Two Level Fractional Factorials

Design of Experiments - Montgomery Sections 8-1 – 8-3

Fractional Factorials

- May not have sources for complete factorial design
- Number of runs required for factorial grows quickly
  - Consider $2^k$ design
  - If $k = 7$ → 128 runs required
  - Can estimate 127 effects
  - Only 7 df for main effects
  - 99 df are for interactions of order $\geq 3$
- Often system driven by low order effects
- Would like design
  - To utilize this sparsity principle
  - Can be projected into larger designs
- Fractional factorials provide these options

Example

- Suppose you were designing a new car for mileage
- Wanted to consider (2 levels each)
  - Engine Size
  - Number of cylinders
  - Drag
  - Weight
  - Automatic vs Manual
  - Shape
  - Tires
  - Suspension
  - Gas Tank Size
- Only have resources for $2^7$ design
  - If you drop two factors for a complete factorial, could discard significant main effects or lower order interactions
  - Want option to keep all nine factors in model
  - Must assume higher order interactions insignificant
  - Are six and seven order interactions meaningful?

Two-Level Fractional Factorials

- Assume certain higher order interactions negligible
- Can then collect more info on lower level effects
- Example: Latin Square ($2^3$ factorial)
  - Let $A=$ Block Factor 1, $B=$ Block Factor 2, and $C=$ Treatment
  - Utilize only four observations instead of eight
  - Observed combinations associated with $ABC$ column
  1. May observe $C$, $B$, $A$, $ABC$
  2. May observe $(1)$, $BC$, $AC$, and $AB$
Since only four obs, certain factors confounded
Use association column to determine confounding
Association column known as the generator
Use generator in the following manner
\[ I = ABC \rightarrow A = BC, B = AC, \text{ and } C = AB \]
Thus if additivity, can estimate A, B, and C

Assumptions and Expectations
- Used when
  - Runs expensive and variance estimates available
  - Screening experiments when many factors considered
  - Sequential design analysis possible (put fractions together)
- Interest in main effects and low order interactions
- Prepared to assume certain interactions are zero
- Not primarily interested in estimate of variance
  Emphasis is on finding small experiments in which a high percentage of df are used for estimation of low order effects
- Notation
  - Full factorial is \( 2^k \)
  - Fractional Factorial is \( 2^{k-p} \)
  - Degree of fraction is \( 2^{-p} \)

Half-Fraction \( 2^k \) Factorials
- This is one half the usual number of runs
- Similar to blocking procedure
  - Choose a generator which divides effects into two
  - Based on pluses and minuses of one factor
  - Defining Relation: \( I = \) generator
- Consider three factor but only 4 runs possible

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
<th>ABC</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>(1)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>b</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>c</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>abc</td>
</tr>
</tbody>
</table>

Select I=ABC, get groups (a,b,c,abc) and (ab,ac,bc,(1))
Select I=A, get groups (a,ab,ac,abc) and ((1),b,c,abc)
Use generator to determine confounded effects

Each effect is estimated as follows
\[ l_A = .5(a - b - c + abc) \quad l_{BC} = .5(a - b - c + abc) \]
\[ l_B = .5(-a + b - c + abc) \quad l_{AC} = .5(-a + b - c + abc) \]
\[ l_C = .5(-a - b + c + abc) \quad l_{AB} = .5(-a - b + c + abc) \]

Cannot differentiate between A and BC, B and AC, and C and AB
Can use defining relation to determine confounding
"Multiply" each side by an effect
\[ (A)I = A = (A)ABC = A^2BC = BC \]
\[ (AB)I = AB = (AB)ABC = A^2B^2C = C \]
Linear combinations estimate
\[ A + BC, B + AC, \text{ and } C + AB \]
Half-Fraction $2^k$ Factorials

- Suppose we use other grouping ($I = -ABC$)

  \[ l_A = .5(ab + ac - bc - (1)) \quad l_{BC} = .5(-ab - ac + be + (1)) \]
  \[ l_B = .5(ab - ac + bc + (1)) \quad l_{AC} = .5(-ab - ac + be + (1)) \]
  \[ l_C = .5(-ab + ac + be - (1)) \quad l_{AB} = .5(ab - ac + bc + (1)) \]

- \( l_A = -l_{BC} \) so combination is estimating \( A - BC \)
- If both fractions were run
  - Could separately estimate effects
  - \( ABC \) is confounded with blocks
- Similar to factorial in two blocks

Can piece together fractional factorial into bigger design if appropriate

Another Example

Consider a $2^{5-1}$ fractional factorial. Usually would have 32 runs, but we will have 16. Let us use the defining relation \( I = ABCDE \). This means we will have the following confounded effects

- Main effects confounded with 3rd order interactions
- 1st order interactions confounded with 2nd order interactions

Main effects confounded with 3rd order interactions, 1st order interactions confounded with 2nd order interactions

Known as a Resolution V Design

Resolution

- A design of resolution \( R \) is one in which no \( p \)-factor effect is confounded with any other effect containing less than \( R - p \) factors
  - Resolution III design does not confound main effects with other main effects.
  - Resolution IV design does not confound main effects with two-factor interactions but does confound two-factor interactions with other two-factor interactions.
  - Resolution V design does not confound any main effects and two-factor interactions with each other.
  - Resolution III - can estimate main effects if you assume no interaction
  - Resolution IV - can estimate main effects without assuming two-factor interactions negligible
  - Resolution V - can estimate main and two-factor interactions if you assume higher order terms negligible.

- Resolution also defined by length of shortest word in defining relation
  - \( I = ABC \) is Resolution III
  - \( I = ABCD \) is Resolution IV
  - \( I = ABCDE \) is Resolution V

Using Yates’ Algorithm

- For \( 2^{k-1} \) design, will set up \( k - 1 \) columns
- Select \( k - 1 \) factors and write in standard order
- Multiply columns to determine sign of last factor
- See Table 8-5 for \( 2^{5-1} \) example

Consider these results from previous example

<table>
<thead>
<tr>
<th>Combination</th>
<th>y</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>A+BCDE</td>
</tr>
<tr>
<td>e</td>
<td>15</td>
<td>25</td>
<td>55</td>
<td>75</td>
<td>35</td>
<td>A+BCDE</td>
</tr>
<tr>
<td>b</td>
<td>5</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>-35</td>
<td>B+ACDE</td>
</tr>
<tr>
<td>abe</td>
<td>25</td>
<td>15</td>
<td>45</td>
<td>15</td>
<td>15</td>
<td>ABC+DE</td>
</tr>
<tr>
<td>c</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>-20</td>
<td>15</td>
<td>C+ABDE</td>
</tr>
<tr>
<td>ace</td>
<td>25</td>
<td>15</td>
<td>5</td>
<td>-15</td>
<td>-15</td>
<td>AC+BD</td>
</tr>
<tr>
<td>bce</td>
<td>10</td>
<td>30</td>
<td>10</td>
<td>10</td>
<td>-45</td>
<td>BC+ADE</td>
</tr>
<tr>
<td>abc</td>
<td>5</td>
<td>15</td>
<td>5</td>
<td>5</td>
<td>-35</td>
<td>ABC+DE</td>
</tr>
<tr>
<td>d</td>
<td>5</td>
<td>-5</td>
<td>5</td>
<td>0</td>
<td>-35</td>
<td>D+ABCE</td>
</tr>
<tr>
<td>ade</td>
<td>10</td>
<td>20</td>
<td>-25</td>
<td>15</td>
<td>-5</td>
<td>AD+BCE</td>
</tr>
<tr>
<td>bde</td>
<td>5</td>
<td>10</td>
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<td>-10</td>
<td>5</td>
<td>BD+ACE</td>
</tr>
<tr>
<td>abd</td>
<td>10</td>
<td>-5</td>
<td>-15</td>
<td>-5</td>
<td>-5</td>
<td>ABD+CE</td>
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<td>cde</td>
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<td>5</td>
<td>25</td>
<td>-30</td>
<td>15</td>
<td>CD+ABE</td>
</tr>
<tr>
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<td>5</td>
<td>-15</td>
<td>-15</td>
<td>5</td>
<td>ACD+BE</td>
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<td>BCD+AE</td>
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<tr>
<td>abcde</td>
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<td>5</td>
<td>5</td>
<td>5</td>
<td>45</td>
<td>ABCD+E</td>
</tr>
</tbody>
</table>
Construction of a $2^{k-1}$ fractional factorial with highest resolution

1. Write a full factorial design for the first $k - 1$ variables
2. Associate the $k$th variable with $\pm$ interaction of $k - 1$

- Consider $2^{k-1}$ design

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D = ABC</th>
<th>Effect</th>
<th>Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>(1)</td>
<td>ad</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>b</td>
<td>bd</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>ab</td>
<td>ab</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>c</td>
<td>cd</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>ac</td>
<td>ac</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>bc</td>
<td>bc</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>abc</td>
<td>abcd</td>
</tr>
</tbody>
</table>

- $I = ABCD \rightarrow$ Resolution 4 design

Alternative View of Half Fraction Design

- Consider any $2^{k-1}$ design
- If we collapse design by omitting variable
- Remaining is a $2^{k}$ full factorial ($k^* = k - 1$)
- Can be shown for resolution $R$, complete factorial for $R - 1$ factors
  - Consider $2^{3-1}$ design (Resolution III)
  - Can view combinations on cube
  - Regardless of direction, you can squeeze cube into square with observation at each corner

Another Example

- Consider the following $2^3$ design

<table>
<thead>
<tr>
<th>Combination</th>
<th>y 1</th>
<th>2</th>
<th>3</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>15</td>
<td>43</td>
<td>81</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>12</td>
<td>26</td>
<td>38</td>
<td>21</td>
</tr>
<tr>
<td>b</td>
<td>10</td>
<td>13</td>
<td>13</td>
<td>21</td>
</tr>
<tr>
<td>ab</td>
<td>16</td>
<td>25</td>
<td>8</td>
<td>-3</td>
</tr>
<tr>
<td>c</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>-5</td>
</tr>
<tr>
<td>ac</td>
<td>9</td>
<td>6</td>
<td>12</td>
<td>-5</td>
</tr>
<tr>
<td>bc</td>
<td>11</td>
<td>5</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>abc</td>
<td>14</td>
<td>3</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

- Consider instead a half fraction with $I = ABC$

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C = AB</th>
<th>Combination</th>
<th>y 1</th>
<th>2</th>
<th>3</th>
<th>Estimate</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>12</td>
<td>24</td>
<td>12</td>
<td>6.0</td>
<td>$A + BC$</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>-</td>
<td>b</td>
<td>10</td>
<td>8</td>
<td>8</td>
<td>4.0</td>
<td>$B + AC$</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>abc</td>
<td>14</td>
<td>4</td>
<td>-4</td>
<td>-2.0</td>
<td>$C + AB$</td>
</tr>
</tbody>
</table>

- Notice effects are sums of previous estimates
- Could ignore $C$ and this becomes full factorial

Sequential Use of Factorial Designs

- Often more efficient to look at half fraction
- Analyze results
- Decide on best set of runs for next experiment
  - Can add or remove factors
  - Change responses
  - Vary factors over new ranges
- If ambiguities, can run remaining half of factorial
- Only lose estimate of highest order interaction
- Important to always randomize order of runs
- Can use Proc Factex to generate design
The General $2^{k-p}$ Fractional Factorial

- Must select $p$ independent generators
- Want to have best alias relationships
- Defining relation based on $2^p - 1$ effects
- Often try to maximize the resolution
- Each effect has $2^p - 1$ aliases
- Often assume higher order interactions zero
- Simplifies alias structure
- Table 8-14 summarizes potential generators
  - $k \leq 15$ and $n \leq 128$
  - Results in highest possible resolution
- Table XII: alias relationships for designs with $n \leq 64$
- Can use Yates’ in similar fashion to obtain estimates

Example

Consider $2^{5-2}$ which consists of eight runs. Suppose we choose $I = ABC$ and $I = BDE$. The defining relation is $I = ABC = BDE = ACDE$ so this is a resolution III design. The factor $A$ is aliased with $BC$, $ABDE$, and $CDE$.

Consider $2^{11-4}$ which has 128 runs. We start with a complete $2^7$. Consider the factors $A, B, C, D, E, G, J$ with remaining factors $F, H, K, L$. Define four generators as

- $F = ABCDE$
- $K = ABFJ$
- $L = AEFGK$
- $H = ACEL$

The defining relation is $I = ABCDEF = ABFJK = AEFGKL = ACEHL$. If multiply these together in pairs, we get $I = CDEJK = BCDGKL = BDFHL = BEGJL = CFGHK$. If we multiply these in triples, we get $I = ABCGHJ = ABDEGHK = ACDFG = ADHJL$ and if we multiply all four together, we get $I = DEFGHJ$. Since the shortest word in relation is of length five, this has resolution V.

Resolution III Designs

- Can use this design to efficiently investigate numerous factors
- Can use resolution III to investigate $N - 1$ factors in $N$ runs
- $N$ must be a multiple of 4
  - 4 runs to investigate 3 factors $\rightarrow 2^{3-1}$ design
  - 8 runs to investigate 7 factors $\rightarrow 2^{7-4}$ design
  - 16 runs to investigate 15 factors $\rightarrow 2^{15-11}$ design
- For $2^{3-1}$ design
  - Each main effect aliased with one two factor interaction
- Introduced fractional factorials with this design
- For $2^{7-4}$ design
  - Each main effect aliased with three two factor interactions
  - Often ignore interactions of order $\geq 3$
- For $2^{15-11}$ design
  - Each main effect aliased with seven two factor interactions
Sequential Assembly of Fractions

- Consider $2^{k-4}$ design with $D = AB$, $E = AC$, $F = BC$, and $G = ABC$
- If factor $D$ important and don’t want it confounded
  - Use same generators except $D = -AB$
  - If all three factor interactions zero, can estimate $D$
  - Can also estimate all interactions concerning $D$

- Can use two $2^{k-4}$ to get resolution IV
  - Instead of flipping one sign, flip sign of all factors
  - Generators of even size flip sign
  - Known as folding over
  - Breaks link between main effects and two-factor interactions

- $D = AB$, $E = AC$, $F = BC$, and $G = ABC$
- $D = -AB$, $E = -AC$, $F = -BC$, $G = ABC$

Example

Consider the $2^{k-1}$ design with $C = AB \rightarrow I = ABC$. This is the trivial example because folding over creates the full factorial.

$$
\begin{array}{cccc}
A & B & C & \text{Combination} \\
- & - & + & c \\
+ & - & - & a \\
+ & + & - & abc \\
\end{array}
$$

Now we switch signs on everything

$$
\begin{array}{cccc}
A & B & C & \text{Combination} \\
- & - & - & ab \\
+ & + & + & bc \\
- & - & + & ac \\
\end{array}
$$

Obtain all estimates except $ABC$

Resolution IV Designs

- Can be obtained by folding over resolution III design
- Thus, Resolution IV can be divided into two blocks
- Must contain at least $2k$ runs
- Minimal design if runs equal $2k$

- Determination of defining relation from fold-over III
  - $L + U$ words used as generators
  - $L$ words of like sign
  - $U$ words of unlike sign
  - Combined design will have $L + U - 1$ generators
    All $L$ words
    Even products of $U$ words

Examples

- $I = ABD = ACE = BCF = ABCG$ and $I = -ABD = -ACE = -BCF = ABCG$
- $L = 1$ and $U = 3$
- $I = ABCG = ABD(ACE) = ABD(BCF)$

- $I = ABCE = BCDF = ACDG = ABDH = ABCDJ$ and $I = ABCE = BCDF = ACDG = ABDH = -ABCDJ$
- $L = 4$ and $U = 1$
- $I = ABCE = BCDF = ACDG = ABDH$