EMPIRICAL BAYES TESTS FOR
SOME NON-EXPONENTIAL DISTRIBUTION FAMILY

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Abstract

Empirical Bayes tests for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ in the distribution family having density $f(x|\theta) = I_{[0 \leq x \leq \theta]} a(x)/A(\theta)$ are considered under the linear loss. We construct the empirical Bayes tests $\delta_n$ and $\tilde{\delta}_n$ and prove that both of them have a good asymptotic property. As a special case, taking $a(x) = 1$ leads to a result obtained by Karunamuni (1999). But our result gets rid of the condition that the critical point $b_G$ falls in some interval $[0, \rho_0]$, where $\rho_0 < \theta_0$ is a known constant.

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§ 1 Introduction

Let $X$ denote a random variable having density function

$$f(x | \theta) = \frac{a(x)}{A(\theta)} I_{[0 \leq x < \theta]}, \quad (1.1)$$

where $a(x)$ is a function on $[0, \infty)$, and continuous, positive for $x > 0$, $A(\theta) = \int_0^\theta a(x)dx < \infty$ for every $\theta \geq 0$, $\theta$ is the parameter, which is distributed according to an unknown prior distribution $G$ on $[0, \infty)$.

We consider the problem of testing the hypotheses $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where $\theta_0 > 0$, is a known constant.

Let $a = i$ be the action in favor of $H_i$. For the parameter $\theta$ and action $a$, we use the loss function

$$l(\theta, a) = a(\theta_0 - \theta) I_{[\theta \leq \theta_0]} + (1 - a)(\theta - \theta_0) I_{[\theta > \theta_0]}. \quad (1.2)$$

Assume that

$$\int_\Omega \theta dG(\theta) < \infty. \quad (1.3)$$

Define

$$\alpha_G(x) = \int_x^\infty \frac{1}{A(\theta)} dG(\theta),$$

and

$$\psi_G(x) = \int_x^\infty \frac{\theta}{A(\theta)} dG(\theta).$$

By Fubini Theorem,

$$\int_0^\infty a(x) \alpha_G(x) dx \quad (1.4)$$

$$= \int_0^\infty a(x) \int_x^\infty \frac{1}{A(\theta)} dG(\theta) dx$$

$$= \int_0^\infty \int_0^\theta \frac{a(x)}{A(\theta)} I_{[0 < x < \theta]} dx dG(\theta) = 1$$

and

$$\int_0^\infty a(x) \psi_G(x) dx \quad (1.5)$$

$$\leq \int_0^\infty a(x) \int_x^\infty \frac{\theta}{A(\theta)} dG(\theta) dx$$

$$= \int_0^\infty \theta \int_0^\theta \frac{a(x)}{A(\theta)} I_{[0 < x < \theta]} dx dG(\theta)$$

$$= \int_0^\theta \theta dG(\theta) < \infty.$$
Let $W(x) = \theta_0 \alpha_G(x) - \psi_G(x)$. Using (1.4) and (1.5), we have

$$
\int_0^\infty |W(x)|a(x)dx < \infty.
$$

A test $\delta(x)$ is defined to be a measurable mapping from $(0, \infty)$ into $[0, 1]$ so that $\delta(x) = P\{\text{accepting } H_1 | X = x\}$, i.e., $\delta(x)$ is the probability of accepting $H_1$ when $X = x$ is observed.

Let $R(G, \delta)$ denote the Bayes risk of the test $\delta$ when $G$ is the prior distribution. Then $R(G, \delta)$ can be expressed as

$$
R(G, \delta) = C_G + \int_0^\infty \int_0^\infty \delta(x)(\theta_0 - \theta)f(x|\theta)dxdG(\theta) \tag{1.7}
$$

$$
= C_G + \int_0^\infty \delta(x) \left[ \int_0^\infty \frac{1}{A(\theta)} dG(\theta) \right] a(x)dx
$$

$$
= C_G + \int_0^\infty \delta(x) \left[ \theta_0 \alpha_G(x) - \psi_G(x) \right] a(x)dx
$$

$$
= C_G + \int_0^\infty \delta(x) W(x) a(x)dx
$$

$$
= C_G + \int_0^\infty \delta(x) (\theta_0 - \phi_G(x)) \alpha_G(x) a(x)dx,
$$

where

$$
C_G = \int_0^\infty (\theta - \theta_0) I_{[\theta > \theta_0]} dG(\theta),
$$

and

$$
\phi_G(x) = \frac{\psi_G(x)}{\alpha_G(x)}.
$$

Here, $\phi_G(x)$ is the posterior mean of $\theta$ given $X = x$. $\phi_G(x)$ is continuous and increasing in $x$.

From (1.7), we see that a Bayes test $\delta_G$ is determined by

$$
\delta_G(x) = \begin{cases} 
1 & \text{if } W(x) \leq 0 \\
0 & \text{if } W(x) > 0
\end{cases} \tag{1.8}
$$

$$
= \begin{cases} 
1 & \text{if } \phi_G(x) \geq \theta_0 \\
0 & \text{if } \phi_G(x) < \theta_0.
\end{cases}
$$

The minimum Bayes risk is

$$
R(G, \delta) = C_G + \int_0^\infty \delta_G(x) W(x) a(x)dx. \tag{1.9}
$$

To exclude trivial cases, we assume that

$$
\begin{cases} 
\phi_G(0) < \theta_0 \\
G(\theta_0) \neq 1
\end{cases} \tag{1.10}
$$

3
From (1.10), we see that \( \phi_G(x) \) is strictly increasing and there exists a unique point \( b_\sigma < \theta_0 \) such that \( \phi_G(b_\sigma) = \theta_0 \). \( \phi_G(x) < \theta_0 \) for \( x < b_\sigma \), and \( \phi_G(x) > \theta_0 \) for \( x > b_\sigma \). Therefore, the Bayes test \( \delta_G \) can be represented as

\[
\delta_G(x) = \begin{cases} 
1 & \text{if } x \geq b_\sigma \\
0 & \text{if } x < b_\sigma.
\end{cases}
\]

We assume that, for some constant \( B > 0 \),

\[
\sup_{0 \leq x \leq \theta_{0+1}} |\alpha_G^{(i)}(x)| \leq B,
\]

where \( i = 0, 1, \ldots, r \) and \( r \geq 1 \). Note that (1.11) implies that \( G'(x) \) exists for \( 0 \leq x \leq \theta_0 + 1 \). Furthermore, we assume that

\[
G'(b_\sigma) \neq 0.
\]

We will deal with this testing problem via the empirical Bayes approach. The empirical Bayes approach was introduced first by Robbins (1956, 1964). Let \( X_1, X_2, \ldots, X_n \) denote the observations from \( n \) independent past experiences. Let \( X \) be the present observation. Denote \( \bar{X}_n = (X_1, X_2, \ldots, X_n) \). An empirical Bayes test \( \delta_n(X, \bar{X}_n) \) is defined to be the probability of accepting \( H_1 \) when \( X \) and \( \bar{X}_n \) are observed. Let \( R(G, \delta_n|\bar{X}_n) \) denote the Bayes risk of \( \delta_n \) conditioning on \( \bar{X}_n \) and \( R(G, \delta) = E[R(G, \delta|\bar{X}_n)] \) the overall (unconditional) Bayes risk of \( \delta_n \).

Since \( R(G, \delta_G) \) is the minimum Bayes risk, \( R(G, \delta_n|\bar{X}_n) - R(G, \delta_G) \geq 0 \) for all \( \bar{X}_n \) and for all \( n \). Thus, the regret \( R(G, \delta_n) - R(G, \delta_G) \geq 0 \) for all \( n \). The nonnegative regret \( R(G, \delta_n) - R(G, \delta_G) \) is often used as a measure of performance of the empirical Bayes test of \( \delta_n \).

Empirical Bayes problem for the Uniform \((0, \theta)\), a special case of (1.1), was studied by a number of authors: Fox (1978), Van Houwelingen (1987), Nogami (1988), Liang (1990) and Karunamuni (1999). For the distribution family having density (1.1), Gupta and Hsiao (1983) considered the empirical Bayes rules in the selection problem formulation, Datta (1991) studied the empirical Bayes rule in the estimation problem formulation.

In this paper, we consider the empirical Bayes rule in the testing problem formulation. For clarity, we consider the different cases of \( a(x) \):

Case 1: \( a(x) \rightarrow a_0 \), where \( 0 < a_0 < \infty \), as \( x \downarrow 0 \),

Case 2: \( a(x) \downarrow 0 \) as \( x \downarrow 0 \),

Case 3: \( a(x) \uparrow \infty \) as \( x \downarrow 0 \),

Case 4: \( a(x) \rightarrow 0 \) as \( x \downarrow 0 \),
Case 5: \( a(x) \to \infty \) as \( x \downarrow 0 \),

where \( \uparrow ( \text{ or } \downarrow ) \) stands for “goes to increasingly” (or “goes to decreasingly”, respectively).

Although Case 2 is the special case of Case 4 and Case 3 is the special case of Case 5, our approach (or result) is a little different between Case 2 and Case 4. Also our approach is different between Case 3 and Case 5. So we treat Case 2 and Case 3 as the separate cases.

Define

\[ G_1 = \{ G : G \text{ satisfies } (1.10), (1.11) \text{ and } (1.12) \} \]

For some \( \xi > 0, L > 0 \) and \( b_0 > 0 \) in Cases 2, 3, 4, 5, \( b_0 = 0 \) in case 1, define

\[ G_2 = \{ G : G \text{ satisfies } (1.10) \text{ and } (1.11), b_G > b_0, \min_{x \in N_{b_G}(\xi)} |W'(x)| \geq L \}, \]

where \( N_{b_G}(\xi) = \{ x : 0 \lor (b_G - \xi) \leq x \leq (b_G + \xi) \land \theta_0 \} \). Then we can construct a Bayes test \( \delta_n \) such that its regret has a convergence rate of order \( O(n^{-\frac{2r}{2r+1}}} \) or \( O(n^{-\frac{2r}{2r+1}}} \epsilon_n^{-1}) \) (different rates in different cases) for any \( G \in G_1 \), where \( \epsilon_n \) is any prespecified (large) positive sequence such that \( \epsilon_n \to 0 \) as \( n \to \infty \). And we can also construct another empirical Bayes test \( \tilde{\delta}_n \) such that its regret has a uniform convergence rate of order \( O(n^{-\frac{2r}{2r+1}}} \) over the class \( G_2 \). Taking \( a(x) = 1 \), we would get that \( \sup_{G \in G_2} R(G, \delta) - R(G, \delta) = O(n^{-\frac{2r}{2r+1}}} \), which is a result obtained by Karunamuni (1999). But we exclude the condition that \( b_G \) falls in some known interval \([0, \rho_0]\), where \( \rho_0 < \theta_0 \) is a known constant.

The paper is organized as follows: §1 gives the introduction; §2 constructs the empirical Bayes test \( \delta_n \); §3 proves that the empirical Bayes test has a good asymptotic property. §4 proves the lemmas stated in §3.

§ 2 Construction of Empirical Bayes Tests

We use the kernel method to construct the empirical Bayes tests. Let \( K_0(y) \) be a Borel-measurable, bounded function vanishing outside the interval \([0, 1]\) such that

\[ \int_0^1 y^j K_0(y)dy = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j = 1, 2, \ldots, r - 1, \end{cases} \quad (2.1) \]

and let

\[ K_1(y) = \int_0^y K_0(s)ds. \quad (2.2) \]

We may let \( B_1 \) be a positive constant such that \( |K_0(y)| \leq B_1 \) for all \( y \in [0, 1] \). Let \( \epsilon_n \) be any (large) positive sequence such that \( \epsilon_n \to 0 \). Without loss of generality, assume
\[ \alpha_n(x) = \frac{1}{nu} \sum_{j=1}^{n} \frac{K_0(\frac{X_j-x}{u})}{a(X_j)}, \]  
(2.3.1)

\[ \psi_n(x) = \frac{x}{nu} \sum_{j=1}^{n} \frac{K_0(\frac{X_j-x}{u})}{a(X_j)} - \frac{1}{n} \sum_{j=1}^{n} \frac{K_1(\frac{X_j-x}{u})}{a(X_j)}. \]  
(2.3.2)

and for Case 3 and Case 5,

\[ \alpha_n(x) = -\frac{1}{nu} \sum_{j=1}^{n} \frac{K_0(\frac{x-X_j}{u})}{a(X_j)}, \]  
(2.4.1)

\[ \psi_n(x) = -\frac{x}{nu} \sum_{j=1}^{n} \frac{K_0(\frac{x-X_j}{u})}{a(X_j)} + \frac{1}{n} \sum_{j=1}^{n} \frac{K_1(\frac{x-X_j}{u})}{a(X_j)}. \]  
(2.4.2)

Let \( W_n(x) = \theta_0 \alpha_n(x) - \psi_n(x) \). We shall show later that \( W_n(x) \) is an asymptotically unbiased and consistent estimators of \( W(x) \) (Lemma 3.1). For any \( G \in \mathcal{G}_1 \), we propose an empirical Bayes test \( \delta_n(x, \overline{X}_n) \) by

\[ \delta_n = \begin{cases} 
1 & \text{if } (x > \theta_0) \text{ or } (d_n \leq x \leq \theta_0 \text{ and } W_n(x) \leq 0), \\
0 & \text{if } (x < d_n) \text{ or } (d_n \leq x \leq \theta_0 \text{ and } W_n(x) > 0),
\end{cases} \]  
(2.5)

where

\[ d_n = \begin{cases} 
0 & \text{for Case 1}, \\
\max \{x : a(x) < \eta_n\} & \text{for Case 2 and Case 4 }, \\
\max \{x : a(x) > \frac{1}{\eta_n}, a(x) > \frac{1}{u}\} + u & \text{for Case 3 and Case 5 },
\end{cases} \]  
(2.6)

and \( \eta_n = \frac{2\pi+1}{\xi_n^{\frac{1}{r}}} \). Since \( \eta_n \to 0 \) as \( n \to \infty \), we know that \( d_n \to 0 \) as \( n \to \infty \). Suppose \( N_0 \) is the smallest positive integer such that \( b_G > d_{N_0} \). Then \( b_G > d_n \) for \( n > N_0 \). When we discuss \( \delta_n \), we always assume that \( n > N_0 \) without further mention in this paper.

Note that \( W(x) \leq 0 \) if \( x \in (0, b_G] \); \( W(x) \geq 0 \) if \( x \in [b_G, \theta_0] \). Then the conditional regret of \( \delta_n \) can be expressed as

\[ R(G, \delta_n | \overline{X}_n) - R(G, \delta) = \int_0^{\infty} (\delta_n - \delta) W(x) a(x) dx \]  
(2.7)

\[ = \int_{d_n}^{b_G} I_{[W_n(x) \leq 0]} W(x) a(x) dx \]

\[ + \int_{\theta_0}^{b_G} I_{[W_n(x) > 0]} W(x) a(x) dx \]
and the unconditional regret of $\delta_n$ becomes

$$R(G, \delta_n) - R(G, \delta) = \int_{b_n}^{b_G} P(W_n(x) \leq 0) W(x) a(x) dx \quad (2.8)$$

$$+ \int_{\theta_0}^{\theta_G} P(W_n(x) > 0) |W(x)| a(x) dx.$$

For $G \in \mathcal{G}_2$, we also propose another empirical Bayes procedure $\delta_n$ by

$$\delta_n = \begin{cases} 
1 & \text{if } (x > \theta_0) \text{ or } (b_0 \leq x \leq \theta_0 \text{ and } W_n(x) \leq 0), \\
0 & \text{if } (x < b_0) \text{ or } (b_0 \leq x \leq \theta_0 \text{ and } W_n(x) > 0). 
\end{cases} \quad (2.9)$$

Then the unconditional regret of $\delta_n$ becomes

$$R(G, \delta_n) - R(G, \delta) = \int_{b_0}^{b_G} P(W_n(x) \leq 0) W(x) a(x) dx \quad (2.10)$$

$$+ \int_{\theta_0}^{\theta_G} P(W_n(x) > 0) |W(x)| a(x) dx.$$

In the following section, the convergence rate of unconditional regret of $\delta_n$ for every $G \in \mathcal{G}_1$ and the uniform convergence rate of unconditional regret of $\delta_n$ over $\mathcal{G}_2$ are considered.

### §3 Asymptotic Optimality of $\delta_n$ and $\delta_n$

The convergence rates of $\delta_n$ and $\delta_n$ depend on the properties of $W(x)$ and $W_n(x)$. The more information about $W(x)$ and $W_n(x)$ (including $a(x)$) is used, the more accurate rate we will get. So firstly, we dig out a few properties of $W_n(x)$ and $W(x)$. That is a few lemmas, whose proofs are left to §4. Then we state a well-known fact. Following that, two theorems about asymptotic optimalities of $\delta_n$ and $\delta_n$ are given.

Note that

$$W_n(x) = \theta_0 \alpha_n(x) - \psi_n(x) = \frac{1}{n} \sum_{j=1}^{n} V(X_j, x, n), \quad (3.1)$$

where, for Case 1, Case 2 and Case 4,

$$V(X_j, x, n) = \frac{\theta_0 - x}{u} \times \frac{K_0 \left( \frac{x - x_j}{u} \right)}{a(X_j)} - \frac{K_1 \left( \frac{x - x_j}{u} \right)}{a(X_j)}, \quad (3.2)$$

and for Case 3 and Case 5,

$$V(X_j, x, n) = - \frac{\theta_0 - x}{u} \times \frac{K_0 \left( \frac{x - x_j}{u} \right)}{a(X_j)} + \frac{K_1 \left( \frac{x - x_j}{u} \right)}{a(X_j)}. \quad (3.3)$$
Let $\bar{W}(x, n) = E[V(X_j, x, n)]$ and $Z_{jn} = V(X_j, x, n) - \bar{W}(x, n)$. Then we have

**Lemma 3.1** $\bar{W}(x, n)$ can be expressed as

$$\bar{W}(x, n) = W(x) + u^r W(x, n),$$  \hspace{1cm} (3.4)

where $W(x, n)$ is some function such that for all $x \in [d_n \land b_0, \theta_0]$,

$$|W(x, n)| \leq \frac{2\theta_0(r + 2) + 1}{(r + 2)!} BB_1 \equiv B_2.$$  \hspace{1cm} (3.5)

**Lemma 3.2** For $x \in [0, \theta_0]$, $|W(x)| \leq 2\theta_0 B$.

**Lemma 3.3** For any fixed $n$, $Z_{jn}$ are i.i.d., and for $x \in [d_n \land b_0, \theta_0]$, $EZ_{jn} = 0$,

$$\sigma^2_n \equiv EZ^2_{jn} \leq \frac{B_3^2}{ua^*(x)},$$  \hspace{1cm} (3.6)

and

$$\frac{E|Z_{jn}|^3}{\sigma^2_n} \leq 2\theta_0 B + B_2 + \frac{2(\theta_0 B_1 + 1)}{ua^*(x)},$$  \hspace{1cm} (3.7)

where $B_3 = (2\theta_0 + 1)B_1\sqrt{B}$, and

$$a^*(x) = \begin{cases} \min\{a(s) : x \leq s \leq x + u\} & \text{for Cases 1, 2, 4}, \\ \min\{a(s) : x - u \leq s \leq x\} & \text{for Cases 3, 5}. \end{cases}$$  \hspace{1cm} (3.8)

From Lemma 3.3, we see that

$$P(W_n(x) > 0) = P(\frac{1}{\sqrt{nEZ^2_{jn}}} \sum_{j=1}^{n} Z_{jn} > -\sqrt{n\sigma_n^{-2}\bar{W}(x, n)}),$$  \hspace{1cm} (3.9)

and

$$P(W_n(x) \leq 0) = P(\frac{1}{\sqrt{nEZ^2_{jn}}} \sum_{j=1}^{n} Z_{jn} \leq -\sqrt{n\sigma_n^{-2}\bar{W}(x, n)}).$$  \hspace{1cm} (3.10)

Based on Lemma 3.1, we obtain the following useful result:

**Lemma 3.4** For $x \in [d_n \land b_0, \theta_0]$,

$$W(x) > 2B_2u^r \implies \bar{W}(x, n) \geq 0 \text{ and } \frac{W(x)}{\bar{W}(x, n)} \leq 2,$$  \hspace{1cm} (3.11)
and
\[
W(x) < -2B_2u^r \implies W(x, n) \leq 0 \text{ and } \left| \frac{W(x)}{W(x, n)} \right| \leq 2. \tag{3.12}
\]

Lemma 3.4 allows us to replace \( W(x, n) \) with \( W(x) \) in (3.9) and (3.10). That makes things a little easier since we will see that \( W(x) \) does not depend on \( n \) and has a few good properties.

The above four hold for any \( G \in G_1 \) or \( G_2 \). Note the bounds in above lemmas do not depend on \( G \). So they are the uniform bounds over \( G_1 \) and \( G_2 \). Next two lemmas give the results related to some \( G \in G_1 \), which will be used only when we consider \( \delta_n \).

**Lemma 3.5** For any \( G \in \mathcal{G}_1 \), there exist \( c_{G_1} \) and \( c_{G_2} \) such that \( 0 < c_{G_1} < b_G < c_{G_2} < \theta_0 \), and for all \( x \in [c_{G_1}, c_{G_2}] \),
\[
|W'(x)| \geq \frac{G'(b_G)}{2A(b_G)}(\theta_0 - b_G) \equiv B_{G_1}^{-1}, \tag{3.13}
\]
for all \( x \in [0, c_{G_1}] \cup [c_{G_2}, \theta_0] \),
\[
|W(x)| \geq W(c_{G_1}) \wedge |W(c_{G_2})| \equiv B_{G_2}. \tag{3.14}
\]

**Lemma 3.6** For any \( G \in \mathcal{G}_1 \), let \( U_G = \max\{a(x) : c_{G_1} \leq x \leq c_{G_2}\} \). Then there exists an integer \( N_1(>N_0) \) such that for \( n > N_1 \),
\[
\int_{d_n}^{\theta_0} I_{[|W(x)| \leq 2B_2u^r]} a(x) dx \leq 4B_2B_{G_1}U_Gu^r. \tag{3.15}
\]
Let \( L_G = \min\{a(x) : \frac{c_{G_1}}{2} \leq x \leq c_{G_2} + \frac{\theta_0 - c_{G_2}}{2}\} \). Then there exists an integer \( N_2(>N_1) \) such that for \( n > N_1 \),
\[
a^*(x) \begin{cases} 
\geq L_G & \text{for } x \in [c_{G_1}, c_{G_2}], \\
> \eta_n & \text{for } x \in [d_n, d_n \vee c_{G_1}] \cup [c_{G_2}, \theta_0].
\end{cases} \tag{3.16}
\]

When we consider the uniform rate of convergence of \( [R(G, \delta_n) - R(G, \delta)] \), we need the following two lemmas.

**Lemma 3.7** For all \( G \in \mathcal{G}_2 \) and all \( x \in [0 \vee (b_G - \xi), (b_G + \xi) \wedge \theta_0] \),
\[
|W'(x)| \geq L, \tag{3.17}
\]

9
for all \( G \in G_2 \) and all \( x \in [0, 0 \vee (b_G - \xi)] \cup [(b_G + \xi) \wedge \theta_0, \theta_0] \)

\[ |W(x)| \geq L\xi. \tag{3.18} \]

**Lemma 3.8** Let \( U_a = \max\{a(x) : b_0 \leq x \leq \theta_0\} \). Then there exists an integer \( \bar{N}_1 \)

such that for all \( n > \bar{N}_1 \),

\[ \int_{b_0}^{\theta_0} I_{|W(x)| \leq 2B_2u^*} a(x) dx \leq \frac{4B_2U_a}{L} u^*. \tag{3.19} \]

Let \( L_a = \min\{a(x) : \frac{b_0}{2} \leq x \leq \theta_0 + 1\} \). Then there exists an integer \( \bar{N}_2 > \bar{N}_1 \)

such that for all \( n > \bar{N}_2 \) and \( x \in [b_0, \theta_0] \)

\[ a^*(x) \geq L_a > 0. \tag{3.20} \]

Next we state a general well-known result. It is about the non-uniform estimate of

the distance between the distribution of a sum of i.i.d. random variables and the normal distribution.

**Result** Let \( X_1, X_2, \ldots, X_n \) be i.i.d random variables, \( EX_1 = 0, EX_1^2 = \sigma^2 > 0, E|X_1|^3 < \infty \). Then for all \( x \)

\[ |F_n(x) - \Psi(x)| \leq A \frac{\rho}{\sqrt{n}(1 + |x|)^3}. \]

Here \( \Psi(x) \) is the c.d.f. of \( N(0, 1) \), \( F_n(x) \) and \( \rho \) are given by

\[ F_n(x) = P\left( \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} X_j \leq x \right), \quad \rho = \frac{E|X_1|^3}{\sigma^3}. \]

**Remark** The above result can be found in Petrov (1975, pp125 Theorem 14) or

Michel (1981). Here \( A \) is independent of \( n \). Michel proved \( A < 30.54 \).

From the above result, we see that, for any fixed \( n \),

if \( x > 0 \),

\[ P\left( \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} X_j > x \right) \tag{3.21} \]

\[ = 1 - F_n(x) \]

\[ \leq 1 - \Psi(x) + A \frac{\rho}{\sqrt{n}(1 + |x|)^3} \]

\[ \leq 1 - \Psi(x) + A \frac{E|X_1|^3}{\sqrt{n}|x|\sigma^3}, \]

10
if $x < 0$,

$$P\left( \frac{1}{\sigma \sqrt{n}} \sum_{j=1}^{n} X_j \leq x \right) = F_n(x) \leq \Psi(x) + A \frac{\rho}{\sqrt{n} (1 + |x|)^3} \leq \Psi(x) + A \frac{E|X_1|^3}{\sqrt{n}|x|^3}. \quad (3.22)$$

Now, we prove our main results. The first one is related to $\delta_n$:

**Theorem 3.9** Let $u = u(n) = n^{-\frac{1}{2r+1}}$ for Cases 1, 2, 3, 5 and $n^{-\frac{1}{2r+1}} \epsilon_n^{-\frac{1}{2r}}$ for Case 4. Then we have, for every $G \in G_1$, as $n \to \infty$, for Case 1, Case 2, Case 3, Case 5,

$$\lim_{n \to \infty} n^{\frac{2r}{2r+1}} \epsilon_n [R(G, \delta_n) - R(G, \delta)] < 16B_2^2 B_G U_G + 2B_G < \infty, \quad (3.23.1)$$

and for Case 4,

$$\lim_{n \to \infty} n^{\frac{2r}{2r+1}} \epsilon_n [R(G, \delta_n) - R(G, \delta)] < 16B_2^2 B_G U_G + 2B_G < \infty, \quad (3.23.2)$$

where

$$B_G = \begin{cases} 
2\theta_0 B A(\theta_0) + \frac{2B_0 B_G^2 U_G}{L_G} + 2A(2\theta_0 B + B_2 + \frac{2\theta_0 B_1 + 1}{a_{\min}}) A(\theta_0) & \text{for Cases 1, 5,} \\
2\theta_0 B A(\theta_0) + \frac{2B_0 B_G^2 U_G}{L_G} + 2A[(2\theta_0 B + B_2) A(\theta_0) + (2\theta_0 B_1 + 1) \theta_0] & \text{for Cases 2, 3,} \\
2\theta_0 B A(\theta_0) + \frac{2B_0 B_G^2 U_G}{L_G} + 2A(2\theta_0 B + B_2 + 2\theta_0 B_1 + 1) A(\theta_0) & \text{for Case 4,}
\end{cases}$$

and $a_{\min} = \min\{a(x) : 0 \leq x \leq \theta_0\} > 0$ in Cases 1, 5.

**Proof.** From (2.8),

$$R(G, \delta_n) - R(G, \delta) = \int_{\text{dn}}^{b_G} P_n(W_n(x) \leq 0) W(x) a(x) I_{[0 < W(x) \leq 2B_2 u^r]} dx \quad (3.24)$$

$$+ \int_{\text{dn}}^{b_G} P_n(W_n(x) > 0) |W(x)| a(x) I_{[-2B_2 u^r < W(x) < 0]} dx$$

$$+ \int_{\text{dn}}^{b_G} P_n(W_n(x) \leq 0) W(x) a(x) I_{[W(x) > 2B_2 u^r]} dx$$

$$+ \int_{\text{dn}}^{b_G} P_n(W_n(x) > 0) |W(x)| a(x) I_{[W(x) < -2B_2 u^r]} dx$$

$$\equiv I + II + III + IV.$$

Part I and Part II are easy to handle, since we have Lemma 3.6. Using (3.15), we have, as $n > N_1$,

$$I \leq 2B_2 u^r \int_{\text{dn}}^{b_G} I_{[0 < W(x) \leq 2B_2 u^r]} a(x) dx \leq 8B_2^2 B_G U_G u^{2r}, \quad (3.25)$$
and
\[ II \leq 2B_2u^* \int_{-2B_2u^*}^{B_2u^*} I_{[-2B_2u^*W(x)<0]} a(x)dx \leq 8B_2^2B_{G1}U_Gu^2. \] (3.26)

Part III and Part IV are a little more complicated. We treat Part III first. Using (3.9), (3.11) and (3.22), we have
\[ III \leq \int_{d_n}^{b_G} P\left(\frac{1}{\sqrt{nE}Z_{jn}^2} \sum_{j=1}^{n} Z_{jn} \leq -\sqrt{n\sigma_n^{-2}W(x,n)}I_{W(x)>2B_2u^*}W(x)a(x)dx \right) \] (3.27)
\[ \leq \int_{d_n}^{b_G} P\left(\frac{1}{\sqrt{nE}Z_{jn}^2} \sum_{j=1}^{n} Z_{jn} \leq -\frac{1}{2}\sqrt{n\sigma_n^{-2}W(x)}I_{W(x)>2B_2u^*}W(x)a(x)dx \right) \]
\[ \leq \int_{d_n}^{b_G} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}W(x)}W(x)a(x)dx + \int_{d_n}^{b_G} \frac{AE|Z_{jn}|^3W(x)a(x)dx}{\sqrt{n} \times | -\frac{1}{2}\sqrt{n\sigma_n^{-2}W(x)}| \times \sigma_n^3} \right) \]
\[ \leq \int_{d_n}^{c_{G1}vd_n} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}W(x)}W(x)a(x)dx \right) \]
\[ + \int_{c_{G1}vd_n}^{b_G} \Psi\left(-\frac{1}{2}\sqrt{n\sigma_n^{-2}W(x)}W(x)a(x)dx \right) \]
\[ + \int_{d_n}^{b_G} \frac{2A}{n} \times \frac{E|Z_{jn}|^3}{\sigma_n^2} a(x)dx \]
\[ \equiv V + VI + VII. \]

Using (3.6), (3.16) and (3.14), we have
\[ V \leq \int_{d_n}^{c_{G1}vd_n} \Phi\left(-\frac{1}{2}\sqrt{n\sqrt{nu}} \frac{B_G}{B_3}W(x)a(x)dx \right) \]
\[ = \Phi\left(-\frac{B_G}{2B_3}\sqrt{nu\eta_n}\right) \int_{d_n}^{c_{G1}vd_n} W(x)a(x)dx \]
\[ \leq \Phi\left(-\frac{B_G}{2B_3}\sqrt{nu\eta_n}\right) \int_{d_n}^{b_G} W(x)a(x)dx \]
\[ \leq \Phi\left(-\frac{B_G}{2B_3}\sqrt{nu\eta_n}\right) \times 2\theta_0BA(\theta_0). \]

Since \(\sqrt{nu\eta_n} > \log n\), there exists an integer \(N_3 > N_2\) such that for \(n > N_3\), \(\Phi\left(-\frac{B_G}{2B_3}\sqrt{nu\eta_n}\right) \leq \frac{1}{n}\) and
\[ V \leq \frac{2\theta_0BA(\theta_0)}{n} \leq \frac{2\theta_0BA(\theta_0)}{nu}. \] (3.28)

As for Part VI, using (3.6) and (3.16), and making change of variable \(y = \frac{1}{2B_3}\sqrt{nuL_G}W(x)\), we obtain
\[ VI \leq \int_{c_{G1}vd_n}^{b_G} \Phi\left(-\frac{1}{2B_3}\sqrt{nu\eta_n}(x)W(x))W(x)a(x)dx \right) \] (3.29)
\[
\begin{align*}
\leq & \int_{cG1v_d}^{bG} U_G \Phi \left( -\frac{1}{2B_3} \sqrt{nuL_G W(x)} \right) W(x) dx \\
\leq & -B_{G1} U_G \int_{cG1v_d}^{bG} \Phi \left( -\frac{1}{2B_3} \sqrt{nuL_G W(x)} \right) W(x) W'(x) dx \\
\leq & \frac{4B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu} \int_0^{\frac{1}{B_3} \sqrt{nuL_G W(cG1v_d)}} \Phi(-y) y dy \\
\leq & \frac{4B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu} \int_0^{\infty} \Phi(-y) y dy \\
\leq & \frac{2B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu}.
\end{align*}
\]

We consider VII in different cases. For Case 1 and Case 5,

\[
\begin{align*}
VII & \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{d_n}^{bG} \frac{2\theta_0 B_1 + 1}{u a(x)} a(x) dx \\
& \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 B_1 + 1)A(\theta_0)}{a_{\min} nu}.
\end{align*}
\]

For Case 2 and Case 3,

\[
\begin{align*}
VII & \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{d_n}^{bG} \frac{2\theta_0 B_1 + 1}{u a(x)} a(x) dx \\
& \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 B_1 + 1)B_1 \theta_0}{nu}.
\end{align*}
\]

For Case 4,

\[
\begin{align*}
VII & \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{d_n}^{bG} \frac{2\theta_0 B_1 + 1}{u \eta_m} a(x) dx \\
& \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + 2A(2\theta_0 B_1 + 1) \times A(\theta_0) \times \frac{1}{nu \eta_m}.
\end{align*}
\]

Combining (3.28), (3.29) and (3.30), we get that when \( n \geq N_3 \),

\[
III \leq \begin{cases} 
B_G \times \frac{1}{nu} \quad \text{for Cases 1, 2, 3, 5,} \\
B_G \times \frac{1}{nu \eta_m} \quad \text{for Case 4.}
\end{cases}
\]

Now we deal with IV. Similar to III, we get

\[
\begin{align*}
IV & \leq \int_{bG}^{t_0} P \left( \frac{1}{\sqrt{n EZ_{jn}^2}} \sum_{j=1}^{n} Z_{jn} \geq -\sqrt{n \sigma_n^{-2} W(x, n)} I_{[W(x) < -2B_2 \mu_2]} W(x) |a(x)| dx \right) \\
& \leq \int_{bG}^{t_0} P \left( \frac{1}{\sqrt{n EZ_{jn}^2}} \sum_{j=1}^{n} Z_{jn} \geq -\frac{1}{2} \sqrt{n \sigma_n^{-2} W(x)} I_{[W(x) < -2B_2 \mu_2]} W(x) |a(x)| dx \right)
\end{align*}
\]

13
\[
\leq \int_{b_G}^{\theta_0} \left[ 1 - \Psi \left( \frac{1}{2} \sqrt{n \sigma_n^{-2}} |W(x)| \right) \right] |W(x)| a(x) \, dx + \int_{b_G}^{\theta_0} \frac{AE|Z_{jn}|^3 |W(x)| a(x) \, dx}{\sqrt{n} \times \frac{1}{2} \sqrt{n \sigma_n^{-2}} |W(x)| \times \sigma_n^3} \\
\leq \int_{c_{G2}}^{\theta_0} \left[ 1 - \Psi \left( \frac{1}{2} \sqrt{n \sigma_n^{-2}} |W(x)| \right) \right] |W(x)| a(x) \, dx \\
+ \int_{b_G}^{c_{G2}} \left[ 1 - \Psi \left( -\frac{1}{2} \sqrt{n \sigma_n^{-2}} |W(x)| \right) \right] |W(x)| a(x) \, dx \\
+ \int_{b_G}^{\theta_0} \frac{2A}{n} \times \frac{E|Z_{jn}|^3}{\sigma_n^2} a(x) \, dx \\
\equiv VIII + IX + X.
\]

Using (3.6), (3.16) and (3.14), we have

\[
VIII \leq \int_{c_{G2}}^{\theta_0} \left[ 1 - \Phi \left( \frac{1}{2} \sqrt{n \sqrt{\eta_n} B_3} \right) \right] |W(x)| a(x) \, dx \\
= \left[ 1 - \Phi \left( \frac{B_{G2}}{2B_3} \sqrt{n \eta_n} \right) \right] \int_{c_{G2}}^{\theta_0} |W(x)| a(x) \, dx \\
\leq \left[ 1 - \Phi \left( \frac{B_{G2}}{2B_3} \sqrt{n \eta_n} \right) \right] \int_{b_G}^{\theta_0} |W(x)| a(x) \, dx \\
\leq \left[ 1 - \Phi \left( \frac{B_{G2}}{2B_3} \sqrt{n \eta_n} \right) \right] \times 2\theta_0 B A(\theta_0).
\]

By the symmetric property of \( \Psi(x) \), for \( n > N_3, 1 - \Phi \left( \frac{B_{G2}}{2B_3} \sqrt{n \eta_n} \right) \leq \frac{1}{n} \) and

\[
VIII \leq \frac{2\theta_0 B A(\theta_0)}{n} \leq \frac{B_0}{nu}.
\] (3.33)

Similar to VII, we obtain

\[
IX \leq \int_{b_G}^{c_{G2}} \left[ 1 - \Phi \left( -\frac{1}{2B_3} \sqrt{nu a^*(x) W(x)} \right) \right] |W(x)| a(x) \, dx \\
\leq \int_{b_G}^{c_{G2}} U_G \left[ 1 - \Phi \left( -\frac{1}{2B_3} \sqrt{nu L_G W(x)} \right) \right] |W(x)| \, dx \\
\leq -B_{G1} U_G \int_{b_G}^{c_{G2}} \left[ 1 - \Phi \left( -\frac{1}{2B_3} \sqrt{nu L_G W(x)} \right) \right] |W(x)| W'(x) \, dx \\
\leq \frac{4B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu} \int_{b_G}^{c_{G2}} \left[ 1 - \Phi(y) \right] y \, dy \\
\leq \frac{4B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu} \int_{0}^{\infty} \left[ 1 - \Phi(y) \right] y \, dy \\
\leq \frac{2B_{G1} B_3^2 U_G}{L_G} \times \frac{1}{nu}.
\] (3.34)

We consider Part VII in different cases. Obviously, for Case 1 and Case 5,

\[
X \leq \frac{2 A(2\theta_0 B + B_2) A(\theta_0)}{nu} + \frac{2 A}{n} \int_{b_G}^{\theta_0} \frac{2\theta_0 B_1 + 1}{u a^*(x)} a(x) \, dx
\] (3.35.1)

14
\[
X \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 B_1 + 1)A(\theta_0)}{a_{\min}nu},
\]
where \(a_{\min} = \min\{a(x) : 0 \leq x \leq \theta_0\}\). For Case 2 and Case 3,

\[
X \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{\theta_0}^{\theta_1} \frac{2\theta_0 B_1 + 1}{ua(x)} a(x) \, dx \quad (3.35.2)
\]

\[
\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A(2\theta_0 B_1 + 1)\theta_0}{nu}.
\]

For Case 4,

\[
X \leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + \frac{2A}{n} \int_{\theta_0}^{\theta_1} \frac{2\theta_0 B_1 + 1}{u\eta_n} a(x) \, dx \quad (3.35.3)
\]

\[
\leq \frac{2A(2\theta_0 B + B_2)A(\theta_0)}{nu} + 2A(2\theta_0 B_1 + 1) A(\theta_0) \times \frac{1}{nu \eta_n}.
\]

Combining (3.33), (3.34) and (3.35), we get that when \(n \geq N_3\),

\[
IV \leq \left\{ \begin{array}{ll}
B_G \times \frac{1}{nu} & \text{for Cases 1, 2, 3, 5,} \\
B_G \times \frac{1}{nu \eta_n} & \text{for Case 4.}
\end{array} \right. \quad (3.36)
\]

From (3.25), (3.26), (3.31) and (3.36), we get, for \(n > N_3\),

\[
R(G, \delta_n) - R(G, \delta) \leq \left\{ \begin{array}{ll}
16B_2^2B_GU_G \times u^{2r} + 2B_G \times \frac{1}{nu} & \text{for Cases 1, 2, 3, 5,} \\
16B_2^2B_GU_G \times u^{2r} + 2B_G \times \frac{1}{nu \eta_n} & \text{for Case 4,}
\end{array} \right. \quad (3.37)
\]

For Cases 1, 2, 3, 5, \(u = n^{-\frac{1}{2r+1}}\). So \(u^{2r} = \frac{1}{nu} = n^{-\frac{2r}{2r+1}}\), and

\[
\lim_{n \to \infty} n^{\frac{2r}{2r+1}} [R(G, \delta_n) - R(G, \delta)] \leq 16B_2^2B_GU_G + 2B_G < \infty.
\]

For Case 4, we have \(u = n^{-\frac{1}{2r+1}} \epsilon_n^{\frac{1}{2r}}, \eta_n = \epsilon_n^{2r} \). So \(u^{2r} = \frac{1}{nu \eta_n} = n^{-\frac{2r}{2r+1}} \epsilon_n^{-1}\), and

\[
\lim_{n \to \infty} n^{\frac{2r}{2r+1}} \epsilon_n [R(G, \delta_n) - R(G, \delta)] \leq 16B_2^2B_GU_G + 2B_G < \infty.
\]

The proof is completed.

Next, we consider the uniform rate of convergence of \(R(G, \overline{\delta}_n) - R(G, \delta)\).

**Theorem 3.10** Take \(u_n = n^{-\frac{1}{2r+1}}\). Then we have, as \(n \to \infty\), for Case 1, Case 2, Case 3, Case 4, Case 5,

\[
\lim_{n \to \infty} n^{\frac{2r}{2r+1}} \sup_{G \in \Theta_2} [R(G, \overline{\delta}_n) - R(G, \delta)] < \frac{4B_2^2U_a}{L} + 2B_a < \infty. \quad (3.38)
\]

15
Proof. In this proof, let \( c_{G_1} = 0 \vee (b_G - \xi) \), \( c_{G_2} = (b_G + \xi) \wedge \theta_0 \). For any \( G \in \mathcal{G} \), like (3.24), we have

\[
R(G, \delta_n) - R(G, \delta) = \int_{b_G}^{b_G} P_n(W_n(x) \leq 0)W(x)a(x)I_{[0 < W(x) \leq 2B_2u^*]}dx + \int_{b_G}^{b_G} P_n(W_n(x) > 0)|W(x)|a(x)I_{[W(x) \leq 2B_2u^* < 0]}dx + \int_{b_G}^{b_G} P_n(W_n(x) \leq 0)W(x)a(x)I_{[W(x) > 2B_2u^*]}dx + \int_{b_G}^{b_G} P_n(W_n(x) > 0)|W(x)|a(x)I_{[W(x) < -2B_2u^*]}dx
\]

\[
\equiv I + II + III + IV.
\]

Using (3.19), we have, as \( n > \bar{N}_1 \),

\[
I \leq 2B_2u^* \int_{b_G}^{b_G} I_{[0 < W(x) \leq 2B_2u^*]}a(x)dx \leq \frac{8B_2^2U_0}{L}u^{2\gamma},
\]

and

\[
II \leq 2B_2u^* \int_{b_G}^{b_G} I_{[-2B_2u^* < W(x) < 0]}a(x)dx \leq \frac{8B_2^2U_0}{L}u^{2\gamma}.
\]

Similar to (3.27)

\[
III \leq \int_{b_G}^{b_G} \Psi(-\frac{1}{2}\sqrt{n}\sigma_n^2W(x))W(x)a(x)dx + \int_{c_{G_1} \vee b_0}^{b_G} \Psi(-\frac{1}{2}\sqrt{n}\sigma_n^2W(x))W(x)a(x)dx + \int_{b_G}^{b_G} \frac{2A}{n} \times \frac{B|Z_{jn}|^3}{\sigma_n^3}a(x)dx
\]

\[
\equiv V + VI + VII.
\]

Consider Part V first. If \( b_0 < c_{G_1} \),

\[
V \leq \int_{b_0}^{c_{G_1} \vee b_0} \Phi\left(-\frac{1}{2}\sqrt{n}\frac{\sqrt{uL_0}}{B_3} L\xi\right)W(x)a(x)dx
\]

\[
= \Phi\left(-\frac{L\xi}{2B_3\sqrt{nuL_0}}\right) \int_{b_0}^{c_{G_1} \vee b_0} W(x)a(x)dx
\]

\[
\leq \Phi\left(-\frac{L\xi}{2B_3\sqrt{nuL_0}}\right) \int_{0}^{\theta_0} W(x)a(x)dx
\]

\[
\leq \Phi\left(-\frac{L\xi}{2B_3\sqrt{nuL_0}}\right)2\theta_0 BA(\theta_0).
\]
Since $\sqrt{nuL_a} > \log n$, there exists an integer $\overline{N}_3 (> \overline{N}_2)$ such that for $n > \overline{N}_3$, $\Phi(-\frac{k}{2B_3} \sqrt{nuL_a}) \leq \frac{1}{n}$ and

$$V \leq \frac{2\theta_0 B}{n} A(\theta_0).$$

(3.42)

Now consider VI.

$$VI \leq \int_{c_1 \sqrt{v_b}}^{b_3} \Phi(-\frac{1}{2B_3} \sqrt{nuL_a} W(x)) W(x)a(x) dx$$

(3.43)

$$\leq U_a \int_{c_1 \sqrt{v_b}}^{b_3} \Phi(-\frac{1}{2B_3} \sqrt{nuL_a} W(x)) W(x) dx$$

$$\leq -L^{-1} U_a \int_{c_1 \sqrt{v_b}}^{b_3} \Phi(-\frac{1}{2B_3} \sqrt{nuL_a} W(x)) W(x) W'(x) dx$$

$$\leq \frac{B_3^2 U_a}{L L_a} \times \frac{1}{nu} \int_0^{\frac{1}{2B_3} \sqrt{nuL_a} W(c_1 \sqrt{v_b})} \Phi(-y) dy$$

$$\leq \frac{4B_3^2 U_a}{L L_a} \times \frac{1}{nu} \int_0^\infty \Phi(-y) dy$$

$$\leq \frac{2B_3^2 U_a}{L L_a} \times \frac{1}{nu}.$$

For VII, using (3.7) and (3.20), we have

$$VII \leq \frac{2A}{n} (2\theta_0 B + B_2) A(\theta_0) \times \frac{1}{n} + \frac{4A\theta_0 B_1 A(\theta_0)}{L_a} \times \frac{1}{nu}.$$  

(3.44)

Combining (3.42), (3.43) and (3.44), we get that when $n \geq \overline{N}_3$,

$$III \leq B_a \times \frac{1}{nu},$$

where

$$B_a = 2\theta_0 BA(\theta_0) + \frac{2B_3^2 U_a}{L L_a} + 2A(2\theta_0 B + B_2) A(\theta_0) + \frac{4A\theta_0 B_1 A(\theta_0)}{L_a}.$$  

Similarly, we have

$$IV \leq B_a \times \frac{1}{nu}. \quad (3.45)$$

From (3.40), (3.41), (3.44) and (3.45), we get, for $n > \overline{N}_3$,

$$R(G, \delta_n) - R(G, \delta) \leq \frac{16B_3^2 U_a}{L} \times u^{2r} + 2B_a \times \frac{1}{nu}$$

Since $u = n^{-\frac{1}{2r+1}}$, $u^{2r} = \frac{1}{nu} = n^{-\frac{2r}{2 r+1}}$. When $n > \overline{N}_3$,

$$n^{\frac{2r}{2 r+1}} \sup_{G \in \mathcal{G}_2} [R(G, \delta_n) - R(G, \delta)] \leq \frac{16B_3^2 U_a}{L} + 2B_a.$$
This completes the proof.

**Remark** For Case 1, \( b_0 = 0 \). Since (1.10) implies \( b_G > b_0 = 0 \), we have

\[
\mathcal{G}_2 = \{ G : \text{G satisfies (1.10) and (1.11), } \min_{x \in N_{k_0}(\varepsilon)} |W(x)| \geq L \}.
\]

Theorem 3.10 tells us that

\[
\lim_{n \to \infty} n^{2r_1+1} \sup_{G \in \mathcal{G}_2} [R(G, \delta_n) - R(G, \delta)] \leq \frac{16B^2U_a}{L} + 2B_a,
\]

which is a result obtained by Karunamuni (1999). But we do not require in \( \mathcal{G}_2 \) that \( b_G \) falls in \([0, \rho_0]\), where \( \rho_0 < \theta_0 \) is a known constant.

§ 4 Proofs of Lemmas

**Proof of Lemma 3.1** From their definitions, \( \alpha_G(x) \) and \( \psi_G(x) \) are monotone functions. Thus \( \alpha_G(x) < \infty \) and \( \psi_G(x) < \infty \) for \( x > 0 \) by (1.4) and (1.5). Since \( \alpha_G(0) < B \), from (1.11), \( \alpha_G(0) < \infty \), and \( \psi_G(0) \leq \int_0^1 \frac{\theta dG(\theta)}{A(\theta)} + \int_1^\infty \frac{\theta dG(\theta)}{A(\theta)} \leq \alpha_G(0) + \psi_G(1) < \infty \). So we have \( \alpha_G(x) < \infty \) and \( \psi_G(x) < \infty \) for all \( x \geq 0 \). Note that

\[
\int_x^\infty \alpha_G(s)ds = \int_0^\infty I_{[s>x]} \int_0^s I_{[\theta>x]} \frac{dG(\theta)}{A(\theta)} ds
\]

\[
= \int_0^\infty I_{[\theta>x]} (\theta - x) \frac{dG(\theta)}{A(\theta)}
\]

\[
= \int_x^\infty (\theta - x) \frac{dG(\theta)}{A(\theta)}
\]

\[
= \psi_G(x) - x\alpha_G(x).
\]

Then

\[
W(x) = (\theta_0 - x)\alpha_G(x) - \int_x^\infty \alpha_G(s)ds.
\]  

In Cases 1, 2, 4. Using Taylor's Theorem, (2.1) and (2.2), a straight-forward computation shows that

\[
E\left[ \frac{K_0\left( \frac{X_j-x}{u} \right)}{ua(X_j)} \right]
\]

\[
= \int_0^\infty \int_0^\infty K_0\left( \frac{y-x}{u} \right) \frac{a(y)}{A(\theta)} I_{[y<\theta]} dy dG(\theta)
\]

\[
= \int_0^\infty \int_0^\infty \frac{1}{u} K_0\left( \frac{y-x}{u} \right) \frac{1}{A(\theta)} I_{[y<\theta]} dy dG(\theta)
\]

18
\[
\int_0^\infty \frac{1}{u} K_0 \left( \frac{y-x}{u} \right) \alpha_G(y) dy \\
= \int_0^1 K_0(t) \alpha_G(x+ut) dt \\
= \int_0^1 K_0(t) \alpha_G(x) dt + \int_0^1 K_0(t) ut \alpha_G^{(1)}(x) dt + \cdots + \int_0^1 K_0(t) \frac{ut^r}{r!} \alpha_G^{(r)}(x+ut^*_t) dt \\
= \alpha_G(x) + u^r \times \frac{1}{r!} \int_0^1 K_0(t) \frac{t^r}{r!} \alpha_G^{(r)}(x+ut^*_t) dt,
\]
where \(0 \leq t^*_t \leq 1\). Also,

\[
E \left[ \frac{K_1 \left( \frac{x-y}{u} \right)}{a(X)} \right] \\
= \int_0^\infty \int_0^\infty K_1 \left( \frac{y-x}{u} \right) \frac{a(y)}{A(\theta)} I_{\{0 < \theta < \theta\}} dy dG(\theta) \\
= \int_0^\infty \int_0^\infty K_1 \left( \frac{y-x}{u} \right) \frac{1}{A(\theta)} I_{\{0 < \theta < \theta\}} dy dG(\theta) \\
= \int_0^\infty \int_0^\infty K_1 \left( \frac{y-x}{u} \right) \alpha_G(y) dy \\
= -\left[ K_1 \left( \frac{y-x}{u} \right) \int_y^\infty \alpha_G(s) ds \right]_{y=x}^\infty + \frac{1}{u} \int_0^\infty \int_y^\infty \alpha_G(s) ds K_0 \left( \frac{y-x}{u} \right) dy \\
= \int_0^\infty K_0(t) \left[ \int_{x+ut}^\infty \alpha_G(s) ds \right] dt \\
+ \int_0^\infty K_0(t) \frac{ut^r+1}{(r+1)!} \alpha_G^{(r)}(x+ut^*_t) dt \\
= \int_x^\infty \alpha_G(s) ds + u^r \times \frac{ut^r+1}{(r+1)!} \int_0^\infty K_0(t) \alpha_G^{(r)}(x+ut^*_t) dt,
\]
where \(0 \leq t^*_t \leq 1\). Let

\[
W(x, n) = \frac{(\theta_0 - x)}{r!} \int_0^1 t^{r+1} K_0(t) \alpha_G^{(r)}(x+ut^*_t) dt \\
- \frac{u}{(r+1)!} \int_0^1 t^{r+1} K_0(t) \alpha_G^{(r)}(x+ut^*_t) dt.
\]

By (3.2), (4.1), (4.3) and (4.4), we have

\[
\bar{W}(x, n) = W(x) + u^r W(x, n).
\]

Since \(\alpha_G^{(r)}(x) < B\) for \(x \in [0, \theta_0 + 1]\), \(x+ut^*_t \leq \theta_0 + 1\) and \(x+ut^*_t \leq \theta_0 + 1\) for \(x \in [d_n \wedge b_0, \theta_0]\),

\[
|W(x, n)| \leq \frac{2\theta_0}{(r+1)!} BB_1 + \frac{BB_1}{(r+2)!} \equiv B_2,
\]

19
for $x \in [d_n \wedge b_0, \theta_0]$. For Cases 3, 5, the similar computation shows that
\[
\bar{W}(x, n) = W(x) + u'W(x, n),
\]
and $|W(x, n)| \leq B_2$ for $x \in [d_n \wedge b_0, \theta_0]$.

**Proof of Lemma 3.2** By (1.10), $\psi_G(0) \leq \theta_0\alpha_G(0) \leq \theta_0 B$. Thus $\psi_G(x) \leq \theta_0 B$ for $x \in [0, \theta_0]$. Then we have
\[
|W(x)| = |\theta_0\alpha_G(x) - \psi_G(x)|
\leq \theta_0\alpha_G(x) + \psi_G(x)
\leq \theta_0\alpha_G(0) + \psi_G(0)
\leq 2\theta_0 B.
\]

**Proof of Lemma 3.3** Obviously, $Z_{jn}$ are i.i.d. for fixed $n$. A few computations show that, for Cases 1, 2, 4,
\[
\sigma_n^2 = EZ_{jn}^2
\leq E[V(X_j, x, n)]^2
\leq \int_0^\infty \int_0^\infty \left[ \frac{\theta_0 - x}{u} \frac{K_0(y - x)}{A(y)} - \frac{K_1(y - x)}{A(y)} \right]^2 a(y) \frac{A(y)}{A(\theta)} I_{[0 < y < \theta]} dy dG(\theta)
\leq \int_0^1 \int_0^\infty \left[ \frac{1}{u^2 a(y)} [(\theta_0 - x)K_0(\frac{y - x}{u}) - uK_1(\frac{y - x}{u})]^2 \alpha_G(x) \right] dy
\leq \int_0^1 \frac{1}{u a(x + ut)} [(\theta_0 - x)K_0(t) - uK_1(t)]^2 \alpha_G(x + ut) dt
\leq \frac{B}{ua^*(x)} (2\theta_0 B_1 + B_1)^2.
\]
Then (3.6) is proved. And also,
\[
|Z_{jn}| = \max_{v \geq 0} \left| \frac{\theta_0 - x}{u} \frac{K_0(\frac{y - x}{u})}{a(y)} - \frac{K_1(\frac{y - x}{u})}{a(y)} + |\bar{W}(x, n)| \right|
\leq \max_{0 \leq t \leq 1} \frac{1}{ua(x + ut)} [(\theta_0 - x)K_0(t) - uK_1(t)] + |W(x)| + B_2
\leq \frac{2\theta_0 B_1 + 1}{ua^*(x)} + 2\theta_0 B + B_2.
\]
For Cases 3, 5, the proof is similar. We omit it here.

**Proof of Lemma 3.4** From Lemma 3.1, $|W(x, n)| \leq B_2$ for all $x \in [d_n \wedge b_0, \theta_0]$. If $W(x) > 2B_2 u^r$,
\[
u^{-r} \bar{W}(x, n) = u^{-r}[W(x) + u'W(x, n)] = u^{-r}W(x) + W(x, n) > 2B_2 - B_2 = B_2 > 0
\]
\[ \frac{W(x)}{W(x, n)} = \frac{u^{-r}W(x)}{u^{-r}W(x) + u^r W(x, n)} \leq \frac{u^{-r}W(x) - 2B_2 + 2B_2}{u^{-r}W(x) - 2B_2 + B_2} \leq 2. \]

Then (3.14) is proved. (3.15) can be proved in a similar way.

**Proof of Lemma 3.5** From (1.11), we know \( G'(x) = g(x) \) exists. Then

\[ W'(x) = \frac{g(x)}{A(x)} (x - \theta_0). \]

Thus we have \( W'(x) \leq 0 \) for \( x \leq \theta_0 \). By (1.12), we know \( W'(b_G) < 0 \). Since \( W'(x) \) is continuous by (1.11), we can choose \( c_{G_1} \) and \( c_{G_2} \) such that \( c_{G_1} < b_G < c_{G_2} < \theta_0 \) and for all \( x \in [c_{G_1}, c_{G_2}] \),

\[ -W'(x) \geq \frac{-W'(b_G)}{2} = B_{G_1}^{-1}. \]

On the other side, let \( B_{G_2} = \min\{W(c_{G_1}), |W(c_{G_2})|\} \). We have

\[ |W(x)| \geq B_{G_2}. \]

**Proof of Lemma 3.6** Since \( u^r \to 0 \) as \( n \to \infty \). There exists an integer \( N_1(> N_0) \) such that for all \( n > N_1 \),

\[ 2B_2u^r < B_{G_2}. \]

Then \( |W(x)| \leq 2B_2u^r \) implies \( x \in [c_{G_1}, c_{G_2}] \). So, using change of variable \( y = \frac{W(x)}{2B_2u^r} \), we have

\[
\int_0^{\theta_0} I_{|W(x)| \leq 2B_2u^r} a(x)dx = \int_{c_{G_1}}^{c_{G_2}} I_{|[W(x)| \leq 2B_2u^r]} a(x)dx \leq U_G B_G \int_{c_{G_1}}^{c_{G_2}} I_{|[W(x)| \leq 2B_2u^r]} [-W'(x)]dx \leq 2U_G B_G B_2u^r \int_{\frac{W(c_{G_1})}{2B_2u^r}}^{\frac{W(c_{G_2})}{2B_2u^r}} I_{[y \leq 1]} dy \leq 4B_2B_1U_Gu^r. \]

(3.11) is proved. As for (3.12), we take \( N_2(> N_1) \) such that \( u < \frac{c_{G_1}}{2} \land \frac{\theta_0 - c_{G_2}}{2} \) and \( \eta_n \leq \min\{a(x) : 0 \leq x \leq \theta_0\} \) for cases 1, 3, 5. For \( x > d_n \), by our definition of \( d_n \), \( a^*(x) \geq \eta_n \) for Cases 2, 4 and \( a^*(x) \geq \min\{a(x) : 0 \leq x \leq \theta_0\} \geq \eta_n \) for Cases 1, 3, 5. For \( x \in [c_{G_1}, c_{G_2}] \), we have \( \frac{c_{G_1}}{2} < x - u < x + u < c_{G_2} + \frac{\theta_0 - c_{G_2}}{2} \). Thus \( a^*(x) \geq L_G \).
Proof of Lemma 3.7 (3.17) is from the definition of $G_2$. We prove (3.18) here. If $b_G - \xi > 0$, for $x \in [0,b_G - \xi]$, by the mean value theorem

$$W(b_G - \xi) - W(b_G) = -W'(x^*)\xi,$$

where $x^* \in [b_G - \xi,b_G]$. Note that $W(b_G) = 0$ and $|W'(x)| \geq L$. Thus $|W(b_G - \xi)| \geq L\xi$. If $b_G + \xi < \theta_0$, $W(b_G + \xi) - W(b_G) = W'(x^{**})\xi$, where $x^{**} \in [b_G,b_G + \xi]$. Thus $|W(b_G + \xi)| \geq L\xi$. So for $x \in [0,0 \vee (b_G - \xi)] \cup [(b_G + \xi) \wedge \theta_0, \theta_0]$, $|W(x)| \geq L\xi$.

Proof of Lemma 3.8

Since $u^r \to 0$ as $n \to \infty$. There exists an integer $\bar{N}_1$ such that for all $n > \bar{N}_1$,

$$2B_2u^r < L\xi.$$

Then $|W(x)| \leq 2B_2u^r$ implies $x \in [0 \vee (b_G - \xi), (b_G + \xi) \wedge \theta_0]$. So, using change of variable $y = \frac{W(x)}{2B_2u^r}$, we have

$$\int_0^{\theta_0} I_{[W(x) \leq 2B_2u^r]} a(x)dx = \int_{0 \vee (b_G - \xi)}^{(b_G + \xi) \wedge \theta_0} I_{[W(x) \leq 2B_2u^r]} a(x)dx$$

$$\leq U_\alpha L^{-1} \int_{0 \vee (b_G - \xi)}^{(b_G + \xi) \wedge \theta_0} I_{[W(x) \leq 2B_2u^r]} [-W'(x)]dx$$

$$\leq \frac{2U_\alpha B_2u^r}{L} \int_{\frac{W((b_G - \xi) \wedge \theta_0)}{2B_2u^r}}^{\frac{W(0 \vee (b_G - \xi))}{2B_2u^r}} I_{[y \leq 1]}dy$$

$$\leq \frac{4B_2U_\alpha}{L} u^r.$$

We prove (3.20) for different cases. For Case 1, $b_0 = 0$, $a^*(x) \geq L_\alpha$. For Cases 2, 3, 4, 5, we can find $\bar{N}_2(> \bar{N}_1)$ such that $u < \frac{b_0}{2} \wedge 1$. For $x \in [b_0, \theta_0]$, $\frac{b_0}{2} < x - u < x + u < \theta_0 + 1$. Thus $a^*(x) \geq L_\alpha$.

References
Theorem. Z.W., 55 109-117.
Empirical Bayes tests for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ in the distribution family having density $f(x|\theta) = I_{[0 \leq x \leq \theta]} a(x)/A(\theta)$ are considered under the linear loss. We construct the empirical Bayes tests $\delta_n$ and $\hat{\delta}_n$ and prove that both of them have a good asymptotic property. As a special case, taking $a(x) = 1$ leads to a result obtained by Karunamuni (1999). But our result gets rid of the condition that the critical point $b_G$ falls in some interval $[0, \rho_0]$, where $\rho_0 < \theta_0$ is a known constant.