Balanced Incomplete Block Design

Design of Experiments - Montgomery
Section 4-4

Balanced Incomplete Block

- Incomplete: cannot fit all trts in each block
- Balanced: each pair of trts occur together \( \lambda \) times
- Balanced: \( \text{Var}(\bar{r}_i - \bar{r}_j) \) is constant

\( a \) trts, \( b \) blocks, \( r \) replicates, and \( k \) trts per block
Total number of obs is \( kb = ar = N \)

So trt \( i \) occurs in \( r \) blocks. To have balance, each other trt is equally likely to be with trt \( i \) in a block. Since there are \( k - 1 \) other units in a block and \( a - 1 \) other trts, the number of times each pair occurs together is

\[ \lambda = r(k - 1)/(a - 1) \]

where \( \lambda \) is an integer. One way to generate this is
- Select \( \binom{a}{2} \) blocks and assign each a diff \( k \) trt combination
- The number of replicates is \( r = \binom{b-1}{k-1} \)
- \( \lambda = \binom{a-2}{k-2} \)
- Sometimes can do this in less than \( \binom{a}{2} \) blocks

Extensive list of BIB designs found in Fisher and Yates (1963) and Cochran and Cox (1957)

Examples

\( a = 3, b = 3, k = 2 \rightarrow r = 2, \lambda = 1 \)

<table>
<thead>
<tr>
<th>BLOCK</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>C</td>
<td></td>
</tr>
</tbody>
</table>

\( a = 4, k = 2, b = 6 \rightarrow r = 3, \lambda = 1 \)

<table>
<thead>
<tr>
<th>BLOCK</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>D</td>
<td>C</td>
<td>D</td>
<td>D</td>
<td></td>
</tr>
</tbody>
</table>

\( a = 4, k = 3, b = 4 \rightarrow r = 3, \lambda = 2 \)

<table>
<thead>
<tr>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>C</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td></td>
</tr>
</tbody>
</table>

Balanced Incomplete Block

- Similar construction as RCBD
- Statistical Model

\[ y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \left\{ \begin{array}{l} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{array} \right. \]

- Not all \( y_{ij} \) exist because of incompleteness
- Additive effect due to block / No Interaction
- Usual treatment and block restrictions

\[ \sum \tau_i = 0 \quad \sum \beta_j = 0 \]

- Nonorthogonality of treatments and blocks

Use Type III Sums of Squares and lsmeans
### Analysis of Variance Table

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>(F_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks</td>
<td>SS(_B)</td>
<td>(b - 1)</td>
<td>MS(_B)</td>
<td></td>
</tr>
<tr>
<td>Treatment</td>
<td>SS(_T)</td>
<td>(a - 1)</td>
<td>MS(_T)</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>SS(_E)</td>
<td>(N - a - b + 1)</td>
<td>MS(_E)</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>SS(_T)</td>
<td>(N - 1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- \(SS_T = \sum \sum y_{ij}^2 - \bar{y}^2 / N\)
- \(SS_{Block} = \frac{1}{k} \sum y_j^2 - \bar{y}^2 / N\)
- \(SS_{Treatment} = \sum Q_i^2 / \lambda a = \sum i^2 / a\)
- If \(F_0 > F_{a-1,N-a-b+1}\) then reject \(H_0\)

### Model Estimates

- Design matrix \(X\) is RCBD with certain rows missing
- Can form normal equations to solve for \(\hat{\mu}, \hat{\tau}_i, \hat{\beta}_j\)

\[
\hat{\mu} = \frac{1}{N} \sum y_i, \\
\hat{\tau}_i = \frac{k}{\lambda a} Q_i, \\
\hat{\beta}_j = \frac{1}{\lambda b} Q_j'
\]

\[
Var(\hat{\mu}) = \frac{1}{N} \text{Var}(y_i) + \frac{1}{k} \text{Var}\left(\frac{1}{\lambda a} \sum n_{ij} y_{ij}\right) - 2\text{Cov}\left(\frac{1}{\lambda a} \sum n_{ij} y_{ij}\right)
\]

\[
= var(Y) + \frac{1}{k} \text{Var}\left(\frac{1}{\lambda a} \sum n_{ij} y_{ij}\right) - 2\text{Cov}\left(\frac{1}{\lambda a} \sum n_{ij} y_{ij}\right)
\]

\[
\sim \frac{1}{\lambda a} \frac{(k-1)\sigma^2}{\lambda^2 a^2}
\]

### Power and Multiple Comparisons

- Power Calculations
  - Assume \(a\) and \(k\) are known
  - Limited to values of \(b\) such that \(\lambda\) an integer
    
    \[
    \text{Non-centrality parameter} \quad \delta = \lambda a \sum \tau_i^2 / k\sigma^2
    \]
    
    Use integer values of \(\lambda\) and solve for \(b\) to get df

  - Can also use confidence interval estimation method
    
    Can show \(\text{Var}(\hat{\mu} - \bar{\mu}) = 2\lambda^2 / \lambda a\)
    
    Want \(a\) level CI to be no larger than \(2D\)
    
    \[
    2\sqrt{q_{1-a/2}^2 \frac{2\lambda^2}{\lambda a}} = D
    \]

- Multiple Comparisons and Contrasts
  - Similar procedures as ANCOVA
  - Must compute adjusted means (lsmeans)
  - Adjusted mean is \(\hat{\mu} + \hat{\tau}_i\)
  - Standard error of adjusted mean is \(\sqrt{\sigma^2 \left(\frac{1}{\lambda a} + \frac{1}{N} \right)}\)
  - Contrasts based on adjusted treatment totals

### SAS Example

```sas
options nocenter p=60 l=75; /* From Table 4-22 */
data example;
    /* From Table 4-22 */
    input trt block resp @@;
    cards;
    1 1 73 1 2 74 1 4 71 2 2 75 2 3 67 2 4 72
    1 3 73 3 2 75 3 3 68 4 1 75 4 3 72 4 4 75
;proc glm;
class block trt;
model resp = block trt;
lsmeans trt / tdiff pdiff adjust=bon stderr;
contrast 'a' trt 1 -1 0 0;
estimate 'b' trt 0 0 1 -1;
```

### Sum of Mean

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Squares</th>
<th>Square</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corrected Total</td>
<td>11</td>
<td>81.000000000</td>
<td>7.583333333</td>
<td>11.67</td>
<td>0.0107</td>
</tr>
<tr>
<td>Source</td>
<td>DF</td>
<td>Type I SS</td>
<td>Mean Square</td>
<td>F Value</td>
<td>Pr &gt; F</td>
</tr>
<tr>
<td>BLOCK</td>
<td>3</td>
<td>55.000000000</td>
<td>19.333333333</td>
<td>28.21</td>
<td>0.0015</td>
</tr>
<tr>
<td>TRT</td>
<td>3</td>
<td>22.750000000</td>
<td>7.583333333</td>
<td>11.67</td>
<td>0.0107</td>
</tr>
</tbody>
</table>

### Power and Multiple Comparisons

- Power Calculations

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```
SAS Example (cont)

<table>
<thead>
<tr>
<th>TRT</th>
<th>RESP</th>
<th>Std Err</th>
<th>Pr &gt;</th>
<th>LSMEAN</th>
<th>LSMEAN</th>
<th>H0:LSMEAN=0</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>71.3750000</td>
<td>0.4868051</td>
<td>0.0001</td>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>71.6250000</td>
<td>0.4868051</td>
<td>0.0001</td>
<td>2</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>72.0000000</td>
<td>0.4868051</td>
<td>0.0001</td>
<td>3</td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>75.0000000</td>
<td>0.4868051</td>
<td>0.0001</td>
<td>4</td>
<td></td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

Least Squares Means : Adjustment for multiple comparisons: Bonferroni

T for H0: LSMEAN(i)=LSMEAN(j) / Pr > |T|

<table>
<thead>
<tr>
<th>i/j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.35806</td>
<td>-0.89514</td>
<td>-5.19183</td>
<td>1.0000</td>
</tr>
<tr>
<td>2</td>
<td>0.358057</td>
<td>-0.53709</td>
<td>-4.83378</td>
<td>1.0000</td>
</tr>
<tr>
<td>3</td>
<td>0.895144</td>
<td>0.537086</td>
<td>-4.29669</td>
<td>1.0000</td>
</tr>
<tr>
<td>4</td>
<td>5.191833</td>
<td>4.833775</td>
<td>4.296689</td>
<td>0.0209</td>
</tr>
</tbody>
</table>

Contrast DF Contrast SS Mean Square F Value Pr > F

| a   | 1  | 0.0833333 | 0.0833333 | 0.13 | 0.7349 |

*** Contrast a compares trt1 and trt2. 13*(-0.35806)*(-0.35806)

*** Different p-values because Bonferroni used in first comparison

Interblock Analysis

Fixed effects analysis is known as intrablock analysis. If blocks are random, we can obtain additional information about \( \tau \)'s by considering the information between block totals. Based on our model, we can write the block totals as

\[
y_{ij} = \mu + \tau_i + \epsilon_{ij}
\]

and compute the least squares estimates of \( \mu \) and \( \tau_i \)

- The least square estimate for \( \tau_i \) is

\[
\hat{\tau}_i = \frac{\sum y_{ij} - k\phi}{r - \lambda}
\]

- Two estimates are uncorrelated \( \rightarrow \) Cov(\( \hat{\tau}_i, \hat{\tau}_j \))=0

- Use weighted combination of estimates

- Weights based on the variances of the two estimates

\[
\sigma^2 = MS_{\text{Blks}} \text{ and } \sigma^2 = \frac{(r-1)MS_{\text{Blocks}}}{\eta - (r-1)}
\]

\[
\tau_i^* = \left\{ \begin{array}{ll}
\frac{\lambda y_{i.} - \phi y_{.i}}{\phi(r-1)} & \text{if } \lambda > 0 \\
\frac{\phi(r-1) - \lambda y_{i.} + \phi y_{.i}}{\phi(r-1)} & \text{if } \lambda = 0
\end{array} \right.
\]

The Mixed Procedure

Dimensions
- Covariance Parameters: 2
- Columns in X: 5
- Columns in Z: 4
- Subjects: 1
- Max Obs Per Subject: 12
- Observations Used: 12
- Observations Not Used: 0
- Total Observations: 12

Model Information
- Data Set: WORK.EXAMPLE
- Covariance Structure: Variance Components
- Estimation Method: REML
- Residual Variance Method: Profile
- Degrees of Freedom Method: Kenward-Roger

Iteration History
- Iterations: 1
- Convergence criteria met.

Covariance Parameter Estimates
- Cov Parm: Estimate
- Block: 8.0167
- Residual: 0.6500

Fit Statistics
- -2 Res Log Likelihood: 34.2
- AIC (smaller is better): 38.2
- AICC (smaller is better): 40.6
- BIC (smaller is better): 37.0
Type 3 Tests of Fixed Effects

<table>
<thead>
<tr>
<th>Effect</th>
<th>Num</th>
<th>Den</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>trt</td>
<td>3</td>
<td>5.03</td>
<td>11.33</td>
<td>0.0112</td>
</tr>
</tbody>
</table>

Estimates

| Label | Estimate | Std Error | DF | t Value | Pr > |t| |
|-------|----------|-----------|----|---------|------|---|
| b     | -2.9705  | 0.6995    | 5.03| -4.25   | 0.0080 |

Contrasts

<table>
<thead>
<tr>
<th>Num</th>
<th>Den</th>
<th>F Value</th>
<th>Pr &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>5.03</td>
<td>0.7829</td>
</tr>
</tbody>
</table>

Least Squares Means

| Effect | trt | Estimate | Std Error | DF | t Value | Pr > |t| |
|--------|-----|----------|-----------|----|---------|------|---|
| trt    | 1   | 71.4131  | 1.4973    | 3.51| 47.70   | <.0001|
| trt    | 2   | 71.6164  | 1.4973    | 3.51| 47.83   | <.0001|
| trt    | 3   | 72.0000  | 1.4973    | 3.51| 48.09   | <.0001|
| trt    | 4   | 74.9705  | 1.4973    | 3.51| 50.07   | <.0001|

Differences of Least Squares Means

| Effect | trt | _trt | Estimate | Std Error | DF | t Value | Pr > |t| |
|--------|-----|------|----------|-----------|----|---------|------|---|
| trt    | 1   | 2    | -0.2033  | 0.6995    | 5.03| -0.29   | 0.7829|
| trt    | 1   | 3    | -0.5869  | 0.6995    | 5.03| -0.84   | 0.4396|
| trt    | 1   | 4    | -3.5574  | 0.6995    | 5.03| -5.09   | 0.0037|
| trt    | 2   | 3    | -0.3836  | 0.6995    | 5.03| -0.55   | 0.6069|
| trt    | 2   | 4    | -3.3641  | 0.6995    | 5.03| -4.79   | 0.0048|
| trt    | 3   | 4    | -2.9705  | 0.6995    | 5.03| -4.26   | 0.0080|

Cyclic Designs

Includes some BIB and PBIB designs

Consider situation where \( r = mk \) and \( b = ma \)

Determine \( m \) initial blocks, generate others by cycling

PBIB example is cyclic design with initial block (1243)

If \( k = a \) and rows also blocks get Latin square

Square, Cubic, and Rectangular Lattices

Square : \( a = k^2 \); Cubic : \( a = k^3 \); Rect: \( a = k(k + 1) \)

Square Example : Consider 9 trts and blocks of size three

```
1 2 3
4 5 6
7 8 9
```

Rep 1 blocks : (123)(456)(789) Using Rows

Rep 2 blocks : (147)(258)(369) Using Columns

Rep 3 blocks : (168)(249)(357) Using Latin Square

Additional Reps obtained from orthogonal squares

Youden Square

Latin Square with one row (col) deleted

Each trt occurs same number of times in each row (col)

Columns (rows) for BIBD

Analysis combination of Latin Square and BIBD

Partially Balanced Incomplete Block Design

 Doesn’t require each pair to occur together \( \lambda \) times

Pair in associate class \( i \) appears together \( \lambda_i \) times

All treatments have same # of \( i \)th associates

Plus additional restrictions on # of associates

Extensive list in Bose, Clatworthy, and Shrikhande (1954)

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