{Y}_{.i}) = \frac{\sigma^2_\beta}{b} + \frac{(a-1)\sigma^2_{\alpha\beta}}{ab} + \frac{\sigma^2}{bn}
\]

\[
\bar{Y}_{.i} - \bar{Y}_{.i'} = \alpha_i - \alpha_{i'} + (\alpha\bar{\beta})_i - (\alpha\bar{\beta})_{i'} + \bar{\varepsilon}_{i.} - \bar{\varepsilon}_{i'.}
\]

\[
\text{Var}(\bar{Y}_{.i} - \bar{Y}_{.i'}) = 2\frac{\sigma^2_\beta}{b} + 2\frac{\sigma^2}{bn}
\]

\[
= 2\left(\frac{a\sigma^2_\beta + \sigma^2}{bn}\right)
\]

- For pairwise comparison, use 2\text{MSAB}/bn
- For \( \bar{Y}_{.i} \), use approximate method

Unrestricted versus Restricted Models

- Test of \( H_0 : \sigma^2 = 0 \)
  - Test over MSAB or MSE
  - Unrestricted considered more conservative test because of DF
- Standard error of Factor A treatment means
  - Use different standard error (slides 6 and 10)
- To decide which model is appropriate, suppose you ran experiment again and sampled some of the same levels of the random effect. Does this mean that the interaction effects for these levels are the same as before? Yes: Restricted No: Unrestricted

Example from Montgomery

- Want to assess variability in a measurement system
- Twenty parts selected from production process
- Gauge used by 3 operators to measure parts
- Each part measured twice by each operator
- Will consider operators fixed
- Will investigate both restricted and unrestricted results
**Gauge Capability Example in Text 12-3**

```plaintext
options nocenter l=75;

data randr;
  input part operator resp @@;
cards;
  1 1 21 1 2 20 1 3 21 2 1 24 2 2 24 2 3 23 2 3 24 3 1 20 3 2 21 3 2 20 3 3 22 4 1 27 4 2 28 4 2 26 4 3 27 4 3 28 . . . .
  proc glm;
  class operator part;
  model resp=operator|part;
  random part operator*part / test;
  means operator / tukey lines E=operator*part;
  lsmeans operator / adjust=tukey E=operator*part tdiff stderr;
  proc mixed alpha=.05 cl covtest;
  class operator part;
  model resp=operator / ddfm=kr;
  random part operator*part;
  lsmeans operator / alpha=.05 cl diff adjust=tukey;
  run;
  quit;
```

---

**Standard Error Issue**

- The SE($\bar{Y}_i$) on the previous page is incorrect. For the unrestricted model, it should be $\sqrt{(\sigma^2 + n\sigma_i^2 + n\sigma_j^2)/bn}$, which can be estimated by $\sqrt{((a-1)MS_{AB} + MS_{B})/(abn)} = .7292$.

- This is due to GLM being a fixed effects procedure.

- We can calculate the correct error from the output but we need to know it is wrong, we’d also need to approximate the degrees of freedom.

- For restricted mixed model, we HAVE TO compute the SE by hand.
The Mixed Procedure

Iteration History

Iteration Evaluations -2 Res Log Likelihood Criterion
0 1 622.7855725
1 2 409.45998838 0.00002843
2 1 409.45716449 0.00000003
3 1 409.45716136 0.00000000

Convergence criteria met.

Covariance Parameter Estimates

Standard Z
Cov Parm Estimate Error Value Pr Z Alpha Lower Upper
part 10.2798 0.05 3.6673 16.8924
operator*part -0.1399 0.05 -0.3789 0.09903
Residual 0.9917 0.05 0.7143 1.4698

Unrestricted Model

Var(\(T_1\)) = (\(\sigma^2 + n\sigma^2_{\mu} + n\sigma^2_{\nu}\))/bn = (0.8832 + 2(0)+ 2(10.2513))/40 = 0.7312.

Var(\(T_1 - T_2\)) = 2(\(\sigma^2 + n\sigma^2_{\mu} + n\sigma^2_{\nu}\))/bn = 0.8832/20 = 0.0442

These estimates are slightly different than GLM because of the zero variance estimate.

proc mixed alpha=.05 c1 nobound;
class operator part;
model resp=operator / ddfm=kr;
random part operator*part;
lsmeans operator / alpha=.05 cl diff adjust=tukey;

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These estimates are slightly different than GLM because of the zero variance estimate.

Multifactor Models

- 3-Factor: Can have 0, 1, 2, or 3 random effects
- Use EMS to determine tests
- In some cases, no straightforward test exists. In other words, there is no single MS for the denominator/numerator
- Must perform approximate F test
- SAS Mixed and random statement use Satterthwaite or Kenward-Roger approximation
**Satterthwaite’s Approximate F-test**

- Involves a linear combination of mean squares
  - To test certain factor, choose numerator and denominator such that the difference in MS is a multiple of the effect of interest
  - Ratio approximately F where
    \[ F_{p,q} = \frac{MS_r \pm \ldots \pm MS_u}{MS_s \pm \ldots \pm MS_v} \]
    \[ p = \frac{(MS_r \pm \ldots \pm MS_u)^2}{MS_r/\nu_r + \ldots + MS_u/\nu_u} \]
    \[ q = \frac{(MS_s \pm \ldots \pm MS_v)^2}{MS_s/\nu_s + \ldots + MS_v/\nu_v} \]
  - \( f_i \) is the degrees of freedom associated with MS_i
  - No MS in both numerator and denominator (indep)
  - Caution when subtraction is used

**3-Factor Mixed Model**

- Denominator for U is leading eligible random term(s)
- Leading: Closest connected random term below U
- Eligible:
  - Unrestricted: Any random term possible
  - Restricted: Any without fixed factor not in U

```
M  \( \rightarrow \) B
  \( \rightarrow \) A
  \( \rightarrow \) C
```

**Construction of Hasse Diagram**

- Described in Oehlert (2000)
- Used for determining tests
- Every term in model is a node
- Terms/nodes placed in layered structure
  - U is above V if all terms in U are in V
- Join nodes based on structure
- Brackets placed around random terms

**Background Reading**

- KNNL Section 25.2-25.6
- KNNL Chapter 22