AN EXPLICIT EXPRESSION FOR THE DISTRIBUTION OF THE SUPREMUM OF BROWNIAN MOTION WITH A CHANGE-POINT*

by

Benzion Boukai

Purdue University
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Department of Statistics

Purdue university

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Abstract

The objective of this paper is to derive explicit expressions to distributions of the supremum of Brownian motion processes having a change-point. Such processes are characterized by a drift parameter which is subjected to a change over time. For that purpose, we make use of several straightforward results on the supremum of general Brownian bridge process, via conditioning arguments. Several different cases are considered and the resulting distributions are illustrated through their p.d.f.'s.

Keywords: **Brownian motion, change-point, Brownian bridge, Wiener process**

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1. Preliminaries.

Brownian motion and associated processes serve as a very useful limiting approximation for many statistical models. Frequently, such approximations appear in models involving partial sums (of r.v.'s) and in a large sample setup. Although, one can find examples in which these type of approximations yield some what poor quantitative results, their great advantage is as a comparison tool of competing procedures. The literature dealing with such approximations is extensive and covers many varieties of applications. For several examples of such applications we refer the reader to Sen (1981) and Siegmund (1985).

Approximation techniques, via processes associated with the Brownian motion process, (Brownian bridge and Bessell processes), can also be applied to many statistical models involving the change-point problem, as shown (amongst others,) by Hawkins (1986), Siegmund (1986) and James, James and Siegmund (1987). When the statistical model under consideration has a change-point, one would expect the approximating Brownian motion process to have a change-point as well. Moreover, in some applications involving the change-point problem, (see for example Boukai (1988)) it is required to express the approximating process in terms of the supremum process of a Brownian motion with a change-point. This type of process is characterized by a drift parameter which is subjected to a change over time. To the best of our knowledge, distributional characterizations of the supremum of such processes are not available in the literature. Such distribution plays an important role in providing the asymptotic power functions of several tests for a change-point, as shown in Boukai (1988) . Therefor The main objective of this paper, is to derive the distribution of the supremum of Brownian motion process having a change-point.

We begin in Section 2 with a short discussion on the important Brownian bridge process. We present several auxiliary results on the supremum of a general Brownian bridge process and provide an expression for the distribution of the supremum of a Brownian motion process with arbitrary drift. Although various formulations of the results in Section 2 can be found in the literature (see for example, Siegmund (1985) Sec. 3.3 and Shepp (1979)), we repersent them here however, with the intention to provide a self contained article and to shed some insights to their construction. These results are then used in Section 3 to derive the main results on the distributions of the supremum of Brownian motion processes having a change-point, as discussed in several cases.

2. The Brownian Bridge process.

Among the stochastic processes related to the Brownian motion process, the Brownian bridge process is considered to be one of great importance. This type of Gaussian process is naturally encountered in situations involving restricted (or rather, conditioned) random walks and similar processes. As will be shown, it also plays an important role in the derivations of our main results.

We begin with some notations. Let $\mathbf{W}_0 = \{W_0(t),\ 0 \le t < \infty\}$ be a standard Brownian motion (Wiener process), defined on some probability space $(\Omega, \mathcal{F}, P)$. We use the conventional term "standard Brownian motion" to refer to a Brownian motion process with mean zero and variance one per time-unit. For $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$, let $\mathbf{W}_\alpha^x = \{W_\alpha^x(t),\ 0 \le t < \infty\}$ denote the Brownian motion process obtained by adding $x + \alpha t$ to $W_0(t)$, $t \ge 0$, that is:

$$W_\alpha^x(t) = x + \alpha t + W_0(t), \quad t \ge 0. \tag{1.1}$$

We will often abbreviate the process W_α^0 by W_α. The process $\mathbf{W}_\alpha$ is usually referred to as Brownian motion with drift of size α.

Consider now a Brownian motion which begins at some arbitrary fixed point x, $x \in \mathbb{R}$. Such Brownian motion is being realized by $\mathbf{W}_0^x$. We wish to "deform" it so that it passes through a fixed point y, $y \in \mathbb{R}$, at a fixed time-point t_0, $(t_0 > 0)$. This type of constrained Brownian motion process is often called a Brownian Bridge or a tied Brownian motion process.

Let us denote by $\mathbf{B}_{t_0}^{(x,y)} = \{ B_{t_0}^{(x,y)}(t),\ 0 \le t \le t_0 \},\ x, y \in \mathbb{R}$, the Brownian bridge process as constructed above. We are interested in obtaining the distribution of the supremum over $[0, t_0]$ of $\mathbf{B}_{t_0}^{(x,y)}$ for any $x, y \in \mathbb{R}$, and a fixed t_0, $t_0 > 0$. In the literature, one can find many discussions on various versions of the Brownian bridge process $\mathbf{B}_{t_0}^{(x,y)}$. In particular, there are many results dealing with the standard version $B_1^{(0,0)}$, as well as results on its supremum. Such results can be found for exapmle, in Robbins and Siegmund (1970), who have used for their derivations martingale properties of certain functional of the process W_0. Here however, we have chosen to take the direct approach, based on a well known result by Doob (1949), to present the distribution of the supremum of the general Brownian bridge process $\mathbf{B}_{t_0}^{(x,y)}$.

As given in Hida (1980, pp. 109) the Brownian Bridge $\mathbf{B}_{t_0}^{(x,y)}$ has a general representation in terms of the standard Brownian motion process $\mathbf{W}_0$ as:

$$B_{t_0}^{(x,y)}(t) = W_0(t) + \frac{(t_0 - t)}{t_0} x - \frac{t}{t_0}(W_0(t_0) - y), \quad x, y \in \mathbb{R}. \tag{1.2}$$

It is clear that $\mathbf{B}_{t_0}^{(x,y)}$ is Gaussian process satisfying the required restrictions $B_{t_0}^{(x,y)}(0) = x$, $B_{t_0}^{(x,y)}(t_0) = y$. Furthermore, it is intuitively understood that a Brownian bridge process is a conditional Brownian motion process. We state that formally in the following trivial lemma.

LEMMA 1. *For arbitrary $x, \alpha \in \mathbb{R}$, let $\mathbf{W}_\alpha^x$ be the Brownian motion (1.1), and for $t_0 > 0$, $y \in \mathbb{R}$, let $\mathbf{B}_{t_0}^{(x,y)}$ be the Brownian bridge defined in (1.2), then*

$$\{ W_\alpha^x(s) \ \ 0 \le s \le t_0 \mid W_\alpha^x(t_0) = y \} \stackrel{\mathcal{D}}{=} \mathbf{B}_{t_0}^{(x,y)}.$$

The following lemma will be of great use in the construction of the distribution of $\sup_{0 \le t \le t_0} B_{t_0}^{(x,y)}(t)$ for any $x, y \in \mathbb{R}$, and $t_0 > 0$.

LEMMA 2. (Doob (1949)) *Let $\mathbf{W}_0$ be a standard Brownian motion and let $a \ge 0$, $b \ge 0$, then*

$$P(\sup_{t \ge 0}(W_0(t) - at - b) > 0) = e^{-2ab}$$

With the use of Lemma 2 we are now able to present the distribution of the supremum of the Brownian bridge process.

THEOREM 1. *Let* $\mathbf{B}_{t_0}^{(x,y)}$ *be the Brownian bridge (1.2) with* $t_0 > 0$, *and* $x, y \in \mathbb{R}$. *Then for* $z \geq 0$ *we have:*

$$P(\sup_{0<t\leq t_0} B_{t_0}^{(x,y)}(t) > z) = \begin{cases} e^{-2(z-x)(z-y)/t_0} & z > \max(x,y) \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $z > \max(x, y)$, from Lemma 1 and (1.1) we have:

$$P(\sup_{0<t\leq t_0} B_{t_0}^{(x,y)}(t) > z) = P(\sup_{0<t\leq t_0} W_\alpha^x(t) > z \mid W_\alpha^x(t_0) = y)$$

writing $t = t_0/(1+s)$, $s \in [0, \infty)$, and using the properties of the standard Brownian motion, the latter probability becomes

$$P(\sup_{s\geq 0}\{ W_0(\frac{t_0}{1+s}) + \frac{\alpha t_0}{1+s} + x\} > z \mid W_0(t_0) = y - x - \alpha t_0) =$$

$$= P(\sup_{s\geq 0}\{ \frac{1}{1+s}(t_0^{1/2} W_0(1+s) + \alpha t_0 - (z-x)(1+s))\} > 0 \mid W_0(1) = \frac{(y-x-\alpha t_0)}{t_0^{-1/2}})$$

$$= P(\sup_{s\geq 0}\{ (t_0^{1/2}\tilde{W}_0(s) + t_0^{1/2} W_0(1) + \alpha t_0 - (z-x)(1+s))\} > 0 \mid W_0(1) = \frac{(y-x-\alpha t_0)}{t_0^{1/2}})$$

where, $\tilde{W}_0(s) \equiv \{W_0(1+s) - W_0(1)\}$ is a standard Brownian motion, independent of $W_0(1)$. Hence, this conditional probability becomes

$$P(\sup_{s\geq 0}\{ (t_0^{1/2}\tilde{W}_0(s) - (z-x)s - (z-y))\} > 0) =$$

$$= P(\sup_{s\geq 0}\{ \tilde{W}_0(s) - \frac{(z-x)}{t_0^{1/2}} s - \frac{(z-y)}{t_0^{1/2}}\} > 0) = e^{-2(z-x)(z-y)/t_0}$$

where the last equality was obtained by using Lemma 2. For the case of $z \leq \max(x, y)$, the result follows by definition. ■

Remark 1. (i) As was mentioned before, Robbins and Siegmund (1970) have obtained the above result by a different approach.

(ii) It should be clear from Lemma 1 (as well as from Theorem 1) that the distribution of $\mathbf{B}_{t_0}^{(x,y)}$ (as well as of $\sup_{0<t\leq t_0} B_{t_0}^{(x,y)}(t)$) does not depend upon the drift parameter α of the original Brownian motion process.

One of the immediate applications of Theorem 1 is in obtaining the distribution of the supremum of a Brownian motion process with an arbitrary drift. We present this important result in the following theorem.

THEOREM 2. *Let $\alpha \in \mathbb{R}$ and let $\mathbf{W}_\alpha = \{W_\alpha(t),\ 0 \leq t < \infty\}$ be the Brownian motion process (1.1). For every $t > 0$ define: $M_\alpha(t) = \sup_{0\leq s\leq t} W_\alpha(s)$, then for $x \geq 0$*

$$P(M_\alpha(t) > x) = 1 - \Phi(\frac{x-\alpha t}{\sqrt{t}}) + e^{2\alpha x}(1 - \Phi(\frac{x+\alpha t}{\sqrt{t}}))\quad t > 0 \tag{1.3}$$

where Φ denotes the standard normal c.d.f. whose density function is given by:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \forall\, z \in \mathbb{R}.$$

Proof. Fix $t > 0$ and $x \geq 0$, by conditioning we have

$$P(M_\alpha(t) > x) = E\{\, P(\sup_{0\leq s\leq t} W_\alpha(s) > x \mid W_\alpha(t))\,\} \tag{1.4}$$

According to Lemma 1 and Theorem 1, the conditional probability at the r.h.s. of (1.4) can be written as:

$$\begin{aligned} P(\sup_{0\leq s\leq t} W_\alpha(s) > x \mid W_\alpha(t) = y) &= P(\sup_{0\leq s\leq t} B_t^{(0,y)}(s) > x) \\ &= \begin{cases} e^{-2x(x-y)/t} & x > y \\ \\ 1 & x \leq y \end{cases} \end{aligned} \tag{1.5}$$

for any given $y \in \mathbb{R}$. Since $W_\alpha(t) \sim N(\alpha t, t)$, we obtain from (1.4) and (1.5) that:

$$P(M_\alpha(t) > x) = \int_x^\infty \phi(\frac{y-\alpha t}{\sqrt{t}})\frac{dy}{\sqrt{t}} + \int_{-\infty}^x e^{-2x(x-y)/t}\phi(\frac{y-\alpha t}{\sqrt{t}})\frac{dy}{\sqrt{t}},$$

where by an appropriate change of variable, yields (1.3) as required. ■

The results presented in the following two corollaries are well-known, yet we show that they can be obtained directly from Theorem 2.

COROLLARY 1. *For any* $\alpha \in \mathbb{R}$ *and* $x \geq 0$*;*

$$\lim_{t\to\infty} P(M_\alpha(t) > x) = \begin{cases} e^{2\alpha x} & \alpha < 0 \\ \\ 1 & \alpha \geq 0 \end{cases}$$

COROLLARY 2. *Let* $\{ M_0(t),\ 0 < t < \infty \}$ *be the supremum process of the standard Brownian motion* $\mathbf{W}_0$*, then for* $x \geq 0$*;*

$$P(M_0(t) > x) = 2(1 - \Phi(\frac{x}{\sqrt{t}})) \quad \forall\, t > 0.$$

Proof. Substitute $\alpha \equiv 0$ in (1.3). ■

Remark 2. (i) Robbins and Siegmund (1970), obtained, by a different approach, an expression similar to (1.3).

(ii) Traditionally, the proof of Corollary 2 uses the reflection principle for Brownian motion.

(iii) One can obtain the result of Corollary 1 directly from Lemma 2, since for $x \geq 0$;

$$P(\sup_{t>0} W_\alpha(t) > x) = P(\sup_{t>0}\{ W_0(t) + \alpha t \} > x) = \begin{cases} e^{2\alpha x} & \alpha < 0 \\ \\ 1 & \alpha \geq 0 \end{cases}$$

Furthermore, if the original process has a constant shift of size z, $z \in \mathbb{R}$, then again; Lemma 2 implies:

$$\begin{aligned} P(\sup_{t>0} W_\alpha^z(t) > x) &= P(\sup_{t>0}\{ W_0(t) + \alpha t + z \} > x) \\ &= \begin{cases} e^{2\alpha(x-z)} & \alpha < 0,\ x > z \\ \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

3. Brownian motion with a change-point.

In the present section we consider a Brownian motion process, which has a change-point at some fixed time. We derive the distribution of the supremum of such a process in several cases as described below.

Let T denote the time interval $[0,1]$ or $[0,\infty)$ and let $\mathbf{W}_0 = \{W_0(t),\ t \in T\}$ be a standard Brownian motion process (over T). Given a fixed instant of time τ, $0 < \tau \in T$, and constants $\alpha_0, \alpha_1 \in \mathbf{R}$, $(\alpha_0 \neq \alpha_1)$, consider the process $\tilde{\mathbf{X}}_\tau = \{\tilde{X}_\tau(t),\ t \in T\}$ satisfying:

$$\tilde{X}_\tau(t) = \begin{cases} \alpha_0 t + W_0(t), & 0 \le t \le \tau \\ \alpha_0 \tau + \alpha_1 (t-\tau) + W_0(t), & \tau < t \in T. \end{cases} \tag{3.1}$$

The process $\tilde{\mathbf{X}}_\tau$ as given in (3.1) is called a Brownian motion process with a change-point (at time τ). Indeed, up to the time instant τ, the process $\tilde{\mathbf{X}}_\tau$ is a Brownian motion process with a drift of size α_0, while, for all times passed the time instant τ, this process has a drift of size α_1. The time instant τ in which the change in the drift occurred, is therefore called the change-point of the process $\tilde{\mathbf{X}}_\tau$. Let $\tilde{M}_\tau$ denote the supremum of the process $\tilde{\mathbf{X}}_\tau$ over T, that is

$$\tilde{M}_\tau = \sup_{t \in T} \tilde{X}_\tau(t). \tag{3.2}$$

We wish to find the distribution of $\tilde{M}_\tau$ for any given fixed τ. This distribution will indeed depend upon the change-point τ and the drifts α_0, α_1 as well as on the time interval T. Accordingly, while constructing the distribution of (3.2), we will differentiate between the two cases: (i) $T = [0,1]$, (ii) $T = [0,\infty)$. These two cases, while being essentially similar, require somewhat different derivations of the distribution of $\tilde{M}_\tau$.

For convenience sake, we consider *at first* a simplified version of the process (3.1) corresponding to the case of $\alpha_0 \equiv 0$. This will enable us to furnish in Theorems 3 and 4 below the results on the distribution of (3.2) in a more presentable fashion. The results corresponding to the process (3.1) in its general form are given at the end of this section. Hence, until stated otherwise, we take in (3.1) $\alpha_0 \equiv 0$ and put $\alpha_1 \equiv \delta$. Accordingly, we will consider the process $\mathbf{X}_\tau$ given by:

$$X_\tau(t) = \delta(t-\tau)^+ + W_0(t), \qquad t \in T \tag{3.3}$$

where $(x)^+ \equiv \max(0,x)$, together with its respective supremum $M_\tau = \sup\{X_\tau(t),\ t \in T\}$. In the following two theorems we present the distribution function of the r.v. M_τ for the two cases; (i) and (ii) respectively.

THEOREM 3. *(case (i)). Suppose that $T = [0,1]$ and consider the process $\mathbf{X}_\tau$ as given in (3.3), with a change-point at time τ, $0 < \tau < 1$. Then the distribution function of M_τ is given by:*

$$P(M_\tau \le z) = \begin{cases} \tilde{Q}(z;\delta,\tau) - \tilde{Q}(-z;\delta,\tau) & z \ge 0 \\ 0 & z < 0 \end{cases}$$

where for any $z \in \mathbb{R}$, $\delta \in \mathbb{R}$ and $0 < \tau < 1$ we define

$$\begin{aligned} \tilde{Q}(z;\delta,\tau) = \Phi(\frac{z}{\sqrt{\tau}}) - \int_{-\infty}^{z/\sqrt{\tau}} \phi(y)\Phi(ay+b_1)dy \\ - e^{2\delta^2\tau+2\delta z} \int_{-\infty}^{(2\delta\tau+z)/\sqrt{\tau}} \phi(y)\Phi(ay+b_2)dy \end{aligned} \tag{3.4}$$

with the quantities a, b_1 and b_2 are given by

$$a = (\frac{\tau}{1-\tau})^{1/2}, \quad b_1 = -(\frac{z-\delta(1-\tau)}{\sqrt{1-\tau}}), \quad b_2 = -(\frac{z+\delta(1+\tau)}{\sqrt{1-\tau}}).$$

THEOREM 4. *(case (ii)). Suppose that $T = [0,\infty)$ and consider the process $\mathbf{X}_\tau$ as given in (3.3) with a change-point at time τ, $\tau > 0$. Then the distribution function of M_τ is given by:*

$$P(M_\tau \le z) = \begin{cases} Q(z;\delta,\tau) - Q(-z;\delta,\tau), & \delta < 0, \ \ z \ge 0 \\ 0, & \delta \ge 0 \text{ or } z < 0 \end{cases}$$

where for all $\rho > 0$, $\mu < 0$ and $y \in \mathbf{R}$, we define:

$$Q(y;\mu,\rho) = \Phi(\frac{y}{\sqrt{\rho}}) - e^{2\mu^2\rho+2\mu y}\Phi(\frac{2\mu\rho + y}{\sqrt{\rho}}). \tag{3.5}$$

Proof of Theorem 3. Let $z \geq 0$, since $T = [0,1]$ and $\tau \in T$ is fixed, we have by conditioning that;

$$P(M_\tau \leq z) = E\{\, P(\sup_{0\leq t\leq \tau} X_\tau(t) \leq z,\ \sup_{0\leq h\leq \rho} X_\tau(\tau+h) \leq z \mid X_\tau(\tau), X_\tau(1))\,\} \tag{3.6}$$

where $\rho = 1 - \tau$. Using (3.3), together with the properties of the standard Brownian motion and Lemma 1, one can easily show that for given $x, y \in \mathbb{R}$

(a) $\sup_{0\leq t\leq \tau} X_\tau(t)\big|_{X_\tau(\tau)=x, X_\tau(1)=y} \stackrel{\mathcal{D}}{\equiv} \sup_{0\leq t\leq \tau} W_0(t)\big|_{W_0(\tau)=x} \stackrel{\mathcal{D}}{\equiv} \sup_{0\leq t\leq \tau} B_\tau^{(0,x)}(t).$

(b) $\sup_{0<h\leq \rho} X_\tau(\tau+h)\big|_{X_\tau(\tau)=x, X_\tau(1)=y} \stackrel{\mathcal{D}}{\equiv} \sup_{0<h\leq \rho} W_\delta^x(h)\big|_{W_\delta^x(\rho)=y} \stackrel{\mathcal{D}}{\equiv} \sup_{0<h\leq \rho} B_\rho^{(x,y)}(W).$

Furthermore, the independent increment property of the Brownian motion process implys that:

(c) given $X_\tau(\tau) = x$, $X_\tau(1) = y$, the two restrictions of the process $\mathbf{X}_\tau$ to the time-intervals $[0,\tau]$ and $(\tau, 1]$ are conditionally, independent.

By using statements (a) - (c) above and Theorem 1 we obtain that for $z \geq \max(x,y)$

$$\begin{aligned} &P(\sup_{0\leq t\leq \tau} X_\tau(t) \leq z,\ \sup_{0<h\leq \rho} (X_\tau(\tau+h) \leq z \mid X_\tau(\tau) = x,\ X_\tau(1) = y) \\ &= (1 - e^{-2z(z-x)/\tau})(1 - e^{-2(z-x)(z-y)/\rho}). \end{aligned} \tag{3.7}$$

Moreover since $X_\tau(\tau) \sim N(0,\tau)$ and $X_\tau(1)\big|_{X_\tau(\tau)=x} \stackrel{\mathcal{D}}{\equiv} W_\delta^x(\rho) \sim N(x + \delta\rho, \rho)$, we have by (3.6) and (3.7) that for $z \leq 0$ $P(M_\tau \leq z) = 0$ while for $z > 0$

(3.8)

$$\begin{aligned} P(M_\tau \leq z) &= \int_{-\infty}^{z}\int_{-\infty}^{z} (1 - e^{-2z(z-x)/\tau})(1 - e^{-2(z-x)(z-y)/\rho})\phi(\frac{x}{\sqrt{\tau}})\phi(\frac{y-\delta\rho-x}{\sqrt{\rho}})\frac{dy}{\sqrt{\rho}}\frac{dx}{\sqrt{\tau}} \\ &\equiv J_1 - J_2 - J_3 + J_4. \end{aligned}$$

Using ordinary change-of-variables techniques the integrals $J_1, \ldots, J_4$ can be presented in an explicit form. Let $a = (\tau/\rho)^{1/2}$, $(\rho = 1 - \tau)$ then we have:

$$J_1 = \Phi(\frac{z}{\sqrt{\tau}}) - \int_{-\infty}^{z/\sqrt{\tau}} \phi(x)\Phi(ax + \frac{\delta\rho - z}{\sqrt{\rho}})dx$$

$$J_2 = \Phi(\frac{-z}{\sqrt{\tau}}) - \int_{-\infty}^{-z/\sqrt{\tau}} \phi(x)\Phi(ax + \frac{(\delta\rho + z)}{\sqrt{\rho}})dx$$

$$J_3 = e^{2\delta^2\tau + 2\delta z} \int_{-\infty}^{(2\delta\tau + z)/\sqrt{\tau}} \phi(x)\Phi(ax - \frac{(\delta(1+\tau) + z)}{\sqrt{\rho}})dx$$

and

$$J_4 = e^{2\delta^2\tau - 2\delta z} \int_{-\infty}^{(2\delta\tau - z)/\sqrt{\tau}} \phi(x)\Phi(ax - \frac{(\delta(1+\tau) - z)}{\sqrt{\rho}})dx.$$

Finally, by collecting the terms of $J_1, \ldots, J_4$ into the form of the function $\tilde{Q}(\cdot)$ in (3.4) the proof is completed. ■

The proof of Theorem 4 is similar to that of Theorem 3. Since in case (ii), $T = [0, \infty)$, we would replace the r.h.s. of (3.6) with $E\{\, P(\sup_{0 \le t \le \tau} X_\tau(t) \le z,\ \sup_{h>0} X_\tau(\tau + h) \le z \mid X_\tau(\tau))\,\}$ and use Theorem 1 and Remark 2 (iii), to establish that for $\delta < 0$ and $z > 0$

$$P(M_\tau \le z) = \int_{-\infty}^{z} (1 - e^{-2z(z-x)/\tau})(1 - e^{2\delta(z-x)})\phi(\frac{x}{\sqrt{\tau}})\frac{dx}{\sqrt{\tau}}$$

$$= Q(z; \delta, \tau) - Q(-z; \delta; \tau)$$

where the function $Q(\cdot)$ is defined in (3.5).

Remark 3. It should be noted that the integrals appearing in definition (3.4) of the $\tilde{Q}(\cdot)$ function could be written in terms of a bivariate normal c.d.f.

The premise of Theorem 3 can be extended easily to a more general case where the time interval T is any closed interval of the form $T = [0, \tau^*]$, for some $\tau^* > 0$, with $\tau^* > \tau$. To obtain the exact distribution of $M_\tau = \sup_{0 \le t \le \tau^*} X_\tau(t)$, one has only to replace the quantities a, b_1 and b_2 which appear in Definition (3.4) of the $\tilde{Q}(\cdot)$ function, with those of

$$a^* = (\frac{\tau}{\tau^* - \tau})^{1/2}, \quad b_1^* = -(\frac{z - \delta(\tau^* - \tau)}{\sqrt{\tau^* - \tau}}), \quad b_2^* = -(\frac{z + \delta(\tau^* + \tau)}{\sqrt{\tau^* - \tau}})$$

and to apply to Theorem 3 appropriately. It should also be noted that case (ii) can be viewed as a limiting case for case (i) of Theorem 3, with $T = [0, \tau^*]$. Indeed, by passing to the limit as $\tau^* \to \infty$, the results of Theorem 3 imply those of Theorem 4, as can be seen directly from the definitions (3.5) and (3.4) of the functions $Q(\cdot)$ and $\tilde{Q}(\cdot)$ (respectively). Furthermore, by assuming that no change in the drift of the Brownian motion $\mathbf{X}_\tau$ has occurred, we would expect that the distribution of its supremum should be independent of the value of τ, - now being just an arbitrary "break" point for the process $\mathbf{X}_\tau$. This is equivalent to assuming that $\delta \equiv 0$, meaning that the Brownian motion $\mathbf{X}_\tau$ retains the same drift before and after the time instant τ. Indeed by using certain properties of the $\tilde{Q}(\cdot)$ function, we obtain that for $\delta \equiv 0$ Theorem 3 yields

$$P(M_\tau \leq z) = 2\Phi(\frac{z}{\sqrt{\tau^*}}) - 1, \qquad z \geq 0$$

for any $\tau \in (0, \tau^*)$. This of course coincides with Corollary 2.

Let $\tau \in (0,1)$, and consider now the supremum $\tilde{M}_\tau$ of the process $\tilde{\mathbf{X}}_\tau = \{\tilde{X}_\tau(t),\ 0 \leq t \leq 1\}$, as defined in (3.1). That is, for $\alpha_0, \alpha_1 \in \mathbb{R}$

$$\tilde{X}_\tau(t) = W_0(t) + \alpha_0(t \wedge \tau) + \alpha_1(t-\tau)^+, \qquad 0 \leq t \leq 1, \tag{3.9}$$

where $(x \wedge y) \equiv \min(x,y)$. We use the proof of Theorem 3 to present the following general result.

THEOREM 5. *For $\alpha_0, \alpha_1 \in \mathbb{R}$, let $\tilde{\mathbf{X}}_\tau$ be defined in (3.9), then the distribution function of the random variable $\tilde{M}_\tau$ is given by:*

$$P(\tilde{M}_\tau \leq z) = \begin{cases} \tilde{Q}(z - \alpha_0\tau, \alpha_1, \tau) - e^{2\alpha_0 z}\tilde{Q}(-z-\alpha_0\tau, \alpha_1, \tau) & z \geq 0 \\ 0 & z < 0. \end{cases} \tag{3.10}$$

Proof. The proof is essentially the one given for Theorem 3, which together with Remark 1 (ii) provide us with a similar expression to (3.7). There is, however, one distinction. Since now, the random variable $\tilde{X}_\tau(\tau)$ is normal with mean $\alpha_0\tau$ and variance τ, one has to substitute $\phi(\frac{x-\alpha_0\tau}{\sqrt{\tau}})$ for $\phi(\frac{x}{\sqrt{\tau}})$ in (3.9). Accordingly we have (for $z \geq 0$);

$$P(\tilde{M}_\tau \leq z) = \int_{-\infty}^{z}\int_{-\infty}^{z}(1-e^{-2(z-x)z/\tau})(1-e^{-2(z-x)(z-y)/\rho})\phi(\frac{x-\alpha_0\tau}{\sqrt{\tau}})$$
$$\cdot\,\phi(\frac{y-x-\alpha_1\rho}{\sqrt{\rho}})\frac{dy}{\sqrt{\rho}}\frac{dx}{\sqrt{\tau}}.$$

The result is then obtained by straightforward integrations and the use of the definition (3.4) for the function $\tilde{Q}(\cdot)$.

Remark 4. (i) The results of Theorem 3 can be obtained from Theorem 5 by substituting $\alpha_0 \equiv 0$ in (3.10).

(ii) Further examination of (3.10) reveals that if $\alpha_0 = \alpha_1 \equiv \alpha$ (say) (that is the process $\tilde{\mathbf{X}}_\tau$ has no change-point and therefore $\tilde{\mathbf{X}}_\tau \equiv \mathbf{W}_\alpha$), then

$$P(\tilde{M}_\tau \leq z) = \Phi(z-\alpha) - e^{2\alpha z}(1-\Phi(z+\alpha)), \qquad z \geq 0.$$

which coincides with the result stated in Theorem 2.

We close this paper with a couple of illustrations. In Figures 1-2 below, we present the plotted graphs of the (approximated) probability density function of the r.v. $\tilde{M}_\tau$ for $\tau = 0.2, 0.5$ and 0.8, and for different choices of α_0 and α_1. Ordinary numerical integration techniques were used in conjunction with Theorem 5 to evaluate the grid points required for these graphs.

Figure 1

the p.d.f. of $\tilde{M}_\tau$ for $\alpha_0 = 1$, $\alpha_1 = -1$

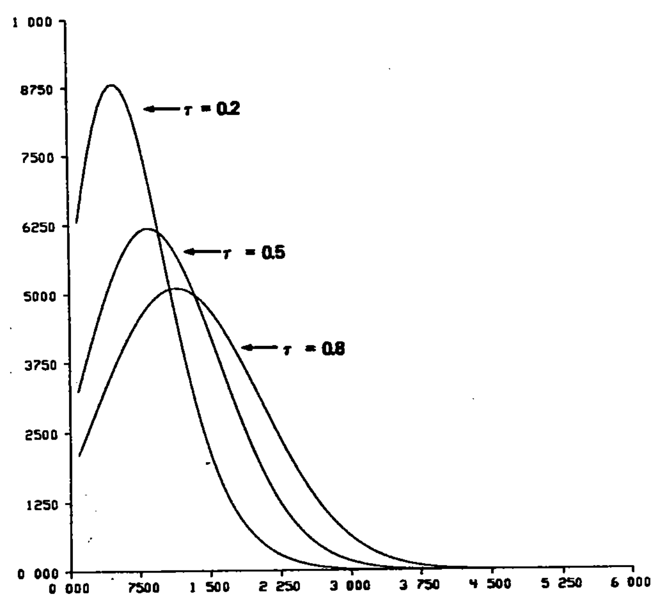
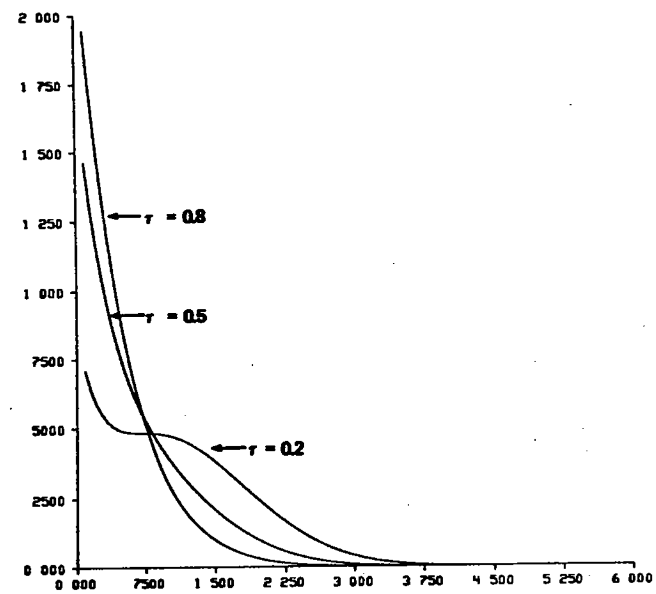


Figure 2

the p.d.f. of $\tilde{M}_\tau$ for $\alpha_0 = -1$, $\alpha_1 = 1$.



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